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RECOVERY TIME OF A REPAIRABLE WARM STANDBY SYSTEM

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Abstract: We analyse the recovery time of a duplex system characterized by warm standby and attended by two general heterogeneous repairmen. In order to describe the random behaviour of the system, we introduce a stochastic process endowed with time-dependent transition measures satisfying coupled partial differential equations. The solution procedure is based on the theory of sectionally holomorphic functions combined with the notion of dual transforms. As a non-standard example, we consider the case of Weibull-Gnedenko repair.

AMS Subject Classification: 60K10
Key Words: standby, **T**-System, recovery time, sectionally holomorphic function, dual transform, Laplace transform, Weibull-Gnedenko repair

1. Introduction

Standby provides a powerful tool to enhance the reliability, availability, quality and safety of operational plants, e.g. Birolini (2007), Epstein and Weissmann (2008), Gnedenko and Ushakov (1995), Ushakov (2012). A frequently employed standby mode is the so-called "cold" standby. The notion of cold standby signifies that a back-up system of an active (online) unit is kept in reserve, with a zero failure rate, until the online unit fails, e.g. Moghaddas and Zuo (2012), Vanderperre and Makhanov (2014a), Wu (2012). An alternative cold standby mode frequently occurs in the management of robot-safety device systems to

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prolong the lifetime of the safety unit, i.e. upon failure of the robot, the safety device is shut off and kept in cold standby until the repair of the robot has been completed, e.g. Vanderperre and Makhanov (2015). Complex engineering systems are often characterized by active redundancy, called "hot" standby. The notion of hot standby means that the failure-free time of the unit in standby has the same distribution as the failure-free time in the operative state. A variant of the classical multiple hot standby system has been introduced by Ozaki, Kara and Cheng (2012). Their results could be applied to the design of integrated circuits consisting of redundant back-up (unrepairable) components. Another practical standby mode is the so-called "warm" standby. The notion of warm standby signifies that the failure-free time of the unit in standby is stochastically larger, see Shaked and Shanthikumar (1990, page 666, Definition 3.1), than the failure-free time in the operative state, e.g. Ruiz-Castro and Fernandez-Villodre (2012). The results obtained by Amari, Pham and Misra (2012) provide a new insight into the warm standby mode. Note that warm (hot) standby is often indispensable to implement a fast (automatic) replacement of the failed online unit. The case of imperfect switching has been considered by Yun and Cha (2010), Yuang and Meng (2012). Industrial applications of warm standby systems have been cited by Shao and Lamberson (1988). In addition, Kim et al. (2012) deal with applications of cold, warm and hot standby redundancy in satellite systems. A particular variant of Birolini's duplex system, Birolini (2007, page 200, chapter 6.4.3), has been introduced by Vanderperre (2008). The modified system (henceforth called the **T**-system) consists of a duplex configuration composed of an active unit (the **o**-unit) sustained by an identical unit in warm standby (the **s**-unit). The **T**-system is supervised by two general heterogeneous repairmen R and R_s. Repairman R is an expert in repairing **o**-failures, whereas repairman R_s is supposed to be skilled in repairing **s**-failures. Both repairmen are jointly occupied if, and only if, the **s**-unit and the **o**-unit are down. Otherwise, at least one repairman is idle. The **T**-system is down if both units are down. Otherwise, the **T**-system is up. Thus, the **T**-system is up if at least one unit is up. By convention, we assume that any **o**-failure is always directed to repairman R, whereas a **s**-failure is always allocated to repairman R_s. Consequently, repairman R_s can be idle if the **T**-system is down.

Finally, note that the entire system act as a closed queuing system evolving in time, i.e. any failed unit goes immediately into repair unless repairman R is busy. In this case, the failed **o**-unit has to queue for repair. On the other hand, any repaired unit lines-up in warm standby if the remaining unit is still available (operative). Otherwise, the repaired unit becomes immediately operative. Any switch from standby to the operative state is assumed to be perfect, changing the failure rate of the **s**-unit into the failure rate of the **o**-unit. The long-run availability and the point availability of the **T**-system have been analysed by Vanderperre (2008, 2012). A numerical approach has been proposed by Vanderperre and Makhanov (2014b). At present, we propose to analyse the recovery time of the **T**-system. Roughly stated, the recovery time is the total (random) amount of time needed by the repair facility to restore the **T**-system from a risky state to the safe state. A precise definition requires the notion of a stopping time (optional time, Markov time), e.g. Brémaud (1990, Chapt. 1, pp. 1-3), Doob (1994, page 191). It should be noted that the methodologies based on Markov processes (such as renewal theory) are totally inadequate to analyse the recovery time of the **T**-system. As a matter of fact, the engineering structure of the warm standby mode and the generality of the repair time distributions are destroying the regenerative nature of the underlying statespace.

Therefore, we analyse the random behaviour of the **T**-system by introducing a stochastic process endowed with time-dependent transition measures satisfying coupled partial differential equations. The solution procedure is based on the theory of sectionally holomorphic functions, e.g. Gakhov (1996), Roos (1969) combined with the notion of dual transforms, Vanderperre (2008). However, it should be remarked that our present equations are substantially different from the former availability equations, e.g. Vanderperre (2012). The delicate difference is due to the presence of an absorbing barrier in the statespace of our (filtered) stochastic process. Finally, as a non-standard example, we consider the case of Weibull-Gnedenko repair.

2. Formulations, Assumptions & Definitions

Consider the **T**-system subjected to the following conditions and assumptions.

- The **o**-unit has a constant failure rate λ and a general repair time distribution $R(\cdot)$, $R(0) = 0$ with finite mean. The failure-free time and the repair time are respectively denoted by f and r.

- The **s**-unit has a constant failure rate $0 < \lambda_s < \lambda$ and a general repair time distribution $R_s(\cdot)$, $R_s(0) = 0$ with finite mean. The failure-free time and the repair time are respectively denoted by f_s and r_s. Note that f_s is stochastically larger than f (as required). The random variables f, r, f_s, r_s are supposed to be *statistically* independent and any repair is perfect.

- The failure rate λ_s of the **s**-unit changes into the failure rate λ upon switch from standby into the operative state.

  Characteristic functions (and their duals) are formulated in terms of a *complex* transform variable. For instance,

$$\mathbf{E}e^{i\omega r} = \int_0^\infty e^{i\omega x} dR(x), \ \operatorname{Im}\omega \geq 0.$$

  Note that

$$\mathbf{E}e^{-i\omega r} = \int_{-\infty}^0 e^{i\omega x} \, d\left\{1 - R((-x)-)\right\}, \ \operatorname{Im}\omega \leq 0.$$

The corresponding Fourier-Stieltjes transforms are called *dual* transforms. Without loss of generality (see Remarks 5.1), we may assume that R and R_s have density functions of bounded variation on $[0, \infty)$.

In order to formulate a precise definition of the (global) recovery time, we employ a stochastic process $\{N_t, \, t \geq 0\}$ with discrete statespace $\{A, \, B, \, C, \, D, \, D_s\}$ characterized by the following mutually exclusive and exhaustive events:

$\{N_t = A\}$: The **T**-system is up and both repairmen are idle at time t.

$\{N_t = B\}$: The **T**-system is up and repairman R_s is busy at time t.

$\{N_t = C\}$: The **T**-system is up and repairman R is busy at time t.

$\{N_t = D\}$: The **T**-system is down and repairman R_s is idle at time t.

$\{N_t = D_s\}$: The **T**-system is down and both repairmen are busy at time t.

Furthermore, we define $\{N_t, \, t \geq 0\}$ on a filtered probability space $\{\Omega, \, \mathcal{A}, \, \mathbf{P}, \, \mathfrak{F}\}$, where the history $\mathfrak{F} := \{\mathfrak{F}_t, \, t \geq 0\}$ satisfies the Dellacherie-conditions

- $\mathfrak{F}_0$ contains the **P**-null sets of $\mathcal{A}$,
- $\forall \, t \geq 0, \, \mathfrak{F}_t = \bigcap_{u>t} \mathfrak{F}_u$, i.e. the family $\mathfrak{F}$ is right-continuous.

Consider the $\mathfrak{F}$-stopping times

$$\Theta_b := \inf\left\{t > 0 : \, N_t = A | N_0 = B, \, V_0 = 0\right\},$$
$$\Theta_c := \inf\left\{t > 0 : \, N_t = A | N_0 = C, \, W_0 = 0\right\},$$

where V_t (respect. W_t) is the elapsed repair time of the **s**-unit (respect. the **o**-unit) being under repair at any instant of time t. The stopping time θ_b (respect. θ_c) measured from $t = 0$ onwards, is called the recovery time of the **T**-system starting with repairman R_s (respect. with repairman R). The random variable

$$\theta := \begin{cases} \theta_b \text{ if } f_s < f, \\ \theta_c \text{ if } f_s > f, \end{cases}$$

will be called the (global) recovery time of the **T**-system. Note that $\mathbf{P}\{f = f_s\} = 0$ and that

$$\theta = \inf\{t > 0 : N_t = A | N_0 = B \text{ or } C\}.$$

Clearly,

$$\begin{aligned} \mathbf{P}\{\theta \le t\} &= \mathbf{P}\{\theta_b \le t\} \mathbf{P}\{f_s < f\} + \mathbf{P}\{\theta_c \le t\} \mathbf{P}\{f_s > f\} \\ &= \mathbf{P}\{\theta_b \le t\} \frac{\lambda_s}{\lambda + \lambda_s} + \mathbf{P}\{\theta_c \le t\} \frac{\lambda}{\lambda + \lambda_s}. \end{aligned}$$

Hence,

$$\mathbf{E}e^{-z\theta} = \mathbf{E}e^{-z\theta_b} \frac{\lambda_s}{\lambda + \lambda_s} + \mathbf{E}e^{-z\theta_c} \frac{\lambda}{\lambda + \lambda_s}. \tag{2.1}$$

Observe that our proposed repair discipline implies that the **T**-system with initial state C and with absorbing state A, behaves like the two-unit cold standby redundant system attended by a single repairman, see e.g. Gnedenko and Ushakov (1995, Ch. 7.3, page 275), henceforth called the **G**-system.

Consequently, θ_c is distributed as the busy (repair) period of the **G**-system starting at the first failure. Moreover, the Markov property of the exponential distribution $1 - e^{-\lambda t}$ implies that θ_c has the same distribution as the busy (service) period of the M/G/1 queue with only one waiting place.

Hence, applying the results obtained by Cohen (1971, Theorem 2, page 825) entails that

$$\mathbf{E}e^{-z\theta_c} = \frac{\mathbf{E}e^{-(z+\lambda)r}}{1 - \mathbf{E}e^{-zr} + \mathbf{E}e^{-(z+\lambda)r}}, \text{ Re } z \ge 0 \tag{2.2}$$

The functional $\mathbf{E}e^{-z\theta_b}$ is by far more difficult to derive. In fact, θ_b depends on f, r, r_s whereas θ_c only depends on f and r. In order to determine the remaining distribution of θ_b, we construct a set of coupled partial differential equations. A Fourier-Laplace transformation of the equations reduces the set into a single functional equation containing the term $\mathbf{E}e^{-z\theta_b}$. A suitable substitution (elimination) transforms the functional equation into a non-trivial Sokhotski-Plemelj problem on the real line solvable by the theory of sectionally holomorphic functions combined with the notion of dual transforms.

First, we consider a (vector) Markov characteristic of the non-Markovian process $\{N_t,\, t \geq 0\}$ with initial state B and absorbing state A, conditionally defined by

$\{N_t\}$ if $N_t = A$ (i.e. if the event $\{N_t = A\}$ occurs). $\{(N_t,\, Y_t)\}$, if $N_t = B$, where Y_t denotes the *remaining* repair time of the failed **s**-unit under progressive repair at time t. $\{(N_t,\, X_t)\}$, if $N_t = C$ or D, where X_t denotes the *remaining* repair time of the failed **o**-unit under progressive repair at time t. $\{(N_t,\, X_t,\, Y_t)\}$, if $N_t = D_s$. The state space of the underlying Markov process is given by

$$\{A\} \bigcup \{(B, y)\} \bigcup \{(C, x)\} \bigcup \{(D, x)\} \bigcup \{(D_s,\, x,\, y)\}\,,$$

where $x \geq 0,\; y \geq 0$.

For $K = A,\, B,\, C,\, D,\, D_s$ let $p_K(t) := \mathbf{P}\{N_t = K\}\,,\, t \geq 0$.

Finally, we introduce the transition measures

$$\begin{aligned} p_B(t,\, y)dy &:= \mathbf{P}\{N_t = B,\, Y_t \in\, dy\}\,, \\ p_C(t,\, x)dx &:= \mathbf{P}\{N_t = C,\, X_t \in\, dx\}\,, \\ p_D(t,\, x)dx &:= \mathbf{P}\{N_t = D,\, X_t \in\, dx\}\,, \\ p_{D_s}(t,\, x,\, y)dxdy &:= \mathbf{P}\{N_t = D_s,\, X_t \in\, dx,\, Y_t \in\, dy\}\,. \end{aligned}$$

Note that, for instance,

$$p_{D_s}(t) = \int_0^\infty \int_0^\infty p_{D_s}(t,\, x,\, y)dx\, dy.$$

3. Notations, Definitions & Properties

- The indicator (function) of an event $\{N_t = K\} \in \mathcal{A}$ is denoted by $\mathbf{1}\{N_t = K\}$.
- The complex plane and the real line are respectively denoted by $\mathbf{C}$ and $\mathbf{R}$ with obvious superscript notations such as $\mathbf{C}^+$ and $\mathbf{C}^-$. For instance, $\mathbf{C}^+ := \{\omega \in \mathbf{C} :\ \operatorname{Im} \omega > 0\}\,.$

- Let $F(t)$ be any probability distribution on $[0, \infty)$. The n-fold convolution of F is denoted by $F^{n*}(t)$. For $n = 0$, $F^{0*}(t)$ represents the Heaviside unit-step function with the unit-jump at $t = 0$, i.e.

$$F^{0*} := \begin{cases} 1, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

- The Laplace transform of any locally integrable and bounded function on $[0, \infty)$ is denoted by the corresponding character marked with an asterisk. For instance,

$$p^*(z) := \int_0^\infty e^{-zt} p(t) dt, \text{ Re } z > 0.$$

Moreover, if $p(t)$ is of bounded variation on $[0, \infty)$, the product rule for Lebesgue-Stieltjes integrals, e.g. Brémaud (1991, Appendix), entails that

$$zp^*(z) = \int_{0-}^\infty e^{-zt} dp(t), \text{ Re } z > 0.$$

- **Property 3.1**

The function $\wp_z^-(\omega) := i\omega + z + \lambda(1 - \mathbf{E}e^{-(i\omega+z)r}),\ \mathrm{Im}\,\omega \leq 0,\ \mathrm{Re}\ z > 0$ has no zeros in $\mathbf{C}^- \bigcup \mathbf{R}$.

Proof

Consider an alternating renewal process, e.g. Shaked and Shanthikumar (1990, page 665) with an up and down state. Let $1 - e^{-\lambda t}$ (respectively $R(t)$) be the sojourn time distribution of the process in the up state (respectively in the down state). Furthermore, let $p_R(t)$ be the probability that the process is up at time t given that it was up at time $t = 0$. Then

$$p_R(t) = \int_{0-}^t e^{-\lambda(t-u)} d \sum_{n=0}^\infty \varphi^{n*}(u),$$

where

$$\varphi(u) := \int_0^u (1 - e^{-\lambda(u-v)}) dR(v)$$

is the distribution function of a cycle. The following properties are valid for an *arbitrary* R

- $p_R(0) = 1,\ 0 < p_R(t) \leq 1,\ p_R(\infty) = (1 + \lambda\,\mathbf{E}r)^{-1}$.

- $p_R(t)$ is Lebesque-absolutely continuous on $(0,\,\infty)$ and of bounded variation on $[0,\,\infty)$.
- $p_R^*(z) = \frac{1}{z+\lambda(1-\mathbf{E}e^{-z r})}$, $\operatorname{Re} z > 0$. (3.1)

For $\operatorname{Im}\omega \leq 0$, z arbitrary but fixed with $\operatorname{Re} z = \delta > 0$, we have by Eq. (3.1),

$$(\wp_z^-(\omega))^{-1} = \int_0^\infty e^{-(i\omega+z)t} p_R\,(t)dt.$$

On the other hand,

$$\left|\int_0^\infty e^{-(i\omega+z)t} p_R\,(t)dt\right| \leq \int_0^\infty e^{-\delta t} p_R\,(t)dt < \delta^{-1}.$$

Consequently, the function $(\wp_z^-(\omega))^{-1}$ is (uniformly) bounded on $\mathbf{C}^-\bigcup\mathbf{R}$. Hence, $\wp_z^-(\omega)$ is zero-free in $\mathbf{C}^-\bigcup\mathbf{R}$.

■ Let $\alpha(\tau)$, $\tau\in\mathbf{R}$ be a bounded and continuous function. $\alpha(\cdot)$ is called Γ-integrable if

$$\lim_{\substack{T\to\infty\\ \varepsilon\downarrow 0}} \int_{\Gamma_{T,\,\varepsilon}} \alpha(\tau)\frac{d\tau}{\tau-u},\ u\in\mathbf{R}$$

exists, where $\Gamma_{T,\,\varepsilon} := (-T,\ u-\varepsilon]\bigcup[u+\varepsilon,\,T)$. The corresponding integral, denoted by

$$\frac{1}{2\pi i}\int_\Gamma \alpha(\tau)\frac{d\tau}{\tau-u}$$

is called a Cauchy principal value in double sense.

■ A function $\alpha(\tau)$, $\tau\in\mathbf{R}$ is Lipschitz-continuous (L-continuous) on $\mathbf{R}$ if $\forall\tau_1,\,\tau_2\in\mathbf{R}$ there exists a constant c such that

$$|\alpha(\tau_2)-\alpha(\tau_1)| \leq c|\tau_2-\tau_1|.$$

The function $\alpha(\tau)$, $\tau\in\mathbf{R}$ is called **L**-continuous at infinity if

$$|\alpha(\tau)| = O\left(\frac{1}{|\tau|}\right),\ |\tau|\to\infty.$$

■ Note that the **L**-continuity of $\alpha(\cdot)$ on **R** and at infinity is sufficient for the existence of the Cauchy-type integral

$$\frac{1}{2\pi i}\int_\Gamma \alpha(\tau)\frac{d\tau}{\tau-\omega},\ \omega\in\mathbf{C}.$$

See Gakhov (1996) for further details.

4. Differential Equations

In order to derive a system of differential equations, we observe the random behaviour of the **T**-system in some time interval $(t,\, t+\Delta),\ \Delta \downarrow 0$. Applying a general birth and death technique and grouping terms of $o(\Delta)$, taking the absorbing state A into account, yields the time-dependent balance equations

$$p_A(t+\Delta) = p_A(t) + p_B(t,0)\Delta + p_C(t,0)\Delta + o(\Delta),$$

$$p_B(t+\Delta,\, y-\Delta) = p_B(t,y)(1-\lambda\Delta) + p_{D_s}(t,0,y)\Delta + o(\Delta),$$

$$p_C(t+\Delta,\, x-\Delta) = p_C(t,x)(1-\lambda\Delta) + p_D(t,0)\frac{dR}{dx}(x)\Delta + p_{D_s}(t,x,0)\Delta + o(\Delta),$$

$$p_D(t+\Delta,\, x-\Delta) = p_D(t,x) + \lambda\, p_C\ (t,x)\,\Delta + o(\Delta),$$

$$p_{D_s}(t+\Delta,\, x-\Delta,\, y-\Delta) = p_{D_s}(t,x,y) + \lambda\, p_B\ (t,y)\,\frac{dR}{dx}(x)\Delta + o(\Delta),$$

where the notation $o(\Delta),\ \Delta \downarrow 0$ stands for any function $\nu(\cdot)$ such that

$$\lim_{\Delta\downarrow 0} \nu(\Delta)/\Delta = 0.$$

Taking the definition of *directional* derivative into account, for instance,

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) p_{D_s}\,(t,x,y) = \lim_{\Delta\downarrow 0} \frac{p_{D_s}(t+\Delta,\, x-\Delta,\, y-\Delta) - p_{D_s}(t,x,y)}{\Delta}$$

yields for $t>0,\ x>0,\ y>0$,

$$\frac{d}{dt}p_A(t) = p_B(t,0) + p_C(t,0),$$

$$\left(\lambda + \frac{\partial}{\partial t} - \frac{\partial}{\partial y}\right) p_B(t,y) = p_{D_s}(t,0,y),$$

$$\left(\lambda + \frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) p_C(t,x) = p_D(t,0)\ \frac{dR}{dx}(x) + p_{D_s}(t,x,0),$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) p_D(t,x) = \lambda\, p_C(t,x),$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) p_{D_s}(t,x,y) = \lambda\, p_B(t,y)\frac{dR}{dx}(x).$$

Note that the derivation of $\mathbf{E}e^{-z\theta_b}$ requires that the recovery process now starts with repairman R_s (state B). Hence, the initial condition of the differential equations is given by $p_B(0,x) = \frac{d}{dx}R_s(x)$. Observe that $p_B(0) = 1$ and that $p_A(t) = \mathbf{P}\{\theta_b \leq t\}, \ t \geq 0$.

Figure 1 displays a right-continuous sample path N_t of the process $\{N_t\}$ with absorbing state A, starting in state B, where $A = 1$, $B = 2$, $C = 3$, $D = 4$, $D_s = 5$. An upwards (respect. downwards) jump corresponds to a failure (respect. repair) of a unit.

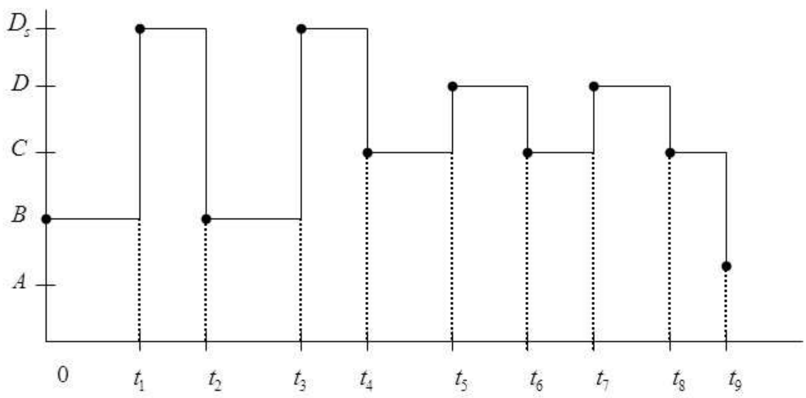

Fig. 1 A right-continuous sample path of the process $\{N_t\}$ with absorbing state A, starting in state B at time $t = 0$.

5. Solution Procedure. Functional Equation

First, we remark that our system of differential equations is well-adapted to a Laplace-Fourier transformation. As a matter of fact, the transition functions are bounded on their appropriate regions and locally integrable with respect to t. Consequently, each Laplace transform exists for $\operatorname{Re} z > 0$. Moreover, the obvious integrability of the repair time density functions and the transition functions with regard to x, y also implies the integrability of the corresponding partial derivatives. Applying a Laplace-Fourier transform technique to the equations and taking the initial condition into account, reveals that for $\operatorname{Re} z > 0, \ \operatorname{Im} \omega \geq 0, \ \operatorname{Im} \eta \geq 0$,

$$z\, p_A^*(z) = p_B^*(z,0) + p_C^*(z,0), \tag{1}$$

$$(z + \lambda + i\eta) \int_0^\infty e^{-zt}\mathbf{E}(e^{i\eta Y_t}\mathbf{1}\{N_t = B\})dt + p_B^*(z,0)$$

$$= \mathbf{E}e^{i\eta r_s} + \int_0^\infty e^{i\eta y} p^*_{D_s}(z,0,y)dy, \quad (2)$$

$$(z+\lambda+i\omega)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}\mathbf{1}\{N_t=C\})dt + p^*_C(z,0)$$
$$= p^*_D(z,0)\mathbf{E}e^{i\omega r} + \int_0^\infty e^{i\omega x}\, p^*_{D_s}(z,x,0)dx, \quad (3)$$

$$(z+i\omega)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}\mathbf{1}\{N_t=D\})dt + p^*_D(z,0)$$
$$= \lambda\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}\mathbf{1}\,\{N_t=C\})dt, \quad (4)$$

$$(z+i\omega+i\eta)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}\, e^{i\eta Y_t}\mathbf{1}\{N_t=D_s\})dt + \int_0^\infty e^{i\omega x} p^*_{D_s}(z,x,0)dx$$
$$+\int_0^\infty e^{i\eta y} p^*_{D_s}(z,0,y)dy = \lambda \mathbf{E}e^{i\omega r}\int_0^\infty e^{-zt}\mathbf{E}(e^{i\eta Y_t}\mathbf{1}\{N_t=B\})dt. \quad (5)$$

Taking $\omega = iz$ in Eq. (5.4) yields $p^*_D(z,0) = \lambda\,\psi^*_C(z)$, where

$$\psi^*_C(z) := \int_0^\infty e^{-zt}\mathbf{E}(e^{-zX_t}\mathbf{1}\{N_t=C\})dt. \quad (6)$$

Adding equations (5.1), (5.2), (5.3), (5.5), taking definition (5.6) into account and noting that $zp^*_A(z) = \mathbf{E}e^{-z\theta_b}$, yields the functional equation

$$\mathbf{E}e^{-z\theta_b} + (z+\lambda(1-\mathbf{E}e^{i\omega r})+i\eta)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\eta Y_t}1\{N_t=B\})dt +$$
$$(z+\lambda+i\omega)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}\mathbf{1}\{N_t=C\})dt +$$
$$(z+i\omega+i\eta)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}e^{i\eta Y_t}\mathbf{1}\{N_t=D_s\})dt = \mathbf{E}e^{i\eta r_s} + \lambda\,\psi^*_C(z)\mathbf{E}e^{i\omega r}, \quad (7)$$

valid for $\operatorname{Re} z > 0,\ \operatorname{Im}\omega \ge 0,\ \operatorname{Im}\eta \ge 0$.

Substituting $\omega = iz$, $\eta = 0$ in Eq. (5.7) reveals that

$$1 - \mathbf{E}e^{-z\theta_b} - \lambda\,\psi^*_C(z)(1-\mathbf{E}e^{-zr}) = (z+\lambda(1-\mathbf{E}e^{-zr}))p^*_B(z). \quad (8)$$

In order to obtain a second (independent) equation, we first eliminate the function

$$\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}e^{i\eta Y_t}\mathbf{1}\{N_t = D_s\})dt,\ \operatorname{Im}\omega \geq 0,\ \operatorname{Im}\eta \geq 0$$

by substituting $\eta = \tau$, $\omega = -\tau + iz$; $\tau \in \mathbf{R}$, z arbitrary but fixed with $\operatorname{Re} z = \delta > 0$ into Eq. (5.7). Noting that $i\omega + i\eta + z = 0$, entails that

$$\begin{aligned}&\mathbf{E}e^{-z\theta_b} + \wp_z^-(\tau)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\tau Y_t}\mathbf{1}\{N_t = B\})dt - \\ &i(\tau + i\lambda)\int_0^\infty e^{-zt}\mathbf{E}(e^{-(i\tau+z)X_t}\mathbf{1}\{N_t = C\})dt = \\ &\lambda\,\psi_C^*(z)\mathbf{E}e^{-(i\tau+z)r} + \mathbf{E}e^{i\eta r_s}\end{aligned} \tag{9}$$

Next, dividing Eq. (5.9) by the factor $\wp_z^-(\tau)$ (justified by Property 3.1) and separating terms according to their appropriate regions of analyticity (marked with a plus or minus superscript) leads to the boundary value equation

$$\phi_z^+(\tau) - \phi_z^-(\tau) = \mathcal{K}_z(\tau),\ \tau \in \mathbf{R} \tag{10}$$

where

$$\phi_z^+(\omega) := \int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega Y_t}\,\mathbf{1}\{N_t = B\})dt,\ \operatorname{Im}\omega \geq 0$$

$$\phi_z^-(\omega) := \frac{\varphi_z^-(\omega)}{\wp_z^-(\omega)},\ \operatorname{Im}\omega \leq 0$$

$$\begin{aligned}\varphi_z^-(\omega) := i(\omega + i\lambda)\int_0^\infty e^{-zt}\mathbf{E}(e^{-(i\omega+z)X_t}\mathbf{1}\{N_t = C\})dt + \\ \lambda\psi_C^*(z)\mathbf{E}e^{-(i\omega+z)r} + 1 - \mathbf{E}e^{-z\theta_b},\ \operatorname{Im}\,\omega \leq 0\end{aligned}$$

and

$$\mathcal{K}_z(\tau) := \frac{\mathbf{E}e^{i\tau r_s} - 1}{\wp_z^-(\tau)},\ \tau \in \mathbf{R}.$$

Eq. (5.10) constitutes a Sokhotski-Plemelj problem on the real line, solvable by the theory of sectionally holomorphic functions, e.g. Gakhov (1996), combined with the notion of dual transforms, Vanderperre (2008). First note that Property 3.1 and the existence of $\mathbf{E}r$ and $\mathbf{E}r_s$ imply that $\frac{d}{d\tau}\mathcal{K}_z(\tau)$ is bounded on $\mathbf{R}$. Consequently, by the mean value theorem, e.g. Apostol (1978, page 110), there exists a constant $k(\delta)$ such that $\forall\,\tau_1,\,\tau_2\ \in\ \mathbf{R}$,

$$\left|\mathcal{K}_z(\tau_2) - \mathcal{K}_z(\tau_1)\right| \leq k(\delta)\left|\tau_2 - \tau_1\right|.$$

Hence, $\mathcal{K}_z(\tau)$ is **L**-continuous on **R**. Finally, note that $\mathcal{K}_z(\tau) = O(|\tau|^{-1})$, $|\tau| \to \infty$. Therefore, $\mathcal{K}_z(\tau)$ is **L**-continuous on **R** and at infinity.

Corollary 5.1

The function

$$\frac{1}{2\pi i}\int_\Gamma \mathcal{K}_z(\tau)\,\frac{d\tau}{\tau-\omega}$$

exist for all $\omega \in \mathbf{C}$ (real or complex) and defines a sectionally holomorphic function in $\mathbf{C}$ vanishing at infinity. In addition,

$$\phi_z^+(\omega) = \frac{1}{2\pi i}\int_\Gamma \mathcal{K}_z(\tau)\frac{d\tau}{\tau-\omega},\ \omega \in \mathbf{C}^+ \tag{11}$$

$$\phi_z^-(\omega) = \frac{1}{2\pi i}\int_\Gamma \mathcal{K}_z(\tau)\frac{d\tau}{\tau-\omega},\ \omega \in \mathbf{C}^- \tag{12}$$

Observe that Eq. (5.11) is only valid for Im $\omega > 0$.

However, note that

$$\lim_{\substack{\omega\to 0\\ \omega\in C^+}} \phi_z^+(\omega) = p_B^*(z). \tag{13}$$

On the other hand, applying the Sokhotski-Plemelj formulas, e.g. Gakhov (1996, page 36, Eq. (6.25)), yields

$$\lim_{\substack{\omega\to 0\\ \omega\in C^+}} \phi_z^+(\omega) = \frac{1}{2}\mathcal{K}_z(0) + \frac{1}{2\pi i}\int_\Gamma \mathcal{K}_z(\tau)\frac{d\tau}{\tau} = \frac{1}{2\pi i}\int_\Gamma \frac{1}{\wp_z^-(\tau)}\frac{\mathbf{E}e^{i\tau r_s}-1}{\tau}d\tau, \tag{14}$$

where

$$\left.\frac{\mathbf{E}e^{i\tau r_s}-1}{\tau}\right|_{\tau=0} := i\mathbf{E}r_s.$$

Combining Eqs. (5.13) and (5.14), reveals that

$$p_B^*(z) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{\wp_z^-(\tau)}\frac{\mathbf{E}e^{i\tau r_s}-1}{\tau}d\tau. \tag{15}$$

Invoking Eqs. (5.8) and (5.15) yields the basic equation

$$1-\mathbf{E}e^{-z\theta_b} - \lambda\psi_C^*(z)(1-\mathbf{E}e^{-zr}) =$$

$$(z+\lambda(1-\mathbf{E}e^{-zr}))\frac{1}{2\pi i}\int_{\Gamma}\frac{1}{\wp_z^-(\tau)}\frac{\mathbf{E}e^{i\tau r_s}-1}{\tau}d\tau. \tag{16}$$

Finally, taking $\omega = -i\lambda$ in Eq. (5.12) and noting that

$$\phi_z^-(-i\lambda) = \frac{\varphi_z^-(-i\lambda)}{\wp_z^-(-i\lambda)} = \frac{1-\mathbf{E}e^{-z\theta_b}+\lambda\psi_C^*(z)\mathbf{E}e^{-(z+\lambda)r}}{z+2\lambda-\lambda\,\mathbf{E}e^{-(z+\lambda)r}}$$

yields the required additional equation

$$1-\mathbf{E}e^{-z\theta_b}+\lambda\,\psi_C^*(z)\mathbf{E}e^{-(z+\lambda)r} =$$
$$(z+2\lambda-\lambda\,\mathbf{E}e^{-(z+\lambda)r})\frac{1}{2\pi i}\int_{\Gamma}\frac{\mathbf{E}e^{i\tau r_s}-1}{\wp_z^-(\tau)}\frac{d\tau}{\tau+i\lambda}. \tag{17}$$

The functional $\mathbf{E}e^{-z\theta_b}$ is now completely determined by Eqs. (5.16) and (5.17).

Remarks 5.1

It should be noted that the kernel $\mathcal{K}_z(\cdot)$ preserves all the relevant properties to ensure the existence of the Cauchy-type integral

$$\frac{1}{2\pi i}\int_{\Gamma}\mathcal{K}_z(\tau)\frac{d\tau}{\tau-\omega},\ \omega\in\mathbf{C}$$

for *arbitrary* repair time distributions with finite mean. First of all, the generality of Property 3.1 ensures that the order relation

$$|\mathcal{K}_z(\tau)| = O(|\tau|^{-1}),\ |\tau|\to\infty$$

also holds for arbitrary characteristic functions. Moreover, the **L**-continuity of $\mathcal{K}_z(\tau)$ does not depend on the canonical structure (Lebesgue decomposition) of R or R_s. For instance, the Lipschitz-inequality

$$|\mathbf{E}e^{i\tau_2 r}-\mathbf{E}e^{i\tau_1 r}|\le\mathbf{E}r|\tau_2-\tau_1|$$

always holds for *any* R with finite mean $\mathbf{E}r$.

Consequently, our initial condition concerning the existence of repair time distributions is totally superfluous to ensure the existence of $\{p_K(t)\}$.

We summarize the following result.

Property 5.1

Let R and R_s be arbitrary repair time distributions with finite mean. Then

$$\mathbf{E}e^{-z\theta} = \frac{\lambda}{\lambda+\lambda_s}\mathbf{E}e^{-z\theta_c} + \frac{\lambda_s}{\lambda+\lambda_s}\mathbf{E}e^{-z\theta_b},$$

$$\frac{1-\mathbf{E}e^{-z\theta_c}}{z} = \frac{\frac{1-\mathbf{E}e^{-zr}}{z}}{1-\mathbf{E}e^{-zr}+\mathbf{E}e^{-(z+\lambda)r}}, \tag{18}$$

$$\mathbf{E}\theta_c = \frac{\mathbf{E}r}{\mathbf{E}e^{-\lambda r}},$$

$$\mathbf{E}e^{-z\theta_c} = \frac{\mathbf{E}e^{-(z+\lambda)r}}{1-\mathbf{E}e^{-zr}+\mathbf{E}e^{-(z+\lambda)r}}, \tag{19}$$

$$\frac{1-\mathbf{E}e^{-z\theta_b}}{z} = \mathbf{E}e^{-z\theta_c}\left(1+\lambda\frac{1-\mathbf{E}e^{-zr}}{z}\right)\frac{1}{2\pi i}\int_\Gamma \frac{1}{\wp_z^-(\tau)}\frac{\mathbf{E}e^{i\tau\, r_s}-1}{\tau}d\tau + \frac{1-\mathbf{E}e^{-z\theta_c}}{z}(z+2\lambda-\lambda\,\mathbf{E}e^{-(z+\lambda)r})\frac{1}{2\pi i}\int_\Gamma \frac{\mathbf{E}e^{i\tau\, r_s}-1}{\wp_z^-(\tau)}\frac{d\tau}{\tau+i\lambda}. \tag{20}$$

6. Application Example

As a non-standard example, we consider the case of Weibull-Gnedenko repair, i.e. let $R_s(\cdot) = W_{1,\beta}(\cdot)$ and $R(\cdot) = W_{\mu,1}(\cdot)$, where $W_{\mu,\beta}(\cdot)$ stands for the Weibull-Gnedenko distribution with scale parameter $\mu > 0$ and with shape parameter $\beta \geq 1$, i.e.

$$W_{\mu,\,\beta}(t) := \begin{cases} 1-e^{-(\mu t)^\beta}, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0, \end{cases}$$

See Ushakov (2012, page 207) for further details.

6.1. Determination of $\mathrm{P}\{\theta_\mathsf{c} \leq t\}$

In order to obtain the distribution of θ_c, we first substitute the functions $\mathbf{E}e^{-zr} = \mu(\mu+z)^{-1}$ and $\mathbf{E}e^{-(z+\lambda)r} = \mu(\mu+\lambda+z)^{-1}$ into Eq. (5.18). Some algebra entails that

$$\frac{1-\mathbf{E}e^{-z\theta_c}}{z} = \frac{z+\lambda+\mu}{(z-z_1)(z-z_2)},$$

where

$$z_1 = -\frac{1}{2}(\lambda + 2\mu - (\lambda^2 + 4\mu)^{\frac{1}{2}}),\ z_2 = -\frac{1}{2}\left(\lambda + 2\mu + (\lambda^2 + 4\mu)^{\frac{1}{2}}\right).$$

Applying the inversion theorem for Laplace transforms, e.g. Apostol (1978, page 342, Ex. 11.39) yields

$$\mathbf{P}\left\{\theta_c > t\right\} = \frac{z_1 + \lambda + \mu}{z_1 - z_2} e^{z_1 t} + \frac{z_2 + \lambda + \mu}{z_2 - z_1} e^{z_2 t},\ t \geq 0. \tag{21}$$

As a numerical example, let $\lambda = 2$, $\mu = 1$. Some calculus entails that the distribution of θ_c is given by

$$\mathbf{P}\left\{\theta_c \leq t\right\} = 1 - \frac{1+\sqrt{2}}{2\sqrt{2}} e^{-(2-\sqrt{2})t} - \frac{\sqrt{2}-1}{2\sqrt{2}} e^{-(2+\sqrt{2})t} \tag{22}$$

with density function

$$\frac{d}{dt}\mathbf{P}\ \left\{\theta_c \leq t\right\} = \frac{1}{2}\left(e^{-(2-\sqrt{2})t} + e^{-(2+\sqrt{2})t}\right). \tag{23}$$

6.2. Determination of $\mathbf{P}\{\theta_b \leq t\}$

In order to derive the distribution of θ_b, we use the identities

$$\frac{1}{2\pi i}\int_\Gamma \frac{1}{\wp_z^-(\tau)} \frac{\mathbf{E}e^{i\tau r_s} - 1}{\tau} d\tau = \frac{\mu}{\lambda+\mu}\frac{1-\mathbf{E}e^{-zr_s}}{z} + \frac{\lambda}{\lambda+\mu}\frac{1-\mathbf{E}e^{-(z+\lambda+\mu)r_s}}{z+\lambda+\mu}, \tag{24}$$

$$\frac{1}{2\pi i}\int_\Gamma \frac{\mathbf{E}e^{i\tau r_s} - 1}{\wp_z^-(\tau)} \frac{d\tau}{\tau + i\lambda} = \frac{\mu\, z}{(z+\lambda)(\lambda+\mu)}\frac{1-\mathbf{E}e^{-zr_s}}{z} + \frac{\lambda}{\lambda+\mu}\frac{1-\mathbf{E}e^{-(z+\lambda+\mu)r_s}}{z+2\lambda+\mu}, \tag{25}$$

Substituting Eq. (6.4) and Eq. (6.5) into Eq. (5.20) yields

$$\frac{1-\mathbf{E}e^{-z\theta_b}}{z} = \frac{\mu}{\lambda+\mu}\frac{1-\mathbf{E}e^{-zr_s}}{z} + \frac{\lambda}{\lambda+\mu}\frac{1-\mathbf{E}e^{-zr_s}}{z}\mathbf{E}e^{-z\theta_c} + \frac{\lambda}{\lambda+\mu}\left(1-\mathbf{E}e^{-(z+\lambda+\mu)r_s}\right)\frac{1-\mathbf{E}e^{-z\theta_c}}{z}. \tag{26}$$

In addition, the existence of $\mathbf{E}r_s$ implies that

$$\mathbf{E}\theta_b = \mathbf{E}r_s + \frac{\lambda}{\mu^2}\left(1 - \mathbf{E}e^{-(\lambda+\mu)r_s}\right).$$

Observe that $\mathbf{P}\{\theta_b > t\}$ is uniquely determined by the Laplace transform

$$\frac{1 - \mathbf{E}e^{-z\theta_b}}{z} = \int_0^\infty e^{-zt}\mathbf{P}\{\theta_b > t\}\,dt,\ \operatorname{Re} z \geq 0.$$

Consequently, invoking the convolution theorem for Laplace transforms, e.g. Apostol (1978, page 342, Ex. 11.37), reveals that

$$\mathbf{P}\{\theta_b > t\} = \frac{\mu}{\lambda+\mu}\mathbf{P}\{r_s > t\} + \frac{\lambda}{\lambda+\mu}\int_0^t \mathbf{P}\{r_s > t-x\}\frac{d}{dx}\mathbf{P}\{\theta_c \leq x\}\,dx$$
$$+\frac{\lambda}{\lambda+\mu}\mathbf{P}\{\theta_c > t\} - \frac{\lambda}{\lambda+\mu}\int_0^t \mathbf{P}\{\theta_c > t-x\}e^{-(\lambda+\mu)x}\frac{d}{dx}\mathbf{P}\{r_s \leq x\}\,dx. \tag{27}$$

As a numerical example, let $\lambda = 2,\ \mu = 1,\ \beta = 2.$

Taking Eqs. (6.1), (6.3) and (6.7) into account, yields

$$\mathbf{P}\{\theta_b \leq t\} = 1 + \frac{2}{3}\int_0^t \left(\frac{\sqrt{2}+1}{\sqrt{2}}e^{-(2-\sqrt{2})(t-x)} + \frac{\sqrt{2}-1}{\sqrt{2}}e^{-(2+\sqrt{2})(t-x)}\right)e^{-3x}xe^{-x^2}dx$$

$$-\frac{1}{3}e^{-t^2} - \frac{1}{3}\int_0^t e^{-(t-x)^2}\left(e^{-(2-\sqrt{2})x} + e^{-(2+\sqrt{2})x}\right)dx$$

$$-\frac{1}{3}\left(\frac{\sqrt{2}+1}{\sqrt{2}}e^{-(2-\sqrt{2})t} + \frac{\sqrt{2}-1}{\sqrt{2}}e^{-(2+\sqrt{2})t}\right).$$

Figure 2 displays the graph of $\mathbf{P}\{\theta_c \leq t\}$ (dashed line) and the graph of $\mathbf{P}\{\theta_b \leq t\}$ (dotted line), $0 \leq t \leq 8$, case $\lambda = 2$, $\mu = 1$, $\beta = 2$.

Figure 3 shows the graph of $\mathbf{P}\{\theta \leq t\}$, case $\lambda = 2$, $\mu = 1$, $\lambda_s = 0.5$, $\beta = 2$. Note that $\mathbf{P}\{\theta \leq t\} = 0.8\,\mathbf{P}\{\theta_c \leq t\} + 0.2\,\mathbf{P}\{\theta_b \leq t\}$.

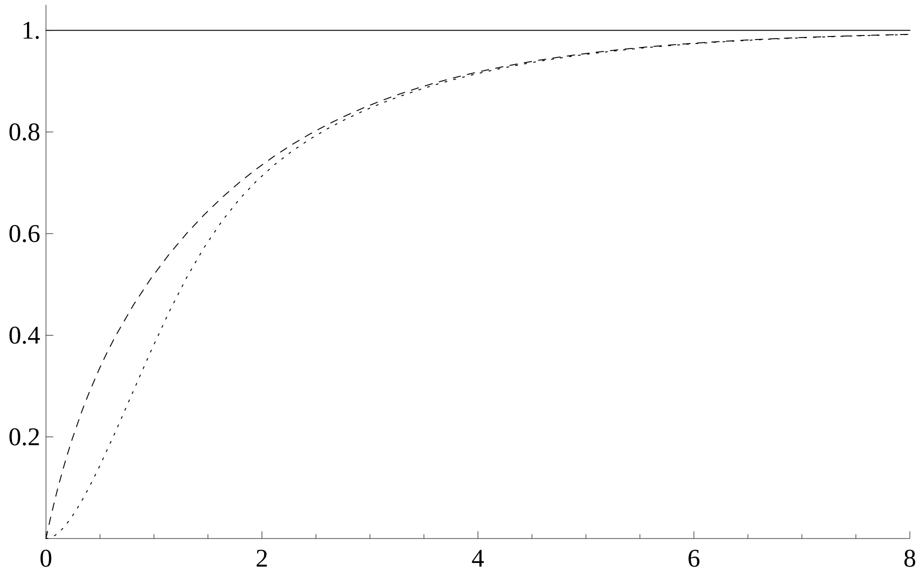

Fig. 2 Graph of $\mathbf{P}\{\theta_c \leq t\}$, (dashed line) versus the graph of $\mathbf{P}\{\theta_b \leq t\}$ (dotted line), $0 \leq t \leq 8$. Case $\lambda = 2$, $\mu = 1$, $\beta = 2$.

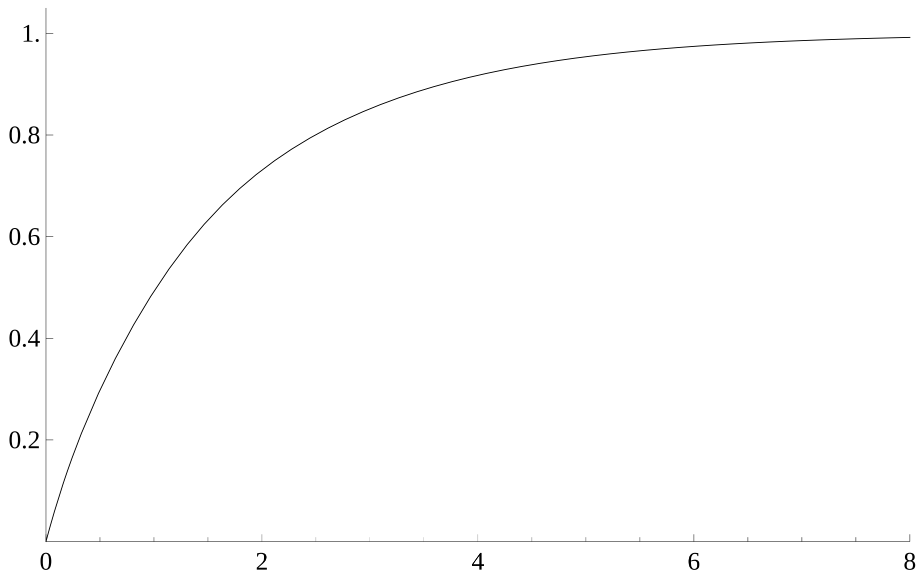

Fig. 3 Graph of $\mathbf{P}\{\theta \leq t\}$. Case $\lambda = 2$, $\mu = 1$, $\beta = 2$, $\lambda_s = 0.5$.

7. Conclusions

The (global) recovery time of the **T**-system is a mixture of the recovery time starting with repairman R and of the recovery time starting with repairman R_s. The functional $\mathbf{E}e^{-z\theta_c}$ follows by a simple application of a result concerning the M/G/1 queue with finite waiting room compared with the two-unit cold standby redundant system attended by a single repairman R. The functional $\mathbf{E}e^{-z\theta_b}$ can be derived as a first passage time distribution of a suitable stochastic process starting in state B and with absorbing state A. The resulting functional equation can be solved by the theory of sectionally holomorphic functions. The non-standard case of Weibull-Gnedenko repair shows that computational results are conceivable by a standard inversion technology based on the inversion theorem for Laplace transforms and on the properties of the convolution integral. Therefore, our proposed methodology allows to solve fairly general problems in statistical reliability engineering.

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