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ON STRICT-DOUBLE-BOUND NUMBERS OF COMPLETE GRAPHS WITHOUT EDGES OF CYCLES

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Abstract: For a poset $P = (X, \leq_P)$, the *strict-double-bound graph* of P is the graph sDB(P) on $V(\text{sDB}(P)) = X$ for which vertices u and v of sDB(P) are adjacent if and only if $u \neq v$ and there exist elements $x, y \in X$ distinct from u and v such that $x \leq_P u \leq_P y$ and $x \leq_P v \leq_P y$. The *strict-double-bound number* $\zeta(G)$ of a graph G is defined as $\min\{\, n\ ;\ \text{sDB}(P) \cong G \cup \overline{K}_n$ for some poset $P\}$. We obtain upper bounds of strict-double-bound numbers of $K_n - E(C_m)$ $(5 \leq m \leq n)$.

AMS Subject Classification: 05C62
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1. Introduction

In this paper we consider graphs without loops and multiple edges. For a graph G and $S \subseteq V(G)$, $\langle S \rangle_V$ is the induced subgraph of S. For a graph G and $v \in V(G)$, $N_G(v) = \{u\ ; uv \in E(G)\}$. For a graph G, $\overline{G}$ is the complement of G. For a graph G and a subgraph H of G, the graph $G-E(H)$ is the graph with the vertex set $V(G-E(H)) = V(G)$ and the edge set $E(G-E(H)) = E(G)-E(H)$.

The *union* $G \cup I$ of two graphs G and I is the graph with the vertex set $V(G \cup I) = V(G) \cup V(I)$ and the edge set $E(G \cup I) = E(G) \cup E(I)$. The *sum* $G+I$ of two graphs G and I is the graph with the vertex set $V(G+I) = V(G) \cup V(I)$ and the edge set $E(G+I) = E(G) \cup E(I) \cup \{uv\ ; u \in V(G), v \in V(I)\}$.

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A *clique* in a graph G is the vertex set of a maximal complete subgraph of G. A family $\mathcal{Q} = \{Q_1, Q_2, ..., Q_l\}$ is an *edge clique cover* of G if each Q_i is a clique of G, and for each $uv \in E(G)$, there exists $Q_i \in \mathcal{Q}$ such that $u, v \in Q_i$.

For a poset P, let $\text{Max}(P)$ be the set of all maximal elements of P and $\text{Min}(P)$ be the set of all minimal elements of P. For a poset P and elements u and v, $u \parallel v$ denotes that u is incomparable with v.

We consider strict-double-bound graphs and strict-double-bound numbers. For a poset $P = (X, \leq_P)$, the *strict-double-bound graph* (*sDB-graph*) of P is the graph $\text{sDB}(P)$ on $V(\text{sDB}(P)) = X$ for which vertices u and v of $\text{sDB}(P)$ are adjacent if and only if $u \neq v$ and there exist elements $x, y \in X$ distinct from u and v such that $x \leq_P u \leq_P y$ and $x \leq_P v \leq_P y$. We say that a graph G is a *strict-double-bound graph* if there exists a poset whose strict-double-bound graph is isomorphic to G. McMorris and Zaslavsky [5] introduced a concept of strict-double-bound graphs. Note that maximal elements and minimal elements of a poset P are isolated vertices of $\text{sDB}(P)$. So a connected graph with $p \geq 2$ vertices is not a strict-double-bound graph. Scott [7] showed as follows.

Theorem 1.1 (Scott [7])**.** *Any graph that is the disjoint union of a non-trivial component and enough number of isolated vertices is a strict-double-bound graph.*

We introduce the strict-double-bound number of a graph. The *strict-double-bound number* $\zeta(G)$ of a graph G is defined as $\min\{\ n\ ;\ \ \text{sDB}(P) \cong G \cup \overline{K}_n$ for some poset $P\}$.

Scott [7] obtained the following result, using a concept of transitive double competition numbers.

Theorem 1.2 (Scott [7])**.** *For a non-trivial connected graph G and a minimal edge clique cover $\mathcal{Q}$ of G,* $\left\lceil 2\sqrt{|\mathcal{Q}|} \right\rceil \leq \zeta(G) \leq |\mathcal{Q}| + 1$.

We already knew the following results.

Proposition 1.3 (Kanada, Kushima, Ogawa, Tagusari and Tsuchiya [2])**.** *Let G be a connected graph with $p \geq 2$ vertices and P a poset with* $\text{sDB}(P) \cong G \cup \overline{K}_{\zeta(G)}$*. Then* $|\text{Max}(P) \cup \text{Min}(P)| = \zeta(G)$*.*

Theorem 1.4 (Konishi, Ogawa, Tagusari and Tsuchiya [4])**.** *For a graph G with $p \geq 2$ vertices and no isolated vertices, $\zeta(K_n + G) = \zeta(G)$ for $n \geq 1$.*

Using these results, we deal with a strict-double-bound number of $G-E(H)$ for some graph H. For example, H is a graph with $|E(G)| \leq 4$ ([1], [6]). And H is K_n, $K_{1,n}$, K_n-*pan* ([2]). In this paper we consider $K_n - E(C_m)$.

2. Cycles

Kim, Park and Sano [3] gave the following result. For a graph G, $\theta_e(G)$ is *the edge clique number* of G, that is, $\theta_e(G) = \min \{|\mathcal{Q}|;\ \mathcal{Q}$ is an edge clique cover of $G\}$.

Theorem 2.1 (Kim, Park and Sano [3]). (1) $\theta_e(\overline{C_6}) = 5$,

(2) $\theta_e(\overline{C_7}) = 7$,

(3) $\theta_e(\overline{C_8}) = 6$,

(4) $\theta_e(\overline{C_m}) \leq \frac{m+5}{2}$ $(m \geq 9, m : odd)$,

(5) $\theta_e(\overline{C_m}) \leq \frac{m+2}{2}$ $(m \geq 10, m : even)$.

Konishi, Ogawa, Tagusari and Tsuchiya [4] obtained the following result.

Proposition 2.2 (Konishi, Ogawa, Tagusari and Tsuchiya [4]). *For a cycle C_n with $n \geq 4$, $\zeta(C_n) = \left\lceil 2\sqrt{n} \right\rceil$.*

We obtain the following results. For a poset P and $v \in V(P)$, $\mathrm{Max}_P(v) = \{u \in \mathrm{Max}(P); v \leq_P u\}$ and $\mathrm{Min}_P(v) = \{u \in \mathrm{Min}(P); u \leq_P v\}$.

Lemma 2.3. *For a graph $K_m - E(C_m)$ and $m \geq 5$, let $P_{K_m-E(C_m)}$ be a poset such that* $\mathrm{sDB}(P_{K_m-E(C_m)}) \cong (K_m - E(C_m)) \cup \overline{K}_{\zeta(K_m-E(C_m))}$.
(1) *For $\forall v \in V(K_m - E(C_m))$, $|\mathrm{Max}_{P_{K_m-E(C_m)}}(v)| \geq 2$ or $|\mathrm{Min}_{P_{K_m-E(C_m)}}(v)| \geq 2$.*
(2) *There exist no element v such that $\mathrm{Max}_{P_{K_m-E(C_m)}}(v) = \mathrm{Max}(P_{K_m-E(C_m)})$ and $\mathrm{Min}_{P_{K_m-E(C_m)}}(v) = \mathrm{Min}(P_{K_m-E(C_m)})$.*

Proof. (1) We assume that there exists $v \in V(K_m - E(C_m))$ such that $\mathrm{Max}_{P_{K_m-E(C_m)}}(v) = \{\alpha\}$ and $\mathrm{Min}_{P_{K_m-E(C_m)}}(v) = \{\beta\}$. Then $\beta \leq_{P_{K_m-E(C_m)}} u \leq_{P_{K_m-E(C_m)}} \alpha$ for $\forall u \in N_{K_m-E(C_m)}(v)$. Thus $N_{K_m-E(C_m)}(v)$ is a clique of $K_m - E(C_m)$. Since $m \geq 5$, we have a contradiction.

(2) We assume that there exists $v \in V(K_m - E(C_m))$ such that

$$\mathrm{Max}_{P_{K_m-E(C_m)}}(v) = \mathrm{Max}(P_{K_m-E(C_m)})$$

and

$$\mathrm{Min}_{P_{K_m-E(C_m)}}(v) = \mathrm{Min}(P_{K_m-E(C_m)}).$$

Thus

$$N_{K_m-E(C_m)}(v) = V(K_m - E(C_m)) - \{v\},$$

which is a contradiction.

Based on these results, we obtain the following results.

Proposition 2.4. *(i)* $\zeta(K_n - E(C_5)) = 5 \ \ (n \geq 5)$,

(ii) $\zeta(K_n - E(C_6)) = 6 \ \ (n \geq 6)$,

(iii) $\zeta(K_n - E(C_7)) = 7 \ \ (n \geq 7)$,

(iv) $\zeta(K_n - E(C_8)) = 7 \ \ (n \geq 8)$,

(v) $\zeta(K_n - E(C_m)) \leq \frac{m+5}{2} + 1 \ \ (n \geq m \geq 9, \ m : \text{odd})$,

(vi) $\zeta(K_n - E(C_m)) \leq \frac{m+2}{2} + 1 \ \ (n \geq m \geq 10, \ m : \text{even})$.

Proof. The graph $K_m - E(C_m)$ is $\overline{C}_m$ and $K_m - E(C_m)$ is a connected graph if $m \geq 5$. And $K_n - E(C_m) = K_{n-m} + (K_m - E(C_m)) = K_{n-m} + \overline{C}_m$. By Theorem 1.4, $\zeta(K_n - E(C_m)) = \zeta(K_m - E(C_m)) = \zeta(\overline{C}_m)$.

(i) Since $\overline{C}_5 = C_5$, $\zeta(C_5) = 5$ by Proposition 2.2. Thus $\zeta(K_n - E(C_5)) = 5$ $(n \geq 5)$.

By Theorem 1.2, $\left\lceil 2\sqrt{\theta_e(\overline{C}_m)} \right\rceil \leq \zeta(\overline{C}_m) \leq \theta_e(\overline{C}_m) + 1$. Using Theorem 2.1, we have the followings:

(ii) $5 \leq \zeta(K_n - E(C_6)) \leq 6 \ (n \geq 6)$,

(iii) $6 \leq \zeta(K_n - E(C_7)) \leq 8 \ (n \geq 7)$,

(iv) $5 \leq \zeta(K_n - E(C_8)) \leq 7 \ (n \geq 8)$,

(v) $\zeta(K_n - E(C_m)) \leq \frac{m+5}{2} + 1 \ (n \geq m \geq 9, m : odd)$,

(vi) $\zeta(K_n - E(C_m)) \leq \frac{m+2}{2} + 1 \ (n \geq m \geq 10, m : even)$.

(ii) $\zeta(K_n - E(C_6)) = 6 \ (n \geq 6)$:

We consider a graph $K_6 - E(C_6)$ (see Figure 1). We assume that there exists a poset P such that $\mathrm{sDB}(P) \cong (K_6 - E(C_6)) \cup \overline{K}_5$. In the case of $|\,\mathrm{Max}(P)| = 4$

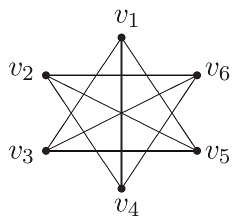

Figure 1: $K_6 - E(C_6)$.

and $|\operatorname{Min}(P)| = 1$ (or $|\operatorname{Max}(P)| = 1$ and $|\operatorname{Min}(P)| = 4$) sDB(P) has at most four cliques. So we consider the case $|\operatorname{Max}(P)| = 3$ and $|\operatorname{Min}(P)| = 2$ by Theorem 2.1. And the case $|\operatorname{Max}(P)| = 2$ and $|\operatorname{Min}(P)| = 3$ is the dual. Let $\operatorname{Max}(P) = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\operatorname{Min}(P) = \{\beta_1, \beta_2\}$.

Case (ii)-1 : there exists an element $v = v_1$ such that $\operatorname{Max}_P(v_1) = \{\alpha_1\}$ and $\operatorname{Min}_P(v_1) = \{\beta_1, \beta_2\}$.

Then for adjacent vertices of v_1, $v_3, v_4, v_5 \leq_P \alpha_1$, $\beta_1 \leq_P v_3, v_5$, and $\beta_2 \leq_P v_4$, because v_4 is not adjacent to v_3 and v_5. Since $N_G(v_2) = \{v_4, v_5, v_6\}$ and $N_G(v_6) = \{v_2, v_3, v_4\}$, $\beta_1 \leq_P v_2, v_6$ and $\beta_2 \leq_P v_2, v_6$. And also $v_2, v_5 \leq_P \alpha_2$ and $v_6, v_3 \leq_P \alpha_3$. Since v_2 is adjacent to v_6, $v_2 \leq_P \alpha_3$ or $v_6 \leq_P \alpha_2$. Then v_2 is adjacent to v_3 or v_6 is adjacent to v_5, which is a contradiction.

Case (ii)-2 : there exists an element $v = v_1$ such that $\operatorname{Max}_P(v_1) = \{\alpha_1, \alpha_2\}$ and $\operatorname{Min}_P(v_1) = \{\beta_1, \beta_2\}$.

Since v_2 is not adjacent to v_1, $\operatorname{Max}_P(v_2) = \{\alpha_3\}$. Then we have Case (ii)-1 or $|\operatorname{Max}_P(v_2)| = |\operatorname{Min}_P(v_2)| = 1$.

Case (ii)-3 : there exists an element $v = v_1$ such that $\operatorname{Max}_P(v_1) = \{\alpha_1, \alpha_2\}$ and $\operatorname{Min}_P(v_1) = \{\beta_1\}$.

Then for adjacent vertices of v_1, $\beta_1 \leq_P v_3, v_4, v_5$, and $v_3, v_5 \leq_P \alpha_1$, and $v_4 \leq_P \alpha_2$, because v_4 is not adjacent to v_3 and v_5. Since v_3 is not adjacent to v_4, $v_3 \parallel \alpha_3$ or $v_4 \parallel \alpha_3$. Thus $|\operatorname{Max}_P(v_3)| = 1$ or $|\operatorname{Max}_P(v_4)| = 1$. So we have Case (ii)-1, $|\operatorname{Max}_P(v_3)| = |\operatorname{Min}_P(v_3)| = 1$ or $|\operatorname{Max}_P(v_4)| = |\operatorname{Min}_P(v_4)| = 1$.

Case (ii)-4 : there exists an element $v = v_1$ such that $\operatorname{Max}_P(v_1) = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\operatorname{Min}_P(v_1) = \{\beta_1\}$.

Then for adjacent vertices of v_1, $\beta_1 \leq_P v_3, v_4, v_5$. Since v_3 is not adjacent to v_4, $|\operatorname{Max}_P(v_3)| = 1$ or $|\operatorname{Max}_P(v_4)| = 1$. Then we have Case (ii)-1,

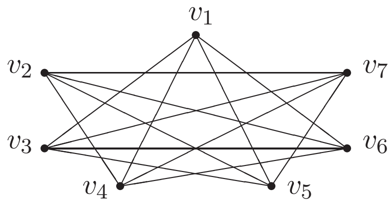

Figure 2: $K_7 - E(C_7)$.

$|\,\mathrm{Max}_P(v_3)| = |\,\mathrm{Min}_P(v_3)| = 1$ or $|\,\mathrm{Max}_P(v_4)| = |\,\mathrm{Min}_P(v_4)| = 1$.

Thus $\zeta(K_6 - E(C_6)) \geq 6$, and $\zeta(K_n - E(C_6)) = \zeta(K_6 - E(C_6)) = 6$.

(iii) $\zeta(K_n - E(C_7)) = 7$ $(n \geq 7)$:

We consider a graph $K_7 - E(C_7)$ (see Figure 2). We assume that there exists a poset P such that $\mathrm{sDB}(P) \cong (K_7 - E(C_7)) \cup \overline{K}_6$. In the case of $|\,\mathrm{Max}(P)| = 5$ and $|\,\mathrm{Min}(P)| = 1$ (or $|\,\mathrm{Max}(P)| = 1$ and $|\,\mathrm{Min}(P)| = 5$), $\mathrm{sDB}(P)$ has at most five cliques. So we consider the case $|\,\mathrm{Max}(P)| = 4$ and $|\,\mathrm{Min}(P)| = 2$, and the case $|\,\mathrm{Max}(P)| = |\,\mathrm{Min}(P)| = 3$ by Theorem 2.1. And the case $|\,\mathrm{Max}(P)| = 2$ and $|\,\mathrm{Min}(P)| = 4$ is the dual. First we consider the case $|\,\mathrm{Max}(P)| = 4$ and $|\,\mathrm{Min}(P)| = 2$. Let $\mathrm{Max}(P) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\mathrm{Min}(P) = \{\beta_1, \beta_2\}$.

Case (iii)-1 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1, \beta_2\}$.

Then for adjacent vertices of v_1, $v_3, v_4, v_5, v_6 \leq_P \alpha_1$. Since v_4 is not adjacent to v_3 and v_5, $\beta_1 \leq_P v_3, v_5$, and $\beta_2 \leq_P v_4$. Since v_6 is adjacent to v_3, $\beta_1 \leq_P v_6$. Thus v_6 is adjacent to v_5, which is a contradiction.

Case (iii)-2 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1\}$.

Then for adjacent vertices of v_1, $\beta_1 \leq_P v_3, v_4, v_5, v_6$. Since v_4 is not adjacent to v_3 and v_5, $v_3, v_5 \leq_P \alpha_1$, and $v_4 \leq_P \alpha_2$. Since v_6 is adjacent to v_1, v_3, v_4 and not adjacent to v_5, $v_6 \leq_P \alpha_2$ and $v_6, v_3 \leq_P \alpha_3$.

Since v_4 is not adjacent to v_5, $\mathrm{Max}_P(v_4) = \{\alpha_2\}$ or $\mathrm{Max}_P(v_5) = \{\alpha_1\}$. T hen we have Case (iii)-1, $|\,\mathrm{Max}_P(v_4)| = |\,\mathrm{Min}_P(v_4)| = 1$ or $|\,\mathrm{Max}_P(v_5)| = |\,\mathrm{Min}_P(v_5)| = 1$.

Case (iii)-3 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1\}$.

Then for adjacent vertices of v_1, $\beta_1 \leq_P v_3, v_4, v_5, v_6$. Since $\langle\{v_5, v_3, v_6, v_4\}\rangle_V$ is a path, we have two cases: (1) $v_5, v_3 \leq_P \alpha_1$, $v_3, v_6 \leq_P \alpha_2$ and $v_6, v_4 \leq_P \alpha_3$, (2) $v_5, v_3 \leq_P \alpha_1$, $v_3, v_6 \leq_P \alpha_4$ and $v_6, v_4 \leq_P \alpha_3$.

In the case $v_5, v_3 \leq_P \alpha_1$, $v_3, v_6 \leq_P \alpha_2$ and $v_6, v_4 \leq_P \alpha_3$, if $\mathrm{Max}_P(v_4) = \{\alpha_3\}$ or $\mathrm{Max}_P(v_5) = \{\alpha_1\}$, we have Case (iii)-1, $|\,\mathrm{Max}(v_4)| = |\,\mathrm{Min}(v_4)| = 1$ or $|\,\mathrm{Max}(v_5)| = |\,\mathrm{Min}(v_5)| = 1$. So $v_4, v_5 \leq_P \alpha_4$. Then v_4 is adjacent to v_5, which is a contradiction.

In the case $v_5, v_3 \leq_P \alpha_1$, $v_3, v_6 \leq_P \alpha_4$ and $v_6, v_4 \leq_P \alpha_3$, if $\mathrm{Max}_P(v_4) = \{\alpha_3\}$ or $\mathrm{Max}_P(v_5) = \{\alpha_1\}$, then we have Case (iii)-1, $|\,\mathrm{Max}(v_4)| = |\,\mathrm{Min}(v_4)| = 1$ or $|\,\mathrm{Max}(v_5)| = |\,\mathrm{Min}(v_5)| = 1$. So $v_4, v_5 \leq_P \alpha_2$, because v_4 is not adjacent to v_3 and v_5 is not adjacent to v_6. Then v_4 is adjacent to v_5, which is a contradiction.

Case (iii)-4 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1\}$.

Then for adjacent vertices of v_1, $\beta_1 \leq_P v_3, v_4, v_5, v_6$. Since $\langle\{v_5, v_3, v_6, v_4\}\rangle_V$ is a path, $v_5, v_3 \leq_P \alpha_1$, and $v_3, v_6 \leq_P \alpha_2$ and $v_6, v_4 \leq_P \alpha_3$,

If $\mathrm{Max}_P(v_4) = \{\alpha_3\}$ or $\mathrm{Max}_P(v_5) = \{\alpha_1\}$, we have Case (iii)-1, $|\,\mathrm{Max}(v_4)| = |\,\mathrm{Min}(v_4)| = 1$ or $|\,\mathrm{Max}(v_5)| = |\,\mathrm{Min}(v_5)| = 1$. So $v_4, v_5 \leq_P \alpha_4$, because v_4 is not adjacent to v_3 and v_5 is not adjacent to v_6. Then v_4 is adjacent to v_5, which is a contradiction.

Case (iii)-5 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1, \beta_2\}$.

If $|\,\mathrm{Min}_P(v_i)| = 1$ $(i = 2, 3, \cdots, 7)$, we have Case (iii)-2, Case (iii)-3, Case (iii)-4, or $|\,\mathrm{Max}_P(v_i)| = |\,\mathrm{Min}_P(v_i)| = 1$. We consider the case $\beta_1 \leq_P v_2, v_3, v_4, v_5, v_6, v_7$ and $\beta_2 \leq_P v_2, v_3, v_4, v_5, v_6, v_7$.

Then $\mathrm{sDB}(P)$ has at most four cliques, $C_i = \{w \notin \mathrm{Max}(P) \cup \mathrm{Min}(P)\ ;\ w \leq_P \alpha_i\}$ $(i = 1, 2, 3, 4)$, which is a contradiction.

Case (iii)-6 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1, \beta_2\}$.

Since v_2 is not adjacent to v_1, $v_2 \leq_P \alpha_4$. We have Case (iii)-1 or $|\,\mathrm{Max}_P(v_2)| = |\,\mathrm{Min}_P(v_2)| = 1$.

Next we consider the case $|\,\mathrm{Max}(P)| = 3$ and $|\,\mathrm{Min}(P)| = 3$. Let $\mathrm{Max}(P) = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathrm{Min}(P) = \{\beta_1, \beta_2, \beta_3\}$.

Case (iii)-7 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1, \beta_2\}$.

Then for adjacent vertices of v_1, $v_3, v_4, v_5, v_6 \leq_P \alpha_1$. Since v_4 is not adjacent to v_3 and v_5, $\beta_1 \leq_P v_3, v_5$ and $\beta_2 \leq_P v_4$. Since v_6 is adjacent to v_1 and v_3, we have two cases: (1) $\beta_1 \leq_P v_6$, and (2) $\beta_2 \leq_P v_6$ and $\beta_3 \leq_P v_3, v_6$.

In the case $\beta_1 \leq_P v_6$, v_6 is adjacent to v_5. We have a contradiction.

In the case $\beta_2 \leq_P v_6$ and $\beta_3 \leq_P v_3, v_6$, the element v_4 is incomparable with β_1, β_3, and v_5 is incomparable with β_2, β_3. Since v_2 is adjacent to v_4 and v_5, β_1, $\beta_2 \leq_P v_2$.

Since v_7 is adjacent to v_4 and v_5, $\beta_1, \beta_2 \leq_P v_7$. And $v_2 \parallel \alpha_1$ and $v_7 \parallel \alpha_1$, because v_2 and v_7 are not adjacent to v_1

If $v_2 \leq_P \alpha_2$ and $v_2 \parallel \alpha_3$, then $v_4, v_5, v_6, v_7 \leq_P \alpha_2$, because v_2 is adjacent to v_4, v_5, v_6 and v_7. Then v_7 is adjacent to v_6, which is a contradiction. So v_2 $\leq_P \alpha_2$ and $v_2 \leq_P \alpha_3$. Since v_7 is adjacent to v_3, $v_7, v_3 \leq_P \alpha_2$ or $v_7, v_3 \leq_P \alpha_3$. Then v_2 is adjacent to v_3, which is a contradiction.

Case (iii)-8 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1, \beta_2, \beta_3\}$.

Then for adjacent vertices of v_1, $v_3, v_4, v_5, v_6 \leq_P \alpha_1$. Since $\langle\{v_5, v_3, v_6, v_4\}\rangle_V$ is a path, $\beta_1 \leq_P v_5, v_3$, $\beta_2 \leq_P v_3, v_6$ and $\beta_3 \leq_P v_6, v_4$.

Since v_2 and v_7 are adjacent to v_4 and v_5, $\beta_1, \beta_3 \leq_P v_2$ and $\beta_1, \beta_3 \leq_P v_7$. And $v_2 \parallel \alpha_1$ and $v_7 \parallel \alpha_1$, because v_2 and v_7 are not adjacent to v_1. If $v_2 \leq_P \alpha_2$ and $v_2 \parallel \alpha_3$, then $v_4, v_5, v_6, v_7 \leq_P \alpha_2$, because v_2 is adjacent to v_4, v_5, v_6 and v_7. Then v_6 is adjacent to v_7, which is a contradiction. So $v_2 \leq_P \alpha_2$ and $v_2 \leq \alpha_3$. Since v_7 is adjacent to v_3, $v_7, v_3 \leq_P \alpha_2$ or $v_7, v_3 \leq_P \alpha_3$. Then v_2 is adjacent to v_3, which is a contradiction.

Case (iii)-9 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1, \beta_2, \beta_3\}$.

Since v_2 is not adjacent to v_1, $\mathrm{Max}_P(v_2) = \{\alpha_3\}$. We have Case (iii)-7, Case (iii)-8, or $|\mathrm{Max}_P(v_2)| = |\mathrm{Min}_P(v_2)| = 1$.

Case (iii)-10 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1\}$.

This case is the dual case of Case (iii)-7.

Case (iii)-11 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1\}$.

This case is the dual case of Case (iii)-8.

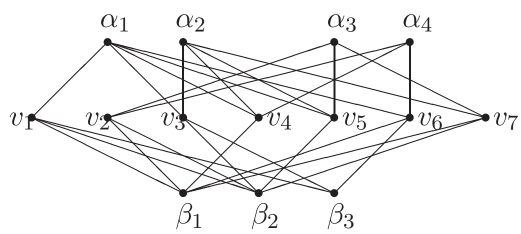

Figure 3: $P_{K_7-E(C_7)}$.

Case (iii)-12 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1, \beta_2\}$.

This case is the dual case of Case (iii)-9.

Case (iii)-13 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1, \beta_2\}$.

If $|\,\mathrm{Max}_P(v_2)| = 1$, we have Case (iii)-7, Case (iii)-8, or $|\,\mathrm{Min}_P(v_2)| = 1$. If $|\,\mathrm{Min}_P(v_2)| = 1$, we have Case (iii)-10, Case (iii)-11, or $|\,\mathrm{Max}_P(v_2)| = 1$. Thus $|\,\mathrm{Max}_P(v_2)| \geq 2$ and $|\,\mathrm{Min}_P(v_2)| \geq 2$. Then v_2 is adjacent to v_1, which is a contradiction.

Thus $\zeta(K_7 - E(C_7)) \geq 7$. We have a poset with $\mathrm{sDB}(P) \cong (K_7 - E(C_7)) \cup \overline{K}_7$ (see Figure 3). Therefore $\zeta(K_n - E(C_7)) = \zeta(K_7 - E(C_7)) = 7$.

(iv) $\zeta(K_n - E(C_8)) = 7$ $(n \geq 8)$:

We consider a graph $K_8 - E(C_8)$ (see Figure 4). We assume that there exists a poset P such that $\mathrm{sDB}(P) \cong (K_8 - E(C_8)) \cup \overline{K}_6$. In the case of $|\,\mathrm{Max}(P)| = 5$ and $|\,\mathrm{Min}(P)| = 1$ (or $|\,\mathrm{Max}(P)| = 1$ and $|\,\mathrm{Min}(P)| = 5$) $\mathrm{sDB}(P)$ has at most five cliques. So we consider the case $|\,\mathrm{Max}(P)| = 4$ and $|\,\mathrm{Min}(P)| = 2$, and the case $|\,\mathrm{Max}(P)| = |\,\mathrm{Min}(P)| = 3$ by Theorem 2.1. And the case $|\,\mathrm{Max}(P)| = 2$ and $|\,\mathrm{Min}(P)| = 4$ is the dual. First we consider the case $|\,\mathrm{Max}(P)| = 4$ and $|\,\mathrm{Min}(P)| = 2$. Let $\mathrm{Max}(P) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\mathrm{Min}(P) = \{\beta_1, \beta_2\}$.

Case (iv)-1 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1, \beta_2\}$.

Then for adjacent vertices of v_1, $v_3, v_4, v_5, v_6 \leq_P \alpha_1$. Since v_4 is not adjacent to v_3 and v_5, $\beta_1 \leq_P v_3, v_5$, and $\beta_2 \leq_P v_4$. Since v_6 is adjacent to v_3, $\beta_1 \leq_P v_6$.

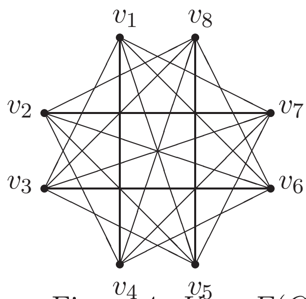

Figure 4: $K_8 - E(C_8)$.

Then v_6 is adjacent to v_5, which is a contradiction.

Case (iv)-2 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1\}$.

Then for adjacent vertices of v_1, $\beta_1 \leq_P v_3, v_4, v_5, v_6$. Since v_4 is not adjacent to v_3 and v_5, $v_3, v_5 \leq_P \alpha_1$, and $v_4 \leq_P \alpha_2$. Since v_6 is adjacent to v_1, v_3, v_4 and not adjacent to v_5, $v_6 \leq_P \alpha_2$ and $v_6, v_3 \leq_P \alpha_3$.

Since v_4 is not adjacent to v_5, $\mathrm{Max}_P(v_4) = \{\alpha_2\}$ or $\mathrm{Max}_P(v_5) = \{\alpha_1\}$. Then we have Case (iv)-1, $|\,\mathrm{Max}_P(v_4)| = |\,\mathrm{Min}_P(v_4)| = 1$ or $|\,\mathrm{Max}_P(v_5)| = |\,\mathrm{Min}_P(v_5)| = 1$.

Case (iv)-3 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1\}$.

Then for adjacent vertices of v_1, $\beta_1 \leq_P v_3, v_4, v_5, v_6$. Since $\langle\{v_5, v_3, v_6, v_4\}\rangle_V$ is a path, we have two cases: (1) $v_5, v_3 \leq_P \alpha_1$, $v_3, v_6 \leq_P \alpha_2$ and $v_6, v_4 \leq_P \alpha_3$, (2) $v_5, v_3 \leq_P \alpha_1$, $v_3, v_6 \leq_P \alpha_4$ and $v_6, v_4 \leq_P \alpha_3$.

In the case $v_5, v_3 \leq_P \alpha_1$, $v_3, v_6 \leq_P \alpha_2$ and $v_6, v_4 \leq_P \alpha_3$, if $\mathrm{Max}_P(v_4) = \{\alpha_3\}$ or $\mathrm{Max}_P(v_5) = \{\alpha_1\}$, we have Case (iv)-1, $|\,\mathrm{Max}(v_4)| = |\,\mathrm{Min}(v_4)| = 1$ or $|\,\mathrm{Max}(v_5)| = |\,\mathrm{Min}(v_5)| = 1$. So $v_4, v_5 \leq_P \alpha_4$, because v_4 is not adjacent to v_3 and v_5 is not adjacent to v_6. Then v_4 is adjacent to v_5, which is a contradiction.

In the case $v_5, v_3 \leq_P \alpha_1$, $v_3, v_6 \leq_P \alpha_4$ and $v_6, v_4 \leq_P \alpha_3$, if $\mathrm{Max}_P(v_4) = \{\alpha_3\}$ or $\mathrm{Max}_P(v_5) = \{\alpha_1\}$, we have Case (iv)-1, $|\,\mathrm{Max}(v_4)| = |\,\mathrm{Min}(v_4)| = 1$ or $|\,\mathrm{Max}(v_5)| = |\,\mathrm{Min}(v_5)| = 1$. So $v_4, v_5 \leq_P \alpha_2$, because v_4 is not adjacent to v_3 and v_5 is not adjacent to v_6. Then v_4 is adjacent to v_5, which is a contradiction.

Case (iv)-4 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1\}$.

Then for adjacent vertices of v_1, $\beta_1 \leq_P v_3, v_4, v_5, v_6$. Since $\langle\{v_5, v_3, v_6, v_4\}\rangle_V$ is a path, $v_5, v_3 \leq_P \alpha_1$, $v_3, v_6 \leq_P \alpha_2$ and $v_6, v_4 \leq_P \alpha_3$.

If $\mathrm{Max}_P(v_4) = \{\alpha_3\}$ or $\mathrm{Max}_P(v_5) = \{\alpha_1\}$, we have Case (iv)-1, $|\mathrm{Max}_P(v_4)| = |\mathrm{Min}_P(v_4)| = 1$ or $|\mathrm{Max}_P(v_5)| = |\mathrm{Min}_P(v_5)| = 1$. So $v_4, v_5 \leq_P \alpha_4$, because v_4 is not adjacent to v_3 and v_5 is not adjacent to v_6. Then v_4 is adjacent to v_5, which is a contradiction.

Case (iv)-5 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1, \beta_2\}$.

If $|\mathrm{Min}_P(v_i)| = 1$ $(i = 2, 3, \cdots, 8)$, we have Case (iv)-2, Case (iv)-3, Case (iv)-4 or $|\mathrm{Max}_P(v_i)| = |\mathrm{Min}_P(v_i)| = 1$. We consider the case $\beta_1 \leq_P v_2, v_3, v_4, v_5, v_6, v_7, v_8$ and $\beta_2 \leq_P v_2, v_3, v_4, v_5, v_6, v_7, v_8$. Then $\mathrm{sDB}(P)$ has at most four cliques, $C_i = \{w \notin \mathrm{Max}(P) \cup \mathrm{Min}(P)\ ;\ w \leq_P \alpha_i\}$ $(i = 1, 2, 3, 4)$, which is a contradiction.

Case (iv)-6 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1, \beta_2\}$.

Since v_2 is not adjacent to v_1, $v_2 \leq_P \alpha_4$. We have Case (iv)-1 or $|\mathrm{Max}_P(v_2)| = |\mathrm{Min}_P(v_2)| = 1$.

Next we consider the case $|\mathrm{Max}(P)| = 3$ and $|\mathrm{Min}(P)| = 3$. Let $\mathrm{Max}(P) = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathrm{Min}(P) = \{\beta_1, \beta_2, \beta_3\}$.

Case (iv)-7 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1, \beta_2\}$.

Then for adjacent vertices of v_1, $v_3, v_4, v_5, v_6, v_7 \leq_P \alpha_1$. Since v_4 is not adjacent to v_3 and v_5, $\beta_1 \leq_P v_3, v_5$ and $\beta_2 \leq_P v_4$. Since v_6 is adjacent to v_1 and v_3, we have two cases: (1) $\beta_1 \leq_P v_6$, and (2) $\beta_2 \leq_P v_6$ and $\beta_3 \leq_P v_3, v_6$.

In the case $\beta_1 \leq_P v_6$, v_6 is adjacent to v_5. We have a contradiction.

In the case $\beta_2 \leq_P v_6$ and $\beta_3 \leq_P v_3, v_6$, $\mathrm{Min}_P(v_4) = \{\beta_2\}$, because v_4 is not adjacent to v_3. Since v_7 is adjacent to v_4, $\beta_2 \leq_P v_7 \leq_P \alpha_1$. Then v_7 is adjacent to v_6, which is a contradiction.

Case (iv)-8 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1, \beta_2, \beta_3\}$.

Then for adjacent vertices of $v_1, v_3, v_4, v_5, v_6, v_7 \leq_P \alpha_1$. Since $\langle\{v_5, v_3, v_6, v_4\}\rangle_V$ is a path, $\beta_1 \leq_P v_5, v_3$, $\beta_2 \leq_P v_3, v_6$ and $\beta_3 \leq_P v_6, v_4$. Since v_4 is not adjacent to v_3, $\mathrm{Min}_P(v_4) = \{\beta_3\}$. Since v_7 is adjacent to v_4, $\beta_3 \leq_P v_7 \leq_P \alpha_1$. Then v_7 is adjacent to v_6, which is a contradiction.

Case (iv)-9 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1, \beta_2, \beta_3\}$.

Since v_2 is not adjacent to v_1, $\mathrm{Max}_P(v_2) = \{\alpha_3\}$. We have Case (iv)-7, or Case (iv)-8, or $|\mathrm{Max}_P(v_2)| = |\mathrm{Min}_P(v_2)| = 1$.

Case (iv)-10 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1\}$.

This case is the dual case of Case (iv)-7.

Case (iv)-11 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1\}$.

This case is the dual case of Case (iv)-8.

Case (iv)-12 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1, \beta_2\}$.

This case is the dual case of Case (iv)-9.

Case (iv)-13 : there exists an element $v = v_1$ such that $\mathrm{Max}_P(v_1) = \{\alpha_1, \alpha_2\}$ and $\mathrm{Min}_P(v_1) = \{\beta_1, \beta_2\}$.

If $|\mathrm{Max}_P(v_2)| = 1$, we have Case (iv)-7, Case (iv)-8, or $|\mathrm{Min}_P(v_2)| = 1$. If $|\mathrm{Min}_P(v_2)| = 1$, we have Case (iv)-10, Case (iv)-11, or $|\mathrm{Max}_P(v_2)| = 1$. Thus $|\mathrm{Max}_P(v_2)| \geq 2$ and $|\mathrm{Min}_P(v_2)| \geq 2$. Then v_2 is adjacent to v_1, which is a contradiction.

Thus $\zeta(K_8 - E(C_8)) \geq 7$, and $\zeta(K_n - E(C_8)) = \zeta(K_8 - E(C_8)) = 7$.

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