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A STUDY ON THE QUASI-HYPERBOLIC ALGEBRA $QHA_5^{(2)}$

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Abstract: In this work a class of indefinite quasi-hyperbolic type of Kac-Moody Algebra $QHA_5^{(2)}$ is considered. A realization for these algebras as a graded Lie Algebras of Kac-Moody type is obtained. Using the techniques of homological theory, the homology modules upto level three are computed and also by spectral sequences theory the sturcture of the components of the maximal ideal upto level four is determined.

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1. Introduction

The theory of Kac-Moody algebra has developed into an area of rich theory and extensive applications to various branches of mathematics and mathematical physics. As far as the indefinite Kac-Moody algebras are concerned, understanding the structure and determination of multiplicities of roots of higher levels is an open problem yet to be explored in depth.

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Feingold and Frenkel [2] has computed level 2 multiplicities for the hyperbolic Kac-Moody Lie algebra $HA_1^{(1)}$. Using homological techniques and spectral sequences theory, Kang [5, 6, 8] has determined the structure and obtain the root multiplicities for roots upto level 5 for $HA_1^{(1)}$ and for $HA_2^{(1)}$ upto level 3. In [7], for the indefinite type of Kac-Moody Lie algebra $HA_n^{(1)}$, some root multiplicities are determined. A new class of extended hyperbolic Kac-Moody algebras was introduced in Sthanumoorthy and Uma Maheswari [12]. Sthanumoorthy and Uma Maheswari [11] have computed the root multiplicities for a particular class of extended hyperbolic Kac-Moody algebra $EHA_1^{(1)}$ and again considered the same algebra generally in Sthanumoorthy et al. [13]. In [15], the more general classes for $EHA_1^{(1)}$ and $EHA_2^{(2)}$ were considered and the root multiplicities upto level 3 were also computed. Another class of indefinite non-hyperbolic Kac-Moody algebras called Quasi-Hyperbolic Kac-Moody Lie algebras was introduced by Uma Maheswari in [16]. In [17, 18] the indefinite non-hyperbolic Kac-Moody type QHG_2, $QHA_2^{(1)}$ where realized as a graded Lie algebra of Kac-Moody type and the structure of the components of the maximal ideal upto level 4 were computed.

In this work, we are going to consider a class of Quasi-Hyperbolic indefinite type of Kac-Moody algebras $QHA_5^{(2)}$. We first give a realization for $QHA_5^{(2)}$ whose associated GCM is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & -a \\ -2 & 2 & -1 & -1 & -a \\ 0 & -1 & 2 & 0 & -a \\ 0 & -1 & 0 & 2 & -a \\ -a & -a & -a & -a & 2 \end{pmatrix}$$

where $a > 2$, $a \in Z^+$ as a graded Lie algebra of Kac-Moody type and then using the homological techniques developed by Benkart et al. [1] and Kang [5, 6, 7, 8], we compute the homology modules upto level three and the structure of the components of the maximal ideal upto level four.

2. Preliminaries

We first recall some results of Kac-Moody algebreas [4, 10, 19] and the general construction of graded Lie algebras of Kac-Moody type (Benkart et al. [1]).

Definition 1. [10] An integer matrix $A = (a_{ij})_{i,j=1}^n$ is a Generalized Cartan Matrix (abbreviated as GCM) if it satisfies the following conditions:

(i) $a_{ii} = 2 \; \forall \; i = 1, 2, \ldots, n$

(ii) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0 \; \forall \; i, j = 1, 2, \ldots, n$

(iii) $a_{ij} \leq 0 \; \forall \; i, j = 1, 2, \ldots, n.$

Let us denote the index set of A by $N = \{1, \ldots, n\}$. A GCM A is said to decomposable if there exist two non-empty subsets $I, J \subset N$ such that $I \cup J = N$ and $a_{ij} = a_{ji} = 0 \; \forall \; i \in I$ and $j \in J$. If A is not decomposable, it is said to be indecomposable.

Definition 2. [4] A realization of a matrix $A = (a_{ij})_{i,j=1}^n$ is a triple (H, Π, Π^v) where l is the rank of A, H is a $2n - l$ dimensional complex vector space, $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ and $\Pi^v = \{\alpha_1^v, \ldots, \alpha_n^v\}$ are linearly independent subsets of H^* and H respectively, satisfying $\alpha_j(\alpha_i^v) = a_{ij}$ for $i, j = 1, \ldots, n$. Π is called the root basis. Elements of Π are called simple roots. The root lattice generated by Π is $Q = \sum_{i=1}^{n} Z\alpha_i$.

Definition 3. [4] The Kac-Moody algebra $g(A)$ associated with a GCM $A = (a_{ij})_{i,j=1}^n$ is the Lie algebra generated by the elements e_i, f_i, $i = 1, 2, \ldots, n$ and H with the following defining relations:

$$[h, h'] = 0, \quad h, h' \in H,$$
$$[e_i, f_j] = \delta_{ij}\alpha_i^v,$$
$$[h, e_j] = \alpha_j(h)e_j,$$
$$[h, f_j] = -\alpha_j(h)f_j, \quad i, j \in N,$$
$$(ade_i)^{1-a_{ij}} e_j = 0,$$
$$(adf_i)^{1-a_{ij}} f_j = 0, \; \forall \; i \neq j, i, j \in N.$$

The Kac-Moody algebra $g(A)$ has the root space decomposition where $g(A) = \underset{\alpha \in Q}{\oplus} g_\alpha(A)$ where $g_\alpha(A) = \{x \in g(A)/[h, x] = \alpha(h)x, \text{ for all } h \in H\}$. An element α, $\alpha \neq 0$ in Q is called a root if $g_\alpha \neq 0$. Let $Q = \sum_{i=1}^{n} Z_+\alpha_i.Q$ has a partial ordering "$\leq$" defined by $\alpha \leq \beta$ if, $\beta - \alpha \in Q$, where $\alpha, \beta \in Q$.

Definition 4. [4] For any $\alpha \in Q$ and $\alpha = \sum_{k=1}^{n} k_i\alpha_i$, define support of α, written as *supp* α, by *supp* $\alpha = \{i \in N/k_i \neq 0\}$. Let $\Delta(= \Delta(A))$ denote the set of all roots of $g(A)$ and Δ, the set of all positive roots of $g(A)$. We have $\Delta_- = -\Delta_+$ and $\Delta = \Delta_+ \cup \Delta_-$.

Definition 5. [4] A GCM A is called symmetrizable if DA is symmetric for some diagonal matrix $D = diag(q_1, \ldots, q_n)$, with $q_i > 0$ and q_i's are rational numbers.

Proposition 6. *[4] A GCM $A = (a_{ij})_{i,j=1}^n$ is symmetrizable if and only if there exists an invariant, bilinear, symmetric, non degenerate form on $g(A)$.*

Definition 7. [4] To every GCM A is associated a Dynkin diagram $S(A)$ defined as follows: (A) has n vertices and vertices i and j are connected by max $\{|a_{ij}|, |a_{ji}|\}$ number of lines if $a_{ij}.a_{ji} \leq 4$ and there is an arrow pointing towards i if $|a_{ij}| > 1$. If $a_{ij}.a_{ji} > 4$, i and j are connected by a bold faced edge, equipped with the ordered pair $(|a_{ij}|, |a_{ji}|)$ of integers.

Theorem 8. *[19] Let A be a real $n \times n$ matrix satisfying $(m1)$, $(m2)$ and $(m3)$.*

(m1) A is indecomposable;

(m2) $a_{ij} \leq 0$ for $i \neq j$;

(m3) $a_{ij} = 0$ implies $a_{ji} = 0$

Then one and only one of the following three possibilities holds for both A and ${}^tA$:

(i) $detA \neq 0$; there exists $u > 0$ such that $Au > 0$; $Av \geq 0$ implies $v > 0$ or $v = 0$;

(ii) co rank $A = 1$; there exists $u > 0$ such that $Au = 0$; $Av \geq 0$ implies $Av = 0$;

(iii) there exists $u > 0$ such that $Au < 0$; $Av \geq 0$, $v \geq 0$ imply $v = 0$.

Then A is of finite, affine or indefinite type iff (i), (ii) or (iii) (respectively) is satisfied.

Definition 9. [19] A Kac-Moody algebra $g(A)$ is said to be of finite, affine or indefinite type if the associated GCM A is of finite, affine or indefinite type respectively.

Definition 10. [16] Let $A = (a_{ij})_{i,j=1}^n$ be an indecomposable GCM of indefinite type. We define the associated Dynkin diagram $S(A)$ to be of Quasi Hyperbolic (QH) type if $S(A)$ has a proper connected sub diagram of hyperbolic types with $n-1$ vertices. The GCM A is of QH type if $S(A)$ is of QH type. We then say the Kac-Moody algebra $g(A)$ is of QH type.

2.1. General Construction of Graded Lie Algebra

We use the following notations:

G - a Lie algebra over a field of characteristic zero.

V, V' - two G-modules.

$\psi : V' \otimes V \to G$ a G-module homomorphism. Consider, $G_0 = G$, $G_{-1} = V$, $G_1 = V'$

$G_+ = \sum_{n \geq 1} G_n$ (resp. $G_- = \sum_{n \geq 1} G_{-n}$) denote the free Lie algebra generated by V' (resp. V).

G_n (resp G_{-n}) for $n > 1$ is the space of all products of n vectors from V' (resp. V).

$G = \sum_{n=-\infty}^{\infty} G_n$ is given a Lie algebra structure by defining the Lie bracket [,] as follows:

For $a, b \in G$, $v \in V$, $w \in V'$ define $[a, v] = a.v = -[v, a]$ and $[a, w] = a.w = -[w, a]$.

For $a, b \in G$ let $[a, b]$ denote the bracket operation in G.

For $w \in V'$, $v \in V$, $[w, v] = \psi(w \otimes v) = -[v, w]$,

Extending this bracket operation,

$G = \sum_{n \in Z} G_n$ becomes a graded Lie algebra which is generated by its local part $G_{-1} + G_0 + G_1$.

For $n \geq 1$ define the subspaces

$I_{\pm} = \{x \in G_{\pm}/[y_1, [\dots[y_{n-1}, x]]\dots] = 0$ for all $y_1, \dots, y_{n-1} \in G_{\mp}\}$. Set $I_+ = \sum_{n>1} I_n$, $I_- = \sum_{n>1} I_{-n}$.

Then I_+ and I_- are ideals of G and the ideal is the largest graded ideal of G trivially intersecting $G_{-1} + G_0 + G_1$.

For $n > 1$, let $L_{\pm n} = G_{\pm n}/I_{\pm n}$. Consider

$L = L(G, V, V', \psi) = G_-/I_- \oplus G_0 \oplus G_+/I_+ = \cdots \oplus L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \oplus \dots$, where $L_0 = G_0$, $L_1 = G_1$, $L_{-1} = G_{-1}$.

Then $L = \oplus_{n \in Z} L_n$ is a graded Lie algebra generated by its local part $V \oplus G \oplus V'$ and $L = G/I$.

By the suitable choice of V (written as the direct sum m irreducible highest weight modules), the contragradient V^*, the basis elements and the homomorphism $\psi : V^* \otimes V \to g^e$, form the graded Lie algebra $L = L(g^e, V, V^*, \psi)$. For details one can refer to [1, 5].

Theorem 11. *[1] L is a Z^{n+m}-graded algebra.*

Theorem 12. *[1] Let $\phi : A(C) \to L$ be the Lie algebra homomorphism sending $E_i \to e_i$, $F_i \to f_i$, $H_i \to h_i$. Then ϕ has kernel as $I(C)$ and $I(C)$ is the* *largest graded ideal of $A(C)$ trivially intersecting the span of $H_1, \ldots, H_{n+m}$. Also $\phi : A(C)/I(C) \to L$ is an isomorphism.*

Proposition 13. *[1] The matrix C has rank $2n-l$ and C is symmetrizable.*

We now recall the definition of homology of Lie algebra and Hochschild-Serre spectral sequence (Kang [5]).

Let $C_q(G, V)$, $q > 0$ be space of all q-dimensional chains of the Lie algebra G with coefficients in V to be $\wedge^q(G) \otimes V$.

The differential $d_q : C_q(G, V) \to C_{q-1}(G, V)$ is defined to be

$$\begin{aligned} & d_q(g_1 \wedge \cdots \wedge g_q \otimes v) \\ & \quad = \sum_{1 \le s \le t \le q} (-1)^{s+t-1}([g_s, g_t]) \wedge g_1 \wedge \cdots \wedge \tilde{g}_s \wedge \cdots \wedge \tilde{g}_t \wedge \cdots \wedge g_q) \otimes v \\ & \qquad + \sum_{1 \le s \le q} (-1)^s (g_1 \wedge \cdots \wedge \tilde{g}_s \wedge \cdots \wedge g_q) \otimes g_s.v, \end{aligned}$$

for $v \in V$, $g_1 \ldots g_q \in G$.

For $q < 0$, define $C_q(G, V) = 0$ and $d_q = 0$. Then $d_q \circ d_{q-1} = 0$.

The homology of the complex $(C, d) = \{C_q(G, V), d_q\}$ is called the homology of the Lie algebra G with coefficients in V and is denoted by $H_q(G, V)$. If $V = C$, we simply write $H_q(G)$ for $H_q(G, C)$.

Let G, V, $C_q(G, V)$ are completely reducible modules in the category O over a Kac-Moody algebra $g(A)$ with d_q having $g(A)$-module homomorphisms. Let I be ideal of G and $L = G/I$.

Define a filtration $\{K_p = K_pC\}$ of the complex $\{C, d\}$ by $K_pC_{p+q} = \{g_1 \wedge g_2 \wedge \cdots \wedge g_{p+q} \otimes v | g_i \in I \text{ for } p+1 \le i \le p+q\}$.

This gives rise to a spectral sequence $\{E^r_{p,q}, d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ such that $E^2_{p,q} \cong H_p(L, H_q(I, V))$, where $E^r_{p,q}$'s are determined by $E^{r+1}_{p,q} = Ker(d_r : E^r_{p,q} \to E^r_{p-r,q+r-1})/Im(d_r : E^r_{p+r,q-r+1} \to E^r_{p,q})$ with boundary homomorphisms $d_{r+1} : E^r_{p,q} \to E^r_{p-r-1,q+r}$.

The modules $E^r_{p,q}$ become stable for r $> max(p, q+1)$ for each (p, q) and the stable module is denoted by $E^\infty_{p,q}$. The spectral sequence $\{E^r_{p,q}, d_r\}$ converges to $H_n(G, V)$ in the following sense: $H_n(G, V) = \bigoplus_{p+q=n} E^\infty_{p,q}$.

Then we get the following Hochschild-Serre five term exact sequence (Kang, [5]):

$H_2(G, V) \to H_2(L, H_0(I, V) \to H_0(L, H_1(I, V)) \to H_1(G, V) \to H_1(L, H_0(I, V)) \to 0$.

Consider $G = \bigoplus_{n \ge 1} G_n$ be the free Lie algebra generated by the subspace G_1 and $I = \bigoplus_{n \ge m} I_n$ be the graded ideal of G generated by the subspace I_m for

$m \geq 2$. Consider the quotient algebra $L = G/I$. Then $L = \underset{n\geq 1}{\oplus} L_n$ is also a graded Lie algebra generated by the subspace $L_1 = G_1$. Let $J = I/[I,I]$. J is an L-module via adjoint action generated by the subspace J_m. As vector spaces, $J_n \cong I_n$ for $m \leq n < 2m$. Suppose that I_m and G_1 are modules over a Kac-Moody algebra $g(A)$.

Then G_n has a $g(A)$-module structure such that $x.[v,w] = [x.v, w] + [v, x.w]$ for $x \in g(A)$, $v \in G$, $w \in G_{n-1}$; I_n also has a similar module structure. We also have the induced module structure of the homogeneous subspaces L_n, J_n. Then we have the following theorem proved in Kang [5].

Theorem 14. *[5] There is an isomorphism of $g(A)$-modules $H_j(L,J) \cong H_{j+2}(L)$, for $j \geq 1$. In particular $I_{m+1} \cong (G_1 \otimes I_m)/H_3(L)_{m+1}$.*

For arbitrary $j \geq m$, set $I^{(j)} = \sum_{n\geq j} I_n$; then $I^{(j)}$ is an ideal of G generated by the subspace I_j. We consider the quotient algebra $L^{(j)} = G/I^{(j)}$. Let $N^{(j)} = I^{(j)}/I^{(j-1)}$. In this notation $L = L^{(m)}$. Then we have an important relation: $I_{j+1} \cong (G_1 \otimes I_j)/H_3(L^{(j)})_{j+1}$. And, there exists a spectral sequence $\{E^r_{p,q}, d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ converging to $H_*(L^{(j)})$ such that and $E^2_{p,q} \cong H_p(L^{(j-1)}) \otimes \wedge^q(I_{j-1})$ and $H_3(L^{(j)}) \cong E^\infty_{3,0} \oplus E^\infty_{2,1} \oplus E^\infty_{1,2} \oplus E^\infty_{0,3}$.

Lemma 15. *[5] In the above notation, $H_2(L) \cong I_m$.*

Let us recall the Kostant's formula for symmetrizable Kac-Moody algebras Liu, [9]:

Let $A = (a_{ij})^n_{i,j=1}$ be a symmetrizable GCM. Let $\Delta \subset \hbar^*$, Δ^+, Δ^- denote the root system of $g(A)$, positive and negative roots, respectively, of $g(A)$. We have the triangular decomposition : $g(A) = n^- \oplus h \oplus n^+$, where $n^\pm = \underset{\alpha\in\Delta^\pm}{\oplus} g_\alpha$.

Let $S = \{1, \ldots, s\}$ be a subset of $N = \{1, \ldots, n\}$ and g_s be the subalgebra of $g(A)$ generated by the elements e_i, f_i, $i = 1, \ldots, s$ and h. Let Δ_s^+ denote the set of positive roots generated by $\alpha_1, \ldots, \alpha_s$ and $\Delta_s^- = -\Delta_s^+$. Then g_s has the corresponding triangular decomposition : $g_s = n_s^- \oplus h \oplus n_s^+$, where $n_s^\pm = \underset{\alpha\in\Delta_s^\pm}{\oplus} g_\alpha$ and $\Delta_s = \Delta_s^+ \cup \Delta_s^-$ is the root system of g_s. Let $\Delta^\pm(s) = \Delta^\pm/\Delta_s^\pm$ and $n^\pm(S) = \underset{\alpha\in\Delta^\pm(S)}{\oplus} g_\alpha$ Then $g(A) = n^-(S) \oplus g_s \oplus n^+(S)$. Let $W(S) = \{w \in W/w\Delta^- \cap \Delta^+ \subset \Delta^+(S)\}$. For $\lambda \in h^*$ denote by $\tilde{V}(\lambda)$, the irreducible highest weight module over $g(A)$ and $V(\lambda)$ the irreducible highest weight module over g_s.

Theorem 16 (Kostant's formula)**.** *[9]*

$$H_j(n^-(S), \tilde{V}(\lambda)) \cong \bigoplus_{\substack{w\in W(S)\\ l(w)=j}} V(w(\lambda+\rho)-\rho).$$

Lemma 17. *[5] Suppose $w = w'r_j$ and $l(w) = l(w')+1$. Then $w \in W(S)$ if and only if $w' \in W(S)$ and $w'(\alpha_j) \in \Delta^+(S)$.*

3. Realization for $QHA_5^{(2)}$

Let us denote by $QHA_5^{(2)}$, the class of Quasi-Hyperbolic Kac-Moody algebras whose associated GCM is $\begin{pmatrix} 2 & -1 & 0 & 0 & -a \\ -2 & 2 & -1 & -1 & -a \\ 0 & -1 & 2 & 0 & -a \\ 0 & -1 & 0 & 2 & -a \\ -a & -a & -a & -a & 2 \end{pmatrix}$, where $a > 2$, $a \in Z^+$.

We start with the Kac-Moody Algebra $A_5^{(2)}$ associated with the GCM $A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$ is symmetrizable and we write $A = DB$ where $D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & -1/2 & -1/2 \\ 0 & -1/2 & 1 & 0 \\ 0 & -1/2 & 0 & 1 \end{pmatrix}$

The associated Dynkin diagram of $QHA_5^{(2)}$ is represented as

Consider the Kac-Moody algebra associated with the GCM $A_5^{(2)}$.

Let $(h, \Pi, \Pi^\vee)$ be the realization of A with $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee, \alpha_4^\vee\}$. Then we have the following bilinear relations $(\alpha_1, \alpha_1) = 2$, $(\alpha_1, \alpha_2) = -1$, $(\alpha_1, \alpha_3) = 0$, $(\alpha_1, \alpha_4) = 0$, $(\alpha_2, \alpha_1) = -1$, $(\alpha_2, \alpha_2) = 1$, $(\alpha_2, \alpha_3) = -1/2$, $(\alpha_2, \alpha_4) = -1/2$, $(\alpha_3, \alpha_1) = 0$, $(\alpha_3, \alpha_2) = -1/2$, $(\alpha_3, \alpha_3) = 1$, $(\alpha_3, \alpha_4) = 0$, $(\alpha_4, \alpha_1) = 0$, $(\alpha_4, \alpha_2) = -1/2$, $(\alpha_4, \alpha_3) = 0$, $(\alpha_4, \alpha_4) = 1$.

Let α_5' be the element in h^* such that $\alpha_5'(\alpha_1^\vee) = 0$, $\alpha_5'(\alpha_2^\vee) = 0$, $\alpha_5'(\alpha_3^\vee) = 0$, $\alpha_5'(\alpha_4^\vee) = 1$ and $(\alpha_5', \alpha_5') = \dfrac{2-6a+11a^2}{9a^2}$. Let us define, $\lambda = \alpha_1 + (2-a)\alpha_2 + \alpha_3 + (1-3a)\alpha_4 + 3a\alpha_5'$. Set $\alpha_5 = -\lambda$. Form the matrix $C = (< \alpha_i, \alpha_j >$

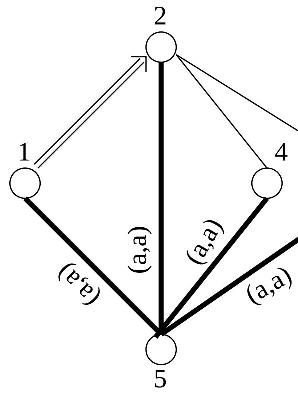

Figure 1

$)_{i,j=1}^{5}$. Then $C = \begin{pmatrix} 2 & -1 & 0 & 0 & -a \\ -2 & 2 & -1 & -1 & -a \\ 0 & -1 & 2 & 0 & -a \\ 0 & -1 & 0 & 2 & -a \\ -a & -a & -a & -a & 2 \end{pmatrix}$ where $a > 2$, $a \in Z^+$ is the symmetrizable GCM of Quasi-Hyperbolic type $QHA_5^{(2)}$.

Let V be the integrable highest weight irreducible module over G with the highest weight λ as defined. Let V^* be the contragradient of V and ψ be the mapping as defined earlier. Let G be the Kac-Moody algebra associated with the GCM $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$. Form the graded Lie algebra $L(G^e, V, V^*, \psi)$. Then $L \cong g(C)$ and L is a symmetrizable Kac-Moody algebra of Quasi-hyperbolic type associated with the GCM C.

Next, we compute the homology modules of the Kac-Moody algebra for $QHA_5^{(2)}$. We note here that, from the realization of $L = QHA_5^{(2)}$ as $L = L_- \oplus L_0 \oplus L_+ = G/I$ and using the involutive automorphism, it suffices to study about the negative part $L_- = G_-/I_-$.

Computation of Homology Modules

Let $S = \{1, 2, 3, 4\} \subset N = \{1, 2, 3, 4, 5\}$. Here g_s is the Kac-Moody Lie algebra $A_5^{(2)}$, $\Delta^+(S) = \{k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4 + k_5\alpha_5' \in \Delta^+/k_5 \neq 0\}$. Δ_s be the root system of g_s.

The only reflection of length 1 in $W(S)$ is r_5
$r_5(\rho) = \rho - \alpha_5$; $r_5(\rho) - \rho = -\alpha_5$. Therefore

$$H_1(L_-) \cong V(-\alpha_5).$$

The reflections of length 2 in $W(S)$ are r_5r_1, r_5r_2, r_5r_3, r_5r_4
$r_5r_1(\rho) - \rho = -(1+a)\alpha_5 - \alpha_1$; $r_5r_2(\rho) - \rho = -(1+a)\alpha_5 - \alpha_2$; $r_5r_3(\rho) - \rho = -(1+a)\alpha_5 - \alpha_3$; $r_5r_4(\rho) - \rho = -(1+a)\alpha_5 - \alpha_4$.

By Kostant's formula,
$H_2(L_-) \cong V(-(1+a)\alpha_5 - \alpha_1) \oplus V(-(1+a)\alpha_5 - \alpha_2)$
$\oplus V(-(1+a)\alpha_5 - \alpha_3) \oplus V(-(1+a)\alpha_5 - \alpha_4)$.

In $W(S)$, the reflections of length 3 are $r_5r_1r_2$, $r_5r_1r_3$, $r_5r_1r_4$, $r_5r_1r_5$, $r_5r_2r_1$, $r_5r_2r_3$, $r_5r_2r_4$, $r_5r_2r_5$, $r_5r_3r_1$, $r_5r_3r_2$, $r_5r_3r_4$, $r_5r_3r_5$, $r_5r_4r_1$, $r_5r_4r_2$, $r_5r_4r_3$, $r_5r_4r_5$.
$r_5r_1r_2(\rho) - \rho = -(1+3a)\alpha_5 - \alpha_2 - 2\alpha_1$;
$r_5r_1r_3(\rho) - \rho = -(1+2a)\alpha_5 - \alpha_3 - \alpha_1$;
$r_5r_1r_4(\rho) - \rho = -(1+2a)\alpha_5 - \alpha_4 - \alpha_1$;
$r_5r_1r_5(\rho) - \rho = -a(1+a)\alpha_5 - (1+a)\alpha_1$;
$r_5r_2r_1(\rho) - \rho = -(1+4a)\alpha_5 - 3\alpha_2 - \alpha_1$;
$r_5r_2r_3(\rho) - \rho = -(1+3a)\alpha_5 - \alpha_3 - 2\alpha_2$;
$r_5r_2r_4(\rho) - \rho = -(1+2a)\alpha_5 - \alpha_4 - \alpha_2$;
$r_5r_2r_5(\rho) - \rho = -a(1+a)\alpha_5 - (1+a)\alpha_2$;
$r_5r_3r_1(\rho) - \rho = -(1+2a)\alpha_5 - \alpha_3 - \alpha_1$;
$r_5r_3r_2(\rho) - \rho = -(1+3a)\alpha_5 - 2\alpha_3 - \alpha_2$;
$r_5r_3r_4(\rho) - \rho = -(1+3a)\alpha_5 - 2\alpha_3 - \alpha_4$;
$r_5r_3r_5(\rho) - \rho = -a(1+a)\alpha_5 - (1+a)\alpha_3$;
$r_5r_4r_1(\rho) - \rho = -(1+2a)\alpha_5 - \alpha_4 - \alpha_1$;
$r_5r_4r_2(\rho) - \rho = -(1+2a)\alpha_5 - \alpha_4 - \alpha_2$;
$r_5r_4r_3(\rho) - \rho = -(1+4a)\alpha_5 - 3\alpha_4 - \alpha_3$;
$r_5r_4r_5(\rho) - \rho = -a(1+a)\alpha_5 - (1+a)\alpha_4$.
Hence, by Kostant formula,

$$\begin{aligned} &H_3(L_-) \\ &\cong \{V(-(1+3a)\alpha_5 - \alpha_2 - 2\alpha_1) \oplus V(-(1+2a)\alpha_5 - \alpha_3 - \alpha_1)\oplus \\ &\quad V(-(1+2a)\alpha_5 - \alpha_4 - \alpha_1) \oplus V(-a(1+a)\alpha_5 - (1+a)\alpha_1)\oplus \\ &\quad V(-(1+4a)\alpha_5 - 3\alpha_2 - \alpha_1) \oplus V(-(1+3a)\alpha_5 - \alpha_3 - 2\alpha_2)\oplus \\ &\quad V(-(1+2a)\alpha_5 - \alpha_4 - \alpha_2) \oplus V(-a(1+a)\alpha_5 - (1+a)\alpha_2)\oplus \end{aligned}$$

$$V(-(1+2a)\alpha_5-\alpha_3-\alpha_1)\oplus V(-(1+3a)\alpha_5-2\alpha_3-\alpha_2)\oplus$$
$$V(-(1+3a)\alpha_5-2\alpha_3-\alpha_4)\oplus V(-a(1+a)\alpha_5-(1+a)\alpha_3)\oplus$$
$$V(-(1+2a)\alpha_5-\alpha_4-\alpha_1)\oplus V(-(1+2a)\alpha_5-\alpha_4-\alpha_2)\oplus$$
$$V(-(1+4a)\alpha_5-3\alpha_4-\alpha_3)\oplus V(-a(1+a)\alpha_5-(1+a)\alpha_4)\} \quad (1)$$

Similarly, we can compute the other homology modules $H_4(L_-)$, $H_5(L_-)$, $H_6(L_-)$ etc.

4. Structure of the Maximal Ideal in $QHA_5^{(2)}$

In this section, we study the structure of the components of maximal ideal upto level 4. We know that the ideal I_- of G_- is generated by the homological subspace I_{-2} and hence we may write $I_-=I_-^{(2)}$. Similarly, for $j\geq 2$, we write $I_-^{(j)}=\sum_{n\geq j}I_-^{(n)}$, $L_-^{(j)}=G/I_-^{(j)}$ and $N_-^{(j)}=I_-^{(j)}/I_-^{(j+1)}$. Using the homological techniques together with the representation theory of Kac-Moody algebra and Hochschild-Serre spectral sequences theory, we shall determine the higher level maximal ideals.

To compute I_{-2}
Since G_- is free and I_- is generated by the subspace I_{-2} from the Hochschild-Serre five term exact sequence and (by Lemma 15) $I_{-2}\cong H_2(L_-)$;
By Kostant's formula,

$$\begin{aligned}H_2(L_-)&\cong\sum_{\substack{w\in W(S)\\ l(w)=1}}V(w\rho-\rho)\\ &\cong V(-(1+a)\alpha_5-\alpha_1)\oplus V(-(1+a)\alpha_5-\alpha_2)\\ &\quad\oplus V(-(1+a)\alpha_5-\alpha_3)\oplus V(-(1+a)\alpha_5-\alpha_4).\end{aligned}$$

Hence, $I_{-2}\cong V(-(1+a)\alpha_5-\alpha_1)\oplus V(-(1+a)\alpha_5-\alpha_2)$
$\oplus V(-(1+a)\alpha_5-\alpha_3)\oplus V(-(1+a)\alpha_5-\alpha_4)$.

To compute I_{-3}
We know that (by Theorem 14) $I_{-(j+1)}\cong(V\otimes I_{-j})/H_3(L_-^{(j)})_{-(j+1)} \quad j\geq 2$.

When $j=2$, $L_-^{(2)}$ coincides with the subspace $\eta^-(S)$ for $S=\{1,2,3,4\}$ and therefore we can compute $H_3(L_-^{(2)})$ using the Kostant formula.

$H_3(L_-^{(2)})$

$$\begin{aligned}\cong \{&V(-(1+3a)\alpha_5-\alpha_2-2\alpha_1)\oplus V(-(1+2a)\alpha_5-\alpha_3-\alpha_1)\oplus\\ &V(-(1+2a)\alpha_5-\alpha_4-\alpha_1)\oplus V(-a(1+a)\alpha_5-(1+a)\alpha_1)\oplus\\ &V(-(1+4a)\alpha_5-3\alpha_2-\alpha_1)\oplus V(-(1+3a)\alpha_5-\alpha_3-2\alpha_2)\oplus\\ &V(-(1+2a)\alpha_5-\alpha_4-\alpha_2)\oplus V(-a(1+a)\alpha_5-(1+a)\alpha_2)\oplus\\ &V(-(1+2a)\alpha_5-\alpha_3-\alpha_1)\oplus V(-(1+3a)\alpha_5-2\alpha_3-\alpha_2)\oplus\\ &V(-(1+3a)\alpha_5-2\alpha_3-\alpha_4)\oplus V(-a(1+a)\alpha_5-(1+a)\alpha_3)\oplus\\ &V(-(1+2a)\alpha_5-\alpha_4-\alpha_1)\oplus V(-(1+2a)\alpha_5-\alpha_4-\alpha_2)\oplus\\ &V(-(1+4a)\alpha_5-3\alpha_4-\alpha_3)\oplus V(-a(1+a)\alpha_5-(1+a)\alpha_4)\}\\ &\quad \text{by using (1)}\end{aligned}$$

Since $a > 2$, $H_3(L_-^{(2)})_{-3} = 0$, $I_{-3} \cong (V \otimes I_{-2})/H_3(L_-^{(2)})_{-3} \cong V \otimes I_{-2}$
To determine I_{-4}:
To find the structure of I_{-4}, we need to find the structure of $H_3(L_-^{(3)})_{-4}$.
Consider the sequence, $0 \to N_-^{(2)} \to L_-^{(3)} \to L_-^{(2)} \to 0$ and the corresponding spectral sequence $\{E^r_{p,q}\}$ converging to $H_*(L_-^{(3)})$ such that $E^2_{p,q} \cong H_p(L_-^{(2)}) \otimes \wedge^q(I_{-2})$.
To compute $H_3(L_-^{(3)})_{-4}$ we start with the sequence $0 \to E^2_{2,0} \xrightarrow{d_2} E^2_{0,1} \to 0$.
Since the spectral sequence converges to $H_*(L_-^{(3)})$, we have $H_1(L_-^{(3)}) \cong E^\infty_{1,0} \oplus E^\infty_{0,1}$.
$H_1(L_-^{(3)}) \cong L_-^{(3)}/[L_-^{(3)}, L_-^{(3)}] \cong L_{-1} = V$.
But $E^\infty_{1,0} = E^2_{1,0} \cong H_1(L_-^{(2)}) \cong L_-^{(2)}/[L_-^{(2)}, L_-^{(2)}] \cong L_{-1} = V$, which implies $E^\infty_{0,1} = E^3_{0,1} = 0$. $\therefore$ d_2 is surjective.
Since $E^2_{2,0} \cong I_{-2}$ and $E^2_{2,0} = E^2_{0,1} = I_{-2}$, d_2 is an isomorphism. Therefore $E^\infty_{2,0} = E^3_{2,0} = 0$.

Now, consider the sequence $0 \to E^2_{3,0} \xrightarrow{d_2} E^2_{1,1} \to 0$. $E^2_{3,0} \cong H_3(L_-^{(2)})$ and $E^2_{1,1} \cong H_1(L_-^{(2)}) \otimes I_{-2} \cong V \otimes I_{-2}$. Since $V \otimes I_{-2}$ is a direct sum of irreducible highest weight modules over $A_5^{(2)}$ of level 3, by comparing the levels of both terms, $d_2 : E^2_{3,0} \to E^2_{1,1}$ is trivial. So $E^3_{3,0} = E^2_{3,0}$ and $E^\infty_{1,1} = E^3_{1,1} = E^2_{1,1} \cong V \otimes I_{-2}$. $I_-^{(3)}$ is generated by I_{-3}.

$\therefore$ $H_2(L_-^{(3)}) \cong I_{-3} = V \otimes I_{-2}$.

But $H_2(L_-^{(2)}) \cong E^\infty_{2,0} \oplus E^\infty_{1,1} \oplus E^\infty_{0,2}$. It follows that $E^\infty_{0,2} = E^4_{0,2} = 0$. Hence we find that either $E^3_{0,2} = 0$ or $d_3 : E^3_{3,0} \to E^3_{0,2}$ is surjective.
First, let us consider $E^3_{0,2} = 0$. Therefore $d_3 : E^3_{3,0} \to E^3_{0,2}$ is trivial and that $d_2 : E^2_{2,1} \to E^2_{0,2}$ is surjective in the sequence $0 \to E^2_{4,0} \xrightarrow{d_2} E^2_{2,1} \xrightarrow{d_2} E^2_{0,2} \to 0$.

Thus $E^\infty_{3,0} = E^4_{3,0} = Ker(d_3 : E^3_{3,0} \to E^3_{0,2})/Im(d_3 : 0 \to E^3_{3,0})$

$= E^3_{3,0} = E^2_{3,0} \cong H_3(L^{(2)}_-)$

By comparing levels, we see that $d_2 : E^2_{4,0} \to E^2_{2,1}$ is trivial. Since $E^2_{0,2} \cong \Lambda^2(I_{-2})$, $E^3_{4,0} = E^2_{4,0}$ and

$E^\infty_{2,1} = E^3_{2,1} = Ker(d_2 : E^2_{2,1} \to E^2_{0,2})/Im(d_2 : E^2_{4,0} \to E^2_{2,1}) \cong Ker(d_2 : E^2_{2,1} \to E^2_{0,2})$. Since $d_2 : E^2_{2,1} \to E^2_{0,2}$ is surjective, $\Lambda^2(I_{-2}) \cong E^2_{0,2} \cong E^2_{2,1}/Ker\ d_2 \cong (I_{-2} \otimes I_{-2})/Ker\ d_2$

Therefore $Ker\ d_2 \cong S^2(I_{-2})$. Hence $E^\infty_{2,1} \cong S^2(I_{-2})$.

Now let us consider $E^3_{0,2}$ is nonzero and $d_3 : E^3_{3,0} \to E^3_{0,2}$ is surjective, then since $E^3_{3,0} = E^2_{3,0}$ is irreducible, $d_3 : E^3_{3,0} \to E^3_{0,2}$ is an isomoprhism. Thus $E^\infty_{3,0} = E^4_{3,0} = 0$ and

$H_3(L^{(2)}_-) \cong E^3_{3,0} \cong E^3_{0,2} \cong E^2_{0,2}/Im(d_2 : E^2_{2,1} \to E^2_{0,2})$

$\cong \Lambda^2(I_{-2})/Im(d_2 : E^2_{2,1} \to E^2_{0,2})$.

Since all the modules are completely reducible over $A_5^{(2)}$, we have $Im(d_2 : E^2_{2,1} \to E^2_{0,2}) \cong \Lambda^2(I_{-2})/H_3(L^{(2)}_-)$. We have seen that $d_2 : E^2_{4,0} \to E^2_{2,1}$ is trivial.

Thus $E^\infty_{2,1} = E^3_{2,1} = Ker(d_2 : E^2_{2,1} \to E^2_{0,2})/Im(d_2 : E^2_{4,0} \to E^2_{2,1}) = Ker(d_2 : E^2_{2,1} \to E^2_{0,2})$.

Since $Im\ d_2 \cong \Lambda^2(I_{-2})/H_3(L^{(2)}_-) \cong E^2_{2,1}/Ker\ d_2 \cong (I_{-2} \otimes I_{-2})/Ker\ d_2$,

we have, $Ker\ d_2 \cong S^2(I_{-2}) \oplus H_3(L^{(2)}_-)$

$\therefore E^\infty_{3,0} \oplus E^\infty_{2,1} \cong S^2(I_{-2}) \oplus H_3(L^{(2)}_-)$

Now consider $0 \to E^2_{5,0} \xrightarrow{d_2} E^2_{3,1} \to 0$. By comparing levels, we see that $d_2 : E^2_{3,1} \to E^2_{1,2}$ is trivial. Thus $E^3_{1,2} = E^2_{1,2} \cong V \otimes \Lambda^2(I_{-2})$. Again by comparing the levels of the terms in the sequence $0 \to E^3_{4,0} \xrightarrow{d_3} E^3_{1,2} \to 0$, we conclude that $d_3 = 0$. Therefore $E^\infty_{1,2} = E^4_{1,2} = E^2_{1,2} \cong V \otimes \Lambda^2(I_{-2})$. Since $E^\infty_{0,3}$ is a sub module of $E^2_{0,3} \cong \Lambda^3(I_{-2})$, we see that, $H_3(L^{(3)}_-) \cong H_3(L^{(2)}_-) \oplus S^2(I_{-2}) \oplus (V \otimes \Lambda^2(I_{-2})) \oplus M$, where M is a direct sum of level 6 irreducible representations of $A_5^{(2)}$. Therefore $H_3(L^{(3)}_-)_{-4} \cong S^2(I_{-2})$ and $I_{-4} \cong (V \otimes I_{-3})/H_3(L^{(3)}_-)_{-4} \cong (V \otimes I_{-3})/S^2(I_{-2})$.

From the above equations we get the structure of the components of the maximal ideal I_- (upto level 4) in the Quasi-Hyperbolic Kac-Moody algebra $QHA_5^{(2)}$. Thus we have proved the following theorem.

Theorem 18. *With the usual notations, let* $L = \underset{n \in Z}{\oplus} L_n$ *be the realization* *of $QHA_5^{(2)}$ associated with the GCM*

$$\begin{pmatrix} 2 & -1 & 0 & 0 & -a \\ -2 & 2 & -1 & -1 & -a \\ 0 & -1 & 2 & 0 & -a \\ 0 & -1 & 0 & 2 & -a \\ -a & -a & -a & -a & 2 \end{pmatrix}$$

where $a > 2$, $a \in Z^+$. Then we have following:

i) $I_{-2} \cong V(-(1+a)\alpha_4 - \alpha_1) \oplus V(-(1+a)\alpha_4 - \alpha_2)$
$\oplus V(-(1+a)\alpha_4 - \alpha_3)$.

ii) $I_{-3} \cong V \otimes I_{-2}$

iii) $I_{-4} \cong (V \otimes I_{-3})/S^2(I_{-2})$.

5. Conclusion

In this work, we have considered a particular class of family of quasi hyperbolic Kac Moody algebras $QHA_5^{(2)}$ and determined the structure of the components in the graded ideals upto level four. This work gives further scope for understanding the structure of the algebra and also to determine the multiplicities of roots.

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