International Journal of Pure and Applied Mathematics
Volume 102 No. 1 2015, 51-56
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v102i1.5

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ON REGULARLY ALMOST COUNTABLY COMPACT SETS IN cpH(i)-SPACES AND RELATED MAPS

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Abstract: We introduce and study "regularly almost countably compact" (henceforth abbreviated as r.a.c.c.) sets and obtain a sufficient condition for a subset of countably para H(i)-set to be r.a.c.c. We also study maps for which image (inverse image) of relatively coutably compact sets is r.a.c.c. We prove that such maps are strictly weaker than r-sequentially subcontinuous maps.

AMS Subject Classification: 54C10, 54C08, 54D20
Key Words: regularly almost countably compact , cpH(i)-set, weakly almost countably compact (preserving) map, sequentially r-subcontinuous map, sequentially θ- clustering

1. Introduction

It is known that a regularly closed subset of almost countably compact set is almost countably compact (Theorem 1.1 below). In particular, it follows therefore that if a set K is relatively countably compact then cl(int(K)) being a regular closed subset of countably compact set cl(K), is almost countably compact. This observation motivates the Definition 2.2 below of regularly almost countably compact (r.a.c.c) sets as sets K for which cl(int(K)) is almost countably compact. We prove (Theorem 2.1 below) that every sequentially θ-clustering subset of countably para H(i)-set is r.a.c.c. The converse of this

Received: October 23, 2014

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result is however not true (example 2.2 below). We study the corresponding maps which are weakly relatively countably compact (preserving). We prove that under appropriate restrictions on the domain and codomain, sequentially r-subcontinuous maps are weakly relatively countably compact preserving.

Let X be topological space, a subset A $\subset$ X is said to be regular closed if it is closure of its interior or, equivalently, if it is closure of some open set. A subset A $\subset$ X is said to be countably compact if every countable open cover of A has a finite subcover. A is said to be relatively countably compact subset of X if cl(A) is countably compact. A map $f : X \to Y$ is said to be relatively countably compact preserving (relatively countably compact) [4] if the image (inverse image) of a relatively countably compact subset of X (Y) is relatively countably compact in Y (X). A space X is said to be Frèchet if for each subset A of X, x $\in$ cl(A) implies there exists a sequence $\{x_n\}$ in A converging to x. A sequence $\{x_n\}$ θ-clusters [3] at x if for every neighbourhood V of x, there are infinitely many natural numbers n such that $x_n \in$ cl(V). A subset K $\subset$ X is said to be almost countably compact [1] if and only if every countable open cover of K has a finite subcover whose closures covers K. Clearly, countably compact implies almost countably compact. A subset K of a topological space X will be called countably para H(i)-set in X if every countable open cover $\mathbf{U}$ of A has a locally finite refinement $\mathbf{V}$ by open subsets of X such that $A \subset \cup\{cl(V) : V \in \mathbf{V}\}$. In [5] such sets are called cpH sets where the assumption of Hausdorff is made. Since we do not assume that the spaces are Hausdorff, therefore we prefer the term countably para H(i), henceforth abbreviated as cpH(i). A map $f : X \to Y$ is r-subcontinuous [3] if each convergent net $\{x_\alpha\}$ in X has a subnet $\{x_{\alpha_\beta}\}$ such that $\{$f$(x_{\alpha_\beta})\}$ θ-converges to point of Y. We define sequentially r-subcontinuous map by replacing nets by sequences in definition of r-subcontinuous maps.

Notation: Throughout this paper, X and Y will denote arbitrary topological spaces. For a subset A of a space X, cl(A), int(A) will denote the closure of A and interior of A respectively.

Theorem 1.1. *[6, Proposition 3.13] For a space X, every regular closed subset of almost countably compact set is almost countably compact in X.*

Theorem 1.2. *[5, Proposition 4] If A is cpH(i)-set in X, B is regular closed and $B \subset A$, then B is cpH(i)-set in X.*

Theorem 1.3. *[4, Lemma 2.1] In a Frèchet, T_1 space, if a sequence $\{x_n\}$ has a cluster point x, then $\{x_n\}$ has a subsequence converging to x.*

2. Results

We begin with the following definitions.

Definition 2.1. A set A $\subset$ X will be called sequentially θ- clustering if every sequence in A has a θ- cluster point in X.

Definition 2.2. A subset K $\subset$ X will be called regularly almost countably compact (r.a.c.c.) if cl(int(K)) is almost countably compact.

Remark 2.1. Every subset of almost countably compact set is regularly almost countably compact by Theorem 1.1.

Remark 2.2. Every relatively countably compact subset of X is regularly almost countably compact by Theorem 1.1. The following example shows that the converse neet not hold.

Example 2.1. Let Deleted Tychonoff plank X = T_∞ is defined to be [0,Ω] $\times$ [0,ω] - {(Ω,ω)} where both ordinal spaces [0,Ω] and [0,ω] are given the interval topology [7]. Example 2 of [[2], p.231] shows that X is completely regular pseudocompact hausdorff space that is not countably compact. By Theorem 3.8 of [2], X is almost countably compact.

Our first Theorem below provides us with many instances of r.a.c.c. sets

Theorem 2.1. *A subset K of a cpH(i)-set is regularly almost countably compact if K is sequentially θ-clustering.*

Proof. Let K be sequentially θ-clustering. Assume cl(int(K)) = K* is not almost countably compact then there exists a countable open cover $\{M_n \cap \mathrm{K}^*\}_{n=1}^{\infty}$ of K* with no finite subcover whose closures cover K*. Let $P_n = \cup_{i=1}^{n}(M_i \cap \mathrm{K}^*)$, then $\{P_n\}_n$ is an increasing sequence of open sets covering K*. Since K* is regular closed subset of cpH(i)-set, therefore by Theorem 1.2 K* is cpH(i)-set, then the open cover $\{P_n\}_{n=1}^{\infty}$ of X has a locally finite open refinement $\{V_n\}_n$ where K*$\subset \cup_n(cl(V_n)|n \in N)$. For every positive integer n, let $H_n = \cup_{j\leq n}(cl(V_j))$. Then H_n is a closed set such that $H_n = \cup_{j\leq n}cl(V_j) \subset \cup_{j\leq n}cl(P_j) = cl(P_n)$ and so $H_n \subset cl(P_n)$. Since $\{V_n\}_n$ is locally finite therefore $\{cl(V_n)\}_n$ is also locally finite, for each x $\in$ K* there exists a neighborhood N(x) of x in K* such that $N(x) \cap (\cup_{j>n}cl(V_j)) = \emptyset$ for some n and so $N(x) \subset \cup_{j\leq n}cl(V_j) = H_n$ for some n. Therefore, for each x $\in$ K*, $x \in int_{K*}(H_n)$ for some n. Let $A_n = int_{K*}(H_n)$. Then $\{A_n\}_n$ is an open cover of K* such that $cl_{K*}(A_n) = cl_{K*}(int_{K*}(H_n)) \subset cl_{K*}(H_n) = H_n \subset cl(P_n)$ and so no finite subfamily of closures of $\{A_n\}_n$ can cover K, since $\{M_n \cap \mathrm{K}^*\}_n$ and therefore, $\{P_n\}_n$ has no finite subcover whose closures covers K*. Therefore, for each n there exists $x_n \in K - \cup_{i=1}^{n}(A_i)$. If x is any point of K*, then x $\in A_n$ for some n and since for each m $\geq$ n, $x_m \notin cl(A_n)$, x is not a θ-cluster point of the sequence $\{x_n\}$ of K. Therefore, the sequence $\{x_n\}$ has no θ-cluster point in K* therefore in X contradicting the fact that K is sequentially θ-clustering. Hence K is regularly almost countably compact. □

Following example shows that converse of above theorem need not hold.

Example 2.2. Let $X = T_\infty = [0, \Omega] \times [0, \omega] - \{(\Omega, \omega)\}$ be the space defined in example 2.1. Now X is almost countably compact implies X is regularly almost countably compact and X is cpH-set. Also X is not countably compact implies there exist a sequence in X which has no θ-cluster point since X is regular. Hence X is not sequentially θ-clustering.

Definition 2.3. A map $f : X \to Y$ will be called weakly relatively countably compact (preserving) if inverse image (image) of every relatively countably compact subset of Y (X) is r.a.c.c. in X(Y).

Remark 2.3. Since cl(int(K)) is regularly closed subsets of cl(K), therefore by Theorem 1.1 every relatively countably compact set is r.a.c.c. It follows therefore that every relatively countably compact (preserving) map is weakly relatively countably compact (weakly relatively countably compact preserving). However, the following example shows that weakly relatively countably compact preserving is relatively countably compact preserving.

Example 2.3. Let $T = [0, \Omega] \times [0, \omega]$ be Tychnoff plank topology and T_∞ be deleted Tychnoff plank topology defined in example 2.1. Define a map $f : T \to T_\infty$ defined by $f(\alpha, \beta) = (\alpha, \beta)$ and $f(\Omega, \omega) = (0, 0)$. Let K be a subset of T where T is countably compact proved in [7]. Therefore, K is relatively countably compact as every subset of countably compact space is relatively countably compact. Now f(K) is subset of T_∞ where T_∞ is almost countably compact by example. Therefore f(K) is regularly almost countably compact. Hence f is weakly relatively countably compact preserving map. Now as T is countably compact implies T is relatively compact but f(T) = T_∞ is not countably compact. Therefore, T_∞ is not relatively countably compact. Hence f is not relatively countably compact preserving map.

Following example shows that weakly relatively countably compact map need not relatively countably compact

Example 2.4. Let X = T_∞ be deleted Tychnoff plank topology. And Y = [0,Ω] be closed ordinal space defined in Example 2.1. Define a map $f : X \to Y$ by $f(\alpha, \beta) = \alpha$. Let K be a subset of Y where Y is countably compact proved in [7]. Therefore, K is relatively countably compact as every subset of countably compact space is relatively countably compact. Now $f^{-1}(K)$ is subset of X where X is almost countably compact by example. Therefore $f^{-1}(K)$ is regularly almost countably compact. Hence f is weakly relatively countably compact map. Now f is not relatively countably compact map, since Y is relatively countably compact but $f^{-1}(Y) =$ X is not relatively countably compact

Theorem 2.2. *If a map $f : X \to Y$, where X is Frèchet, T_1- space and Y is cpH(i)-set is sequentially r-subcontinuous, then f is weakly relatively countably compact preserving map.*

Proof. Let A be a relatively countably compact subset of X. By Theorem 2.1, it is sufficient to prove that f (A) is sequentially θ-clustering. Let {f (x_n)} be any sequence in f (A). Then $\{x_n\}$ can be taken to be a sequence in A where A is relatively countably compact. Therefore, $\{x_n\}$ has a cluster point in X say x where X is Frèchet, T_1- space implies $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to x by Theorem 1.3 and since f is sequentially r-subcontinuous, there exist a subsequence $\{x_{n_{K*}}\}$ of $\{x_{n_K}\}$ such that $\{f(x_{n_{K*}})\}$ θ-converges to some point in Y. Therefore, $\{f(x_n)\}$ has a θ-cluster point in Y then proving that f (A) is sequentially θ-clustering. Hence f is weakly relatively countably compact preserving map. □

Using above Theorem and Theorem 2.2 of [3] we get following corollary

Corollary 2.1. *Let $f : X \to Y$ be a weakly continuous map with a strongly closed graph where X is Frèchet, T_1- space and Y is cpH(i)-set then f is weakly relatively countably compact preserving map.*

Theorem 2.3. *If a map $f : X \to Y$, where Y is Frèchet, T_1- space and X is cpH(i)-set is inversely sequentially r-subcontinuous, then f is weakly relatively countably compact map.*

Proof. Let A be a relatively countably compact subset of Y. For proving $f^{-1}(A)$ is r.a.c.c, it is sufficient to prove that $f^{-1}(A)$ is inversely sequentially θ-clustering by Theorem 2.1. Let $\{x_n\}$ be any sequence in $f^{-1}(A)$, then {f (x_n)} can be taken to be a sequence in A where A is relatively countably compact. Therefore, {f (x_n)} has a cluster point in Y say y as X is Frèchet, T_1-space implies $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $\{f(x_{n_k})\}$ converging to y and since f is inversely sequentially r-subcontinuous, there exists a subsequence $\{x_{n_{K*}}\}$ of $\{x_{n_k}\}$ which θ-converges to some point in X. Therefore, $\{x_n\}$ has a θ-cluster point in X proving that $f^{-1}(A)$ is sequentially θ-clustering. Hence f is weakly relatively countably compact map. □

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