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STRUCTURE OF INDEFINITE QUASI-AFFINE KAC-MOODY ALGEBRAS $QAD_3^{(2)}$

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Abstract: In this work a class of indefinite quasi-affine type of Kac-Moody algebras $QAD_3^{(2)}$ is defined. To understand the structure of these Quasi-Affine algebras, first, these algebras are realized as graded Lie algebra of kac-Moody type. Then using the homological techniques and spectral sequences theory, the homology modules are computed upto level 3 and then the structure of the components of the maximal ideals upto level 5 is computed.

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Key Words: Kac-Moody algebra, finite, affine, indefinite, extended hyperbolic, quasi hyperbolic, homology modules, spectral sequences

1. Introduction

Kac-Moody algebras, a modern field of Mathematical research has been rapidly growing and drawing the attention of researchers, due to its connection and application to various branches of Mathematics and Mathematical Physics. Kac-Moody algebras are broadly classified into finite, affine and indefinite types. A lot of theory has been developed for finite and affine cases, while the nature of indefinite Kac-Moody algebas remains to be explored completely. Understanding the structure and explicit determination of root multiplicities of the indefinite Kac-Moody algebras is still an open problem.

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In [1], Benkart et al. gave the general construction of graded Lie algebra of Kac-Moody type. In [2], Feingold and Frenkel has computed level 2 root multiplicities for the hyperbolic Kac-Moody Lie algebra $HA_1^{(1)}$. In [5], [6], [8] Kang has determined the structure and obtain the root multiplicities for roots upto level 5 for $HA_1^{(1)}$ and for $HA_2^{(1)}$ upto level 3 using homological techniques and spectral sequences theory. The indefinite of Kac-Moody Lie algebras $HA_n^{(1)}$, were studied in [7]. Sthanumoorthy and Uma Maheswari introduced a new class called extended hyperbolic Kac-Moody algebras in [12] and studied the structure and have computed the root multiplicities for a specific class of extended hyperbolic Kac-Moody algebra $EHA_1^{(1)}$ in [11] and again considered the general cases in [13]. More general classes of the families $EHA_1^{(1)}$ and $EHA_2^{(2)}$ were considered in [15]. Uma Mahaeswari introduced another class of indefinite non-hyperbolic Kac-Moody algebras called Quasi-Hyperbolic Kac-Moody Lie algebras in [16].The indefinite non-hyperbolic Kac-Moody algebras QHG_2 and $QHA_2^{(1)}$ where realized as a graded Lie algebra of Kac-Moody type and the structure of the components of the maximal ideal upto level 4 were computed in [19], [20]. The indefinite quasi-affine Kac-Moody algebras were introduced by A.Uma Maheswari in [17] and two specific classes namely $QAG_2^{(1)}$ and $QAC_2^{(1)}$ were studied in [17], [18].

In this work, a class of indefinite Quasi-Affine Kac-Moody algebras, generally represented as $QAD_3^{(2)}$ is to be considered. The GCM associated with $QAD_3^{(2)}$ is given by

$$\begin{pmatrix} 2 & -2 & 0 & -p \\ -1 & 2 & -1 & -q \\ 0 & -2 & 2 & -r \\ -u & -v & -w & 2 \end{pmatrix},$$

where p, q, r, u, v, w are non-negative integers. In this work, we shall consider $p, q, r, u, v, w \in Z^+$. First we obtain a realization for these algebras as graded Lie algebra of Kac-Moody type.Then using the homological techniques developed by Benkart et al. [1] and Kang [5], [6], [7], [8], the homology modules upto level three are computed. Using the generalized form of Kostants formula, the structure of the components of the maximal ideal upto level five is determined.

2. Preliminaries

We first recall some results of Kac-Moody algebreas [4], [10], [21] and the general construction of graded Lie algebras of Kac-Moody type (Benkart et al. [1]).

Definition 1. [10] Let $A = (a_{ij})_{i,j=1}^{n}$ is a Generalized Cartan Matrix (abbreviated as GCM).Let (H, Π, Π^v) be a realization of A, where l is the rank of A, H is a $2n - l$ dimensional complex vector space, $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ and $\Pi^v = \{\alpha_1^v, \ldots, \alpha_n^v\}$ are linearly independent subsets of H^* and H respectively, satisfying $\alpha_j(\alpha_i^v) = a_{ij}$ for $i, j = 1, \ldots, n$. Π is called the root basis. Elements of Π are called simple roots.

The Kac-Moody algebra $g(A)$ associated with a GCM $A = (a_{ij})_{i,j=1}^{n}$ is the Lie algebra generated by the elements e_i, f_i, $i = 1, 2, \ldots, n$ and H with the following defining relations:

$$[h, h'] = 0, \quad h, h' \in H; \quad [e_i, f_j] = \delta_{ij}\alpha_i^v; \quad [h, e_j] = \alpha_j(h)e_j,$$

$$[h, f_j] = -\alpha_j(h)f_j, \ i, j \in N; \quad (ade_i)^{1-a_{ij}}e_j = 0;$$

$$(adf_i)^{1-a_{ij}}f_j = 0, \ \forall \ i \neq j, i, j \in N.$$

The Kac-Moody algebra $g(A)$ has the root space decomposition where $g(A) = \underset{\alpha \in Q}{\oplus} g_\alpha(A)$, where $g_\alpha(A) = \{x \in g(A)/[h, x] = \alpha(h)x, \text{ for all } h \in H\}$. An element α, $\alpha \neq 0$ in Q is called a root if $g_\alpha \neq 0$.

Definition 2. [4] For any $\alpha \in Q$ and $\alpha = \sum_{k=1}^{n} k_i\alpha_i$, define support of α, written as *supp* α, by *supp* $\alpha = \{i \in N/k_i \neq 0\}$. Let $\Delta(= \Delta(A))$ denote the set of all roots of $g(A)$ and Δ, the set of all positive roots of $g(A)$. We have $\Delta_- = -\Delta_+$ and $\Delta = \Delta_+ \cup \Delta_-$.

Definition 3. [4] A GCM A is called symmetrizable if DA is symmetric for some diagonal matrix $D = diag(q_1, \ldots, q_n)$, with $q_i > 0$ and q_i's are rational numbers.

Proposition 4. *[4] A GCM $A = (a_{ij})_{i,j=1}^{n}$ is symmetrizable if and only if there exists an invariant, bilinear, symmetric, non degenerate form on $g(A)$.*

Definition 5. [4] To every GCM A is associated a Dynkin diagram $S(A)$ defined as follows: (A) has n vertices and vertices i and j are connected by max $\{|a_{ij}|, |a_{ji}|\}$ number of lines if $a_{ij}a_{ji} \leq 4$ and there is an arrow pointing towards i if $|a_{ij}| > 1$. If $a_{ij}.a_{ji} > 4$, i and j are connected by a bold faced edge, equipped with the ordered pair $(|a_{ij}|, |a_{ji}|)$ of integers.

Theorem 6. *[21] Let A be a real $n \times n$ matrix satisfying $(m1)$, $(m2)$ and $(m3)$.*

(m1) A is indecomposable;

(m2) $a_{ij} \leq 0$ *for* $i \neq j$*;*

(m3) $a_{ij} = 0$ *implies* $a_{ji} = 0$

Then one and only one of the following three possibilities holds for both A *and* ${}^tA$*:*

(i) $detA \neq 0$*; there exists* $u > 0$ *such that* $Au > 0$*;* $Av \geq 0$ *implies* $v > 0$ *or* $v = 0$*;*

(ii) co rank $A = 1$*; there exists* $u > 0$ *such that* $Au = 0$*;* $Av \geq 0$ *implies* $Av = 0$*;*

(iii) there exists $u > 0$ *such that* $Au < 0$*;* $Av \geq 0$, $v \geq 0$ *imply* $v = 0$.

Then A *is of finite, affine or indefinite type iff (i), (ii) or (iii) (respectively) is satisfied.*

Definition 7. [21] A Kac-Moody algebra $g(A)$ is said to be of finite, affine or indefinite type if the associated GCM A is of finite, affine or indefinite type respectively.

Definition 8. [17] Let $A = (a_{ij})_{i,j=1}^{n}$ be an indecomposable GCM of indefinite type. We define the associated Dynkin diagram $S(A)$ to be of Quasi Affine (QA) type if $S(A)$ has a proper connected sub diagram of Affine types with $n-1$ vertices. The GCM A is of QA type if $S(A)$ is of QA type. We then say the Kac-Moody algebra $g(A)$ is of QA type.

2.1. General Construction of Graded Lie Algebra

Throughout the sections, we follow these notations:

G - a Lie algebra over a field of characteristic zero.

V, V' - two G-modules.

$\psi : V' \otimes V \to G$ a G-module homomorphism.

$G_0 = G$, $G_{-1} = V$, $G_1 = V'$

$G_+ = \sum_{n \geq 1} G_n$ (resp. $G_- = \sum_{n \geq 1} G_{-n}$) denote the free Lie algebra generated by V' (resp. V).

G_n (resp G_{-n}) for $n > 1$ is the space of all products of n vectors from V' (resp. V).

Now, $G = \sum_{n=-\infty}^{\infty} G_n$ is given a Lie algebra structure by defining the Lie bracket [,] as follows:

For $a, b \in G$, $v \in V$, $w \in V'$ define $[a, v] = a.v = -[v, a]$ and $[a, w] = a.w = -[w, a]$.

For $a, b \in G$ let $[a, b]$ denote the bracket operation in G.

For $w \in V'$, $v \in V$, $[w, v] = \psi(w \otimes v) = -[v, w]$,

Extending this bracket operation, $G = \sum_{n \in Z} G_n$ becomes a graded Lie algebra which is generated by its local part $G_{-1} + G_0 + G_1$.

For $n \geq 1$ define the subspaces

$I_{\pm} = \{x \in G_{\pm}/[y_1, [\dots[y_{n-1}, x]]\dots] = 0$ for all $y_1, \dots, y_{n-1} \in G_{\mp}\}$. Set $I_+ = \sum_{n>1} I_n$, $I_- = \sum_{n>1} I_{-n}$.

Then I_+ and I_- are ideals of G and the ideal is the largest graded ideal of G trivially intersecting $G_{-1} + G_0 + G_1$. For $n > 1$, let $L_{\pm n} = G_{\pm n}/I_{\pm n}$. Consider $L = L(G, V, V', \psi) = G_-/I_- \oplus G_0 \oplus G_+/I_+ = \dots \oplus L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \oplus \dots$, where $L_0 = G_0$, $L_1 = G_1$, $L_{-1} = G_{-1}$.

Then $L = \oplus_{n \in Z} L_n$ is a graded Lie algebra generated by its local part $V \oplus G \oplus V'$ and $L = G/I$.

For the suitable choice of V (written as the direct sum of irreducible highest weight modules), the contragradient V^*, the basis elements and the homomorphism $\psi : V^* \otimes V \to g^e$, form the graded Lie algebra $L = L(g^e, V, V^*, \psi)$. For further details one can refer to [1], [5].

Theorem 9. *[1]*

(i) L is a Z^{n+m}-graded algebra.

(ii) Let $\phi : A(C) \to L$ be the Lie algebra homomorphism sending $E_i \to e_i$, $F_i \to f_i$, $H_i \to h_i$. Then ϕ has kernel as $I(C)$ and $I(C)$ is the largest graded ideal of $A(C)$ trivially intersecting the span of $H_1, \dots, H_{n+m}$. Also $\phi : A(C)/I(C) \to L$ is an isomorphism.

(iii) The matrix C has rank $2n - l$ and C is symmetrizable.

Next, we recall the definition of homology of Lie algebra and the Hochschild-Serre spectral sequence (Kang [5]).

Let $C_q(G, V)$, $q > 0$ be space of all q-dimensional chains of the Lie algebra G with coefficients in V, $\wedge^q(G) \otimes V$.

The differential $d_q : C_q(G, V) \to C_{q-1}(G, V)$ is defined to be

$$\begin{aligned} &d_q(g_1 \wedge \dots \wedge g_q \otimes v) \\ &\quad = \sum_{1 \leq s \leq t \leq q} (-1)^{s+t-1}([g_s, g_t]) \wedge g_1 \wedge \dots \wedge \tilde{g}_s \wedge \dots \wedge \tilde{g}_t \wedge \dots \wedge g_q) \otimes v \\ &\quad + \sum_{1 \leq s \leq q} (-1)^s (g_1 \wedge \dots \wedge \tilde{g}_s \wedge \dots \wedge g_q) \otimes g_s.v, \end{aligned}$$

for $v \in V$, $g_1 \dots g_q \in G$.

For $q < 0$, define $C_q(G,V) = 0$ and $d_q = 0$. Then $d_q \circ d_{q-1} = 0$.

The homology of the complex $(C,d) = \{C_q(G,V), d_q\}$ is called the homology of the Lie algebra G with coefficients in V and is denoted by $H_q(G,V)$. If $V = C$, we simply write $H_q(G)$ for $H_q(G,C)$.

Let G, V, $C_q(G,V)$ are completely reducible modules in the category O over a Kac-Moody algebra $g(A)$ with d_q having $g(A)$-module homomorphisms. Let I be ideal of G and $L = G/I$.

Define a filtration $\{K_p = K_pC\}$ of the complex $\{C,d\}$ by $K_pC_{p+q} = \{g_1 \wedge g_2 \wedge \cdots \wedge g_{p+q} \otimes v | g_i \in I \text{ for } p+1 \leq i \leq p+q\}$.

This gives rise to a spectral sequence $\{E^r_{p,q}, d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ such that $E^2_{p,q} \cong H_p(L, H_q(I,V))$, where $E^r_{p,q}$'s are determined by $E^{r+1}_{p,q} = Ker(d_r : E^r_{p,q} \to E^r_{p-r,q+r-1})/Im(d_r : E^r_{p+r,q-r+1} \to E^r_{p,q})$ with boundary homomorphisms $d_{r+1} : E^r_{p,q} \to E^r_{p-r-1,q+r}$.

The modules $E^r_{p,q}$ become stable for r $> max(p, q+1)$ for each (p,q) and the stable module is denoted by $E^\infty_{p,q}$. The spectral sequence $\{E^r_{p,q}, d_r\}$ converges to $H_n(G,V)$ in the following sense:

$$H_n(G,V) = \bigoplus_{p+q=n} E^\infty_{p,q}.$$

We get the following Hochschild-Serre five term exact sequence (Kang, [5]):

$$H_2(G,V) \to H_2(L, H_0(I,V) \to H_0(L, H_1(I,V)) \\ \to H_1(G,V) \to H_1(L, H_0(I,V)) \to 0.$$

Let $G = \bigoplus_{n \geq 1} G_n$ be the free Lie algebra generated by the subspace G_1 and $I = \bigoplus_{n \geq m} I_n$ be the graded ideal of G generated by the subspace I_m for $m \geq 2$. Consider the quotient algebra $L = G/I$. Then $L = \bigoplus_{n \geq 1} L_n$ is also a graded Lie algebra generated by the subspace $L_1 = G_1$. Let $J = I/[I,I]$. J is an L-module via adjoint action generated by the subspace J_m. As vector spaces, $J_n \cong I_n$ for $m \leq n < 2m$.

Suppose that $I_m$and G_1 are modules over a Kac-Moody algebra $g(A)$. Then G_n has a $g(A)$-module structure such that $x.[v,w] = [x.v,w] + [v,x.w]$ for $x \in g(A)$, $v \in G$, $w \in G_{n-1}$; I_n also has a similar module structure. We also have the induced module structure of the homogeneous subspaces L_n, J_n.

Theorem 10. *[5] There is an isomorphism of $g(A)$-modules $H_j(L,J) \cong H_{j+2}(L)$, for $j \geq 1$. In particular $I_{m+1} \cong (G_1 \otimes I_m)/H_3(L)_{m+1}$.*

For arbitrary $j \geq m$, set $I^{(j)} = \sum_{n \geq j} I_n$; then $I^{(j)}$ is an ideal of G generated by the subspace I_j. We consider the quotient algebra $L^{(j)} = G/I^{(j)}$. Let $N^{(j)} =$

$I^{(j)}/I^{(j-1)}$. In this notation $L = L^{(m)}$. We have an important relation: $I_{j+1} \cong (G_1 \otimes I_j)/H_3(L^{(j)})_{j+1}$. There exists a spectral sequence $\{E^r_{p,q}, d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ converging to $H_*(L^{(j)})$ such that and $E^2_{p,q} \cong H_p(L^{(j-1)}) \otimes \wedge^q(I_{j-1})$ and $H_3(L^{(j)}) \cong E^\infty_{3,0} \oplus E^\infty_{2,1} \oplus E^\infty_{1,2} \oplus E^\infty_{0,3}$.

Lemma 11. *[5] In the above notation, $H_2(L) \cong I_m$.*

Let us recall the Kostant's formula for symmetrizable Kac-Moody algebras Liu, [9]:

Let $A = (a_{ij})^n_{i,j=1}$ be a symmetrizable GCM.

Let $S = \{1, \ldots, s\}$ be a subset of $N = \{1, \ldots, n\}$ and g_s be the subalgebra of $g(A)$ generated by the elements e_i, f_i, $i = 1, \ldots, s$ and h. Let Δ^+_s denote the set of positive roots generated by $\alpha_1, \ldots, \alpha_s$ and $\Delta^-_s = -\Delta^+_s$. Then g_s has the corresponding triangular decomposition : $g_s = n^-_s \oplus h \oplus n^+_s$, where $n^\pm_s = \bigoplus_{\alpha \in \Delta^\pm_s} g_\alpha$ and $\Delta_s = \Delta^+_s \cup \Delta^-_s$ is the root system of g_s. Let $\Delta^\pm(s) = \Delta^\pm/\Delta^\pm_s$ and $n^\pm(S) = \bigoplus_{\alpha \in \Delta^\pm(S)} g_\alpha$ Then $g(A) = n^-(S) \oplus g_s \oplus n^+(S)$. Let $W(S) = \{w \in W/w\Delta^- \cap \Delta^+ \subset \Delta^+(S)\}$. For $\lambda \in h^*$ denote by $\tilde{V}(\lambda)$, the irreducible highest weight module over $g(A)$ and $V(\lambda)$ the irreducible highest weight module over g_s.

Theorem 12 (Kostant's formula). *[9]*

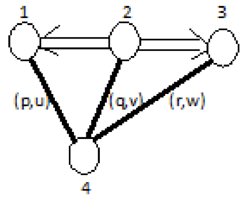
$$H_j(n^-(S), \tilde{V}(\lambda)) \cong \bigoplus_{\substack{w \in W(S) \\ l(w)=j}} V(w(\lambda + \rho) - \rho).$$

Lemma 13. *[5] Suppose $w = w'r_j$ and $l(w) = l(w') + 1$. Then $w \in W(S)$ if and only if $w' \in W(S)$ and $w'(\alpha_j) \in \Delta^+(S)$.*

3. Realization for $QAD_3^{(2)}$

Let us denote by $QAD_3^{(2)}$, the class of Quasi-Affine Kac-Moody algebras whose associated GCM is $\begin{pmatrix} 2 & -2 & 0 & -p \\ -1 & 2 & -1 & -q \\ 0 & -2 & 2 & -r \\ -u & -v & -w & 2 \end{pmatrix}$, where $p, q, r, u, v, w \in Z^+$. We start with the Kac-Moody algebra $D_3^{(2)}$ associated with the GCM $A = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$ is symmetrizable and we write $A = DB$ where $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix}$

The Dynkin diagram associated with $QAD_3^{(2)}$ is



We begin with the Kac-Moody algebra $D_3^{(2)}$ associated with the GCM A.

Let $(h, \Pi, \Pi^\vee)$ be the realization of A with $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\}$. Then we have the following bilinear relations $(\alpha_1, \alpha_1) = 2$, $(\alpha_1, \alpha_2) = -2$, $(\alpha_1, \alpha_3) = 0$, $(\alpha_2, \alpha_1) = -2$, $(\alpha_2, \alpha_2) = 4$, $(\alpha_2, \alpha_3) = -2$, $(\alpha_3, \alpha_1) = 0$, $(\alpha_3, \alpha_2) = -2$, $(\alpha_3, \alpha_3) = 2$

Let α_4' be the element in h^* such that $\alpha_4'(\alpha_1^\vee) = 0$, $\alpha_4'(\alpha_2^\vee) = 0$, $\alpha_4'(\alpha_3^\vee) = 1$.

Let us define $\lambda = p\alpha_1 + p\alpha_2 + (p - q)\alpha_3 + (2q + r)\alpha_4' \in h^*$. Set $\alpha_4 = -\lambda$. Form the matrix $C = (< \alpha_i, \alpha_j >)_{i,j=1}^4$. Then $C = \begin{pmatrix} 2 & -2 & 0 & -p \\ -1 & 2 & -1 & -q \\ 0 & -2 & 2 & -r \\ -u & -v & -w & 2 \end{pmatrix}$ where $p, q, r, u, v, w \in Z^+$ is the symmetrizable GCM of Quasi-Affine type, generally represented as $QAD_3^{(2)}$.

Let V be the integrable highest weight irreducible module over G with the highest weight λ as defined. Let V^* be the contragradient of V and ψ be the mapping as defined earlier. Let G be the Kac-Moody algebra associated with the GCM $\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$. Form the graded Lie algebra $L(G^e, V, V^*, \psi)$. Then $L \cong g(C)$ and L is a symmetrizable Kac-Moody algebra of Quasi-Affine type associated with the GCM C.

Now, we compute the homology modules of the Kac-Moody algebra for $QAD_3^{(2)}$.We first note that, from the realization of $L = QAD_3^{(2)}$ as $L = L_- \oplus L_0 \oplus L_+ = G/I$ and using the involutive automorphism, it suffices to consider only the negative part $L_- = G_-/I_-$.

3.1. Computation of Homology Modules

Let $S = \{1, 2, 3\} \subset N = \{1, 2, 3, 4\}$. Let g_s be the Kac-Moody Lie algebra $D_3^{(2)}$, $\Delta^+(S) = \{k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4' + /k_4 \neq 0\}$. Δ_s be the root system of g_s.

The only reflection of length 1 in $W(S)$ is r_4.

$r_4(\rho) = \rho - \alpha_4$; $r_4(\rho) - \rho = -\alpha_4 \therefore H_1(L_-) \cong V(-\alpha_4)$.

The reflections of length 2 in $W(S)$ are r_4r_1, r_4r_2, r_4r_3.

$r_4r_1(\rho)-\rho=-(u+1)\alpha_4-\alpha_1$; $r_4r_2(\rho)-\rho=-(v+1)\alpha_4-\alpha_2$; $r_4r_3(\rho)-\rho=-(w+1)\alpha_4-\alpha_3$.

Using Kostant's formula,
$H_2(L_-)\cong\{V(-(u+1)\alpha_4-\alpha_1)\oplus V(-(v+1)\alpha_4-\alpha_2)$
$\oplus(-(w+1)\alpha_4-\alpha_3)\}$.

The reflections of length 3 in $W(S)$ are $r_4r_1r_2, r_4r_1r_3$, $r_4r_1r_4$, $r_4r_2r_1$, $r_4r_2r_3$, $r_4r_2r_4$, $r_4r_3r_1$, $r_4r_3r_2$, $r_4r_3r_4$.
$r_4r_1r_2(\rho)-\rho=-(v+3u+1)\alpha_4-\alpha_2-3\alpha_1$;
$r_4r_1r_3(\rho)-\rho=-(u+w+1)\alpha_4-\alpha_3-\alpha_1$;
$r_4r_1r_4(\rho)-\rho=-(pu+u)\alpha_4-(p+1)\alpha_1$;
$r_4r_2r_1(\rho)-\rho=-(u+2v+1)\alpha_4-2\alpha_2-\alpha_1$;
$r_4r_2r_3(\rho)-\rho=-(2v+w+1)\alpha_4-2\alpha_2-\alpha_3$;
$r_4r_2r_4(\rho)-\rho=-(qv+v)\alpha_4-(q+1)\alpha_2$;
$r_4r_3r_1(\rho)-\rho=-(u+w+1)\alpha_4-\alpha_3-\alpha_1$;
$r_4r_3r_2(\rho)-\rho=-(v+3w+1)\alpha_4-3\alpha_3-\alpha_2$;
$r_4r_3r_4(\rho)-\rho=-(rw+w-1)\alpha_4-(r+1)\alpha_3$.
Hence, by Kostant formula,
$H_3(L_-)\cong V(-(v+3u+1)\alpha_4-\alpha_2-3\alpha_1)\oplus$
$V(-(u+w+1)\alpha_4-\alpha_3-\alpha_1)\oplus V(-(pu+u)\alpha_4-(p+1)\alpha_1)\oplus V(-(u+2v+1)\alpha_4-2\alpha_2-\alpha_1)\oplus V(-(2v+w+1)\alpha_4-2\alpha_2-\alpha_3)\oplus V(-(qv+v)\alpha_4-(q+1)\alpha_2)\oplus V(-(u+w+1)\alpha_4-\alpha_3-\alpha_1)\oplus V(-(v+3w+1)\alpha_4-3\alpha_3-\alpha_2)\oplus V(rw+w-1)\alpha_4-(r+1)\alpha_3)$

Other homology modules $H_4(L_-)$, $H_5(L_-)$, $H_6(L_-)$ etc. can be computed in a similar manner.

4. Structure of the Maximal Ideal in $QAD_3^{(2)}$

In this section, using the homological techniques together with the representation theory of Kac-Moody algebra and Hochschild-Serre spectral sequences theory, as developed in [1], [5], we shall determine the structure of the components of maximal ideals, upto level 5 for $QAD_3^{(2)}$.

We know that the ideal I_- of G_- is generated by the homological subspace I_{-2} and hence we may write $I_-=I_-^{(2)}$. Similarly, for $j\geq 2$, we write

$$I_-^{(j)}=\sum_{n\geq j}I_-^{(n)},\quad L_-^{(j)}=G/I_-^{(j)},$$

and $N_-^{(j)}=I_-^{(j)}/I_-^{(j+1)}$.

To compute I_{-2}:

Since G_- is free and I_- is generated by the subspace I_{-2} from the Hochschild-Serre five term exact sequence and (by Lemma 11) $I_{-2} \cong H_2(L_-)$; By Kostant's formula,

$$\begin{aligned} H_2(L_-) &\cong \sum_{\substack{w \in W(S) \\ l(w)=1}} V(w\rho - \rho) \\ &\cong \{V(-(u+1)\alpha_4 - \alpha_1) \oplus V(-(v+1)\alpha_4 - \alpha_2) \oplus V(-(w+1)\alpha_4 - \alpha_3)\} \end{aligned}$$

Hence

$$I_{-2} \cong \{V(-(u+1)\alpha_4 - \alpha_1) \oplus V(-(v+1)\alpha_4 - \alpha_2) \\ \oplus V(-(w+1)\alpha_4 - \alpha_3)\}.$$

To compute I_{-3}:

By Theorem 10, $I_{-(j+1)} \cong (V \otimes I_{-j})/H_3(L_-^{(j)})_{-(j+1)} \quad j \geq 2$.

When $j = 2$, $L_-^{(2)}$ coincides with the subspace $\eta^-(S)$ for $S = \{1, 2, 3, 4\}$. From $H_3(L_-^{(2)})$ computed in Section 3, we see that $H_3(L_-^{(2)})_{-3} = 0$ and hence $I_{-3} \cong (V \otimes I_{-2})/H_3(L_-^{(2)})_{-3} \cong V \otimes I_{-2}$

To determine I_{-4}:

Consider the sequence $0 \to N_-^{(2)} \to L_-^{(3)} \to L_-^{(2)} \to 0$ and the corresponding spectral sequence $\{E^r_{p,q}\}$ converging to $H_*(L_-^{(3)})$ such that $E^2_{p,q} \cong H_p(L_-^{(2)}) \otimes \wedge^q(I_{-2})$.

To compute $H_3(L_-^{(3)})_{-4}$ we start with the sequence

$$0 \to E^2_{2,0} \xrightarrow{d_2} E^2_{0,1} \to 0.$$

Since the spectral sequence converges to $H_*(L_-^{(3)})$,

$$H_1(L_-^{(3)}) \cong E^\infty_{1,0} \oplus E^\infty_{0,1}.$$

Also,

$$H_1(L_-^{(3)}) \cong L_-^{(3)}/[L_-^{(3)}, L_-^{(3)}] \cong L_{-1} = V.$$

But $E^\infty_{1,0} = E^2_{1,0} \cong H_1(L_-^{(2)}) \cong L_-^{(2)}/[L_-^{(2)}, L_-^{(2)}] \cong L_{-1} = V$, and hence $E^\infty_{0,1} = E^3_{0,1} = 0$. $\therefore$ d_2 is surjective.

Since $E^2_{2,0} \cong I_{-2}$ and $E^2_{2,0} = E^2_{0,1} = I_{-2}$, d_2 is an isomorphism. So $E^\infty_{2,0} = E^3_{2,0} = 0$.

Next, consider the sequence $0 \to E^2_{3,0} \xrightarrow{d_2} E^2_{1,1} \to 0$. $E^2_{3,0} \cong H_3(L^{(2)}_-)$ and $E^2_{1,1} \cong H_1(L^{(2)}_-) \otimes I_{-2} \cong V \otimes I_{-2}$. By comparing the levels of both terms, we see that $d_2 : E^2_{3,0} \to E^2_{1,1}$ is trivial. So $E^3_{3,0} = E^2_{3,0}$ and $E^\infty_{1,1} = E^3_{1,1} = E^2_{1,1} \cong V \otimes I_{-2}$. Note that $I^{(3)}_-$ is generated by I_{-3}. $\therefore H_2(L^{(3)}_-) \cong I_{-3} = V \otimes I_{-2}$.

But $H_2(L^{(2)}_-) \cong E^\infty_{2,0} \oplus E^\infty_{1,1} \oplus E^\infty_{0,2}$. It follows that $E^\infty_{0,2} = E^4_{0,2} = 0$. Hence we find that either $E^3_{0,2} = 0$ or $d_3 : E^3_{3,0} \to E^3_{0,2}$ is surjective.

First, let us consider $E^3_{0,2} = 0$. $d_3 : E^3_{3,0} \to E^3_{0,2}$ is trivial and $d_2 : E^2_{2,1} \to E^2_{0,2}$ is surjective in the sequence $0 \to E^2_{4,0} \xrightarrow{d_2} E^2_{2,1} \xrightarrow{d_2} E^2_{0,2} \to 0$.

Thus

$$\begin{aligned} E^\infty_{3,0} = E^4_{3,0} = Ker(d_3 : E^3_{3,0} \to E^3_{0,2})/Im(d_3 : 0 \to E^3_{3,0}) \\ = E^3_{3,0} = E^2_{3,0} \cong H_3(L^{(2)}_-). \end{aligned}$$

By comparing levels, we see that $d_2 : E^2_{4,0} \to E^2_{2,1}$ is trivial. Since $E^2_{0,2} \cong \Lambda^2(I_{-2})$, $E^3_{4,0} = E^2_{4,0}$ and

$$\begin{aligned} E^\infty_{2,1} = E^3_{2,1} = Ker(d_2 : E^2_{2,1} \to E^2_{0,2})/Im(d_2 : E^2_{4,0} \to E^2_{2,1}) \\ \cong Ker(d_2 : E^2_{2,1} \to E^2_{0,2}). \end{aligned}$$

Since $d_2 : E^2_{2,1} \to E^2_{0,2}$ is surjective, $\Lambda^2(I_{-2}) \cong E^2_{0,2} \cong E^2_{2,1}/Ker\ d_2 \cong (I_{-2} \otimes I_{-2})/Ker\ d_2$.

Therefore $Ker\ d_2 \cong S^2(I_{-2})$. Hence

$$E^\infty_{2,1} \cong S^2(I_{-2}).$$

Now let us consider $E^3_{0,2}$ is nonzero and $d_3 : E^3_{3,0} \to E^3_{0,2}$ is surjective, then since $E^3_{3,0} = E^2_{3,0}$ is irreducible, $d_3 : E^3_{3,0} \to E^3_{0,2}$ is an isomoprhism. Thus

$$E^\infty_{3,0} = E^4_{3,0} = 0$$

and

$$\begin{aligned} H_3(L^{(2)}_-) \cong E^3_{3,0} \cong E^3_{0,2} \cong E^2_{0,2}/Im(d_2 : E^2_{2,1} \to E^2_{0,2}) \\ \cong \Lambda^2(I_{-2})/Im(d_2 : E^2_{2,1} \to E^2_{0,2}). \end{aligned}$$

Since all the modules are completely reducible over $D^{(2)}_3$, we have $Im(d_2 : E^2_{2,1} \to E^2_{0,2}) \cong \Lambda^2(I_{-2})/H_3(L^{(2)}_-)$. We have seen that $d_2 : E^2_{4,0} \to E^2_{2,1}$ is trivial.

Thus

$$E^{\infty}_{2,1} = E^3_{2,1} = Ker(d_2 : E^2_{2,1} \to E^2_{0,2})/Im(d_2 : E^2_{4,0} \to E^2_{2,1})$$
$$= Ker(d_2 : E^2_{2,1} \to E^2_{0,2}).$$

Since

$$Im\ d_2 \cong \Lambda^2(I_{-2})/H_3(L^{(2)}_-) \cong E^2_{2,1}/Ker\ d_2 \cong (I_{-2} \otimes I_{-2})/Ker\ d_2,$$

we have

$$Ker\ d_2 \cong S^2(I_{-2}) \oplus H_3(L^{(2)}_-)$$
$$\therefore\ E^{\infty}_{3,0} \oplus E^{\infty}_{2,1} \cong S^2(I_{-2}) \oplus H_3(L^{(2)}_-).$$

Now consider $0 \to E^2_{5,0} \xrightarrow{d_2} E^2_{3,1} \to 0$. By comparing levels, we see that $d_2 : E^2_{3,1} \to E^2_{1,2}$ is trivial. Thus $E^3_{1,2} = E^2_{1,2} \cong V \otimes \Lambda^2(I_{-2})$. Again by comparing the levels of the terms in the sequence $0 \to E^3_{4,0} \xrightarrow{d_3} E^3_{1,2} \to 0$, we conclude that $d_3 = 0$. Therefore $E^{\infty}_{1,2} = E^4_{1,2} = E^2_{1,2} \cong V \otimes \Lambda^2(I_{-2}).E^{\infty}_{0,3}$ is a sub module of $E^2_{0,3} \cong \Lambda^3(I_{-2})$. We get $H_3(L^{(3)}_-)_{-4} = 0$ and

$$I_{-4} \cong (V \otimes I_{-3})/H_3(L^{(3)}_-)_{-4} \cong V \otimes I_{-3}.$$

Now we shall determine I_{-5}: We start with the exact sequence, $0 \to N^{(3)}_- \to L^{(4)}_- \to L^{(3)}_- \to 0$ and the corresponding spectral sequence $\{E^r_{p,q}\}$ converging to $H_*(L^{(4)}_-)$ such that $E^2_{p,q} \cong H_p(L^{(3)}_-) \otimes \wedge^q(I_{-3})$.

We shall compute $H_3(L^{(4)}_-)_{-5}$ from this spectral sequence. Clearly $d_2 : E^2_{2,0} \to E^2_{0,1}$ is an isomorphism and $E^{\infty}_{2,0} = 0$. Consider the sequence $0 \to E^2_{3,0} \xrightarrow{d_2} E^2_{1,1} \to 0$.

From equation, we get $E^2_{3,0} \cong H_3(L^{(2)}_-) \oplus S^2(I_{-2}) \oplus V \otimes \wedge^2(I\_2) \oplus M$ and, $E^2_{1,1} \cong H_1(L^{(3)}_-) \times I_{-3} \cong V \otimes I\_3$.

By elementary linear algebra, we have

$$E^2_{3,0} \cong Ker d_2 \oplus Im d_2.$$

Consider $d_2 : E^2_{3,0} \to E^2_{1,1}$ and by comparing the levels of the terms above, we get that d_2 maps $E^2_{3,0}$ to 0. This implies $Imd_2 = 0$.

Hence

$$H_2(L^{(4)}_-) \cong I_{-4} \cong V \otimes I_{-3}.$$

We have

$$E^{\infty}_{1,1} = E^{3}_{1,1} = E^{2}_{1,1}/Imd_2 \cong V \otimes I_{-3}/Imd_2.$$

Hence we conclude that $Imd_2 = 0$.

$\therefore E^{3}_{3,0} \cong E^{2}_{3,0}$ and $E^{\infty}_{1,1} = E^{3}_{1,1} \cong I_{-4}$. This implies that $E^{\infty}_{0,2} = E^{4}_{0,2} = 0$. Thus the homomorphism d_3 is surjective in the sequence $0 \to E^{3}_{3,0} \xrightarrow{d_3} E^{3}_{0,2} \to 0$. Since $E^{3}_{0,2}$ is a submodule of $E^{2}_{0,2} \cong \wedge^2(I_{-3})$, by comparing levels of $E^{3}_{3,0}$ and $E^{3}_{0,2}$, we see that $Kerd_3$ must contain $V \otimes \wedge^2(I_{-2})$. It follows that $E^{\infty}_{3,0} = E^{4}_{3,0} \cong V \otimes \wedge^2(I_{-2}) \oplus M'$ where M' is a direct sum of level > 6 irreducible highest weight representation of $D_3^{(2)}$. Therefore $(E^{\infty}_{3,0})_{-5} = 0$.

It is easy to see that

$$(E^{\infty}_{2,1})_{-5} = (E^{\infty}_{1,2})_{-5} = (E^{\infty}_{0,3})_{-5} = 0.$$

Therefore we have $H_3(L_-^{(4)})_{-5} \cong 0$.

Hence

$$I_{-5} \cong V \otimes I_{-4}/H_{-3}(L_-^{(4)})_{-5} \cong V \otimes I_{-4}.$$

Thus we have obtained the following structure theorem. □

Theorem 14. *With the usual notations, let* $L = \bigoplus_{n\in Z} L_n$ *be the realization of* $QAD_3^{(2)}$ *associated with the GCM*

$$\begin{pmatrix} 2 & -2 & 0 & -p \\ -1 & 2 & -1 & -q \\ 0 & -2 & 2 & -r \\ -u & -v & -w & 2 \end{pmatrix}$$

where $p, q, r, u, v, w \in Z^+$. *Then we have following:*

i) $I_{-2} \cong \{V(-(u+1)\alpha_4 - \alpha_1) \oplus V(-(v+1)\alpha_4 - \alpha_2)$
$\oplus V(-(1+a)\alpha_4 - \alpha_3)\}$.

ii) $I_{-3} \cong V \otimes I_{-2}$

iii) $I_{-4} \cong V \otimes I_{-3}$.

iv) $I_{-5} \cong V \otimes I_{-4}$.

5. Conclusion

A particular class of indefinite, quasi-affine Kac-Moody algebras $QAD_3^{(2)}$ is studied in this work and the structure of the components of the graded ideals upto level five have been determined. These results give us further scope for understanding the structure of these algebras in depth.

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