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# SOME FIXED POINT THEOREMS OF MULTI-VALUED MAPPINGS IN $G$-METRIC SPACE

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**Abstract:** In this paper, we prove some fixed point theorems concerning multi-valued mappings using $A^{*}$-type contraction in complete symmetric $G$-metric space.

Our results generalizes and improved many results in the existing literature.

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**Key Words:** $G$-metric, $A^{*}$-type contraction, symmetric $G$-metric, weakly commuting mappings

## 1. Introduction

The simplicity and usefulness of fixed point theory has inspired many re-

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searchers to analyze it in multi-valued mappings. The work of B. Fisher and K. Iseiki [5] and Rhodes [11] is worth mentioning. M. S. Khan and I. Kubiaczyk [6] generalized the results of Chang [4], singh and Whitefield [13] and Khan [6]. F. U. Rehman and B. Ahmed [12] presents some fixed point results on fixed point theorems which improves the results of M. S. Khan and I. Kubiaczyk [6].

In 2005, Mustafa and Sims [9] introduced a new structure of generalized metric space which are called $G$-metric as a generalization of metric space.

M. Akram, A. A. Siddiqui and A. A. Zafar in [1] introduced a general class of contractions, called $A$-contractions. This class properly contains contractions originally studied by R. Kannan [8], M. S. Khan [7], Bianchini [3] and Reich [10] for detail see [1].

In 2003, M. Akram, A. A. Siddiqui, and A. A. Zafar have proved some fixed point theorem concerning multi-valued mappings using $A^{\square}$-contractions in metric space, for detail see [2] . Here, we prove some fixed point theorems concerning multi-valued mappings using $A^{\square}$-type contraction in $G$-metric space.

Throughout sequel $X$ will denote a complete symmetric $G$-metric space, $\mathbb{R}_+$ denote the set of non-negative real numbers.

## 2. Preliminaries

**Definition 2.1.** [9] Let $X$ be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}_+$ be a function satisfying the following properties,

(i) $G(x, y, z) = 0$ if $x = y = z$,

(ii) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(iii) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$,

(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, (symmetry in all three variables),

(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangular inequality).

Then the function $G$ is called a generalized metric or more specifically, a $G$-metric on $X$, and the pair $(X, G)$ is called a $G$-metric space.

**Definition 2.2.** [9] A $G$-metric space $(X, G)$ is called symmetric $G$-metric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

**Proposition 2.3.** *[9] Let $(X, G)$ be a G-metric space, then the following are equivalent,*

*(i)* $\{x_n\}$ *is G-convergent to* $x$,

*(ii)* $G(x_n, x_n, x) \to 0$, *as* $n \to \infty$,

*(iii)* $G(x_n, x, x) \to 0$, *as* $n \to \infty$,

*(iv)* $G(x_m, x_n, x) \to 0$, *as* $m, n \to \infty$.

**Proposition 2.4.** *[9] Let $(X, G)$ be a G-metric space, then the following are equivalent,*

*(i)* $\{x_n\}$ *is G-Cauchy,*

*(ii) for* $\epsilon > 0$, *there exist* $N \in \mathbb{N}$ *such that* $G(x_n, x_m, x_m) < \epsilon$, *for all* $n, m \geq N$.

**Definition 2.5.** Let $G$ be a $G$-metric on a set $X$. For subsets $A$ and $B$ of a $G$-metric space $X$,

$$D(A, B, B) = inf\{G(a, b, b) : a \in A, b \in B\}.$$

$$\delta(A, B, B) = Sup\{G(a, b, b) : a \in A, b \in B\}.$$

$$H(A, B, B) = max\{sup\{D(a, a, B) : a \in A\}, sup\{D(A, b, b) : b \in B\}\}.$$

**Definition 2.6.** A mapping $T : X \to Y$ associating with each element $x$ of $X$, a subset $T(x)$ of set $Y$, that is, if for each $x \in X$ the subset $T(x)$ of set $Y$ is called a multi-valued mapping.

**Definition 2.7.** Let $T : X \to CB(X)$ be a mapping, a point $x \in X$ is said to be a fixed point of a multi-valued mapping $T$ provided $x \in Tx$.

**Definition 2.8.** In a $G$-metric space $X$, the mapping $f : X \to X$ and $T : X \to CB(X)$ are said to be weekly commuting if for each $x$ in $X$ and $fTx$ in $CB(X)$,

$$H(fTx, Tfy, Tfy) \leq D(Tx, fy, fy).$$

**Definition 2.9.** [2] Let $A$ stands for the set of all functions $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ satisfying,

(i) $\alpha$ is continuous on the set $\mathbb{R}^3_+$ of all triplets of nonnegative reals (with respect to the Euclidean $G$-metric on $\mathbb{R}^3_+$ ).

(ii) $\alpha$ is non-decreasing in each coordinate variable.

(iii) $a \leq kb$, for some $k \in [0,1)$ whenever $a \leq \alpha(a,b,b)$ or $a \leq \alpha(b,a,b)$ or $a \leq \alpha(b,b,a)$, for all $a,b \in \mathbb{R}_+$.

**Definition 2.10.** **$A^{\square}$-type Contraction.** A map $T : X \to 2^X$ is said to be $A^{\square}$-type contraction of $X$ if there exists $\alpha \in A$ such that

$$\delta(Tx,Ty,Ty) \leq \alpha(\delta(x,y,y),\delta(x,Tx,Tx),\delta(y,Ty,Ty)),$$

for all $x,y$ in $X$.

## 3. Some fixed point theorems

**Theorem 3.1.** *Let $S,T : X \to B(X)$ are mappings such that*

$$\delta(Sx,Ty,Ty) \leq \alpha(\delta(x,y,y),\delta(x,Sx,Sx),\delta(y,Ty,Ty))...(1),$$

*where $\alpha \in A$, then $S$ and $T$ have a common fixed point.*

*Proof.* Let $x_0 \in X$. Choose a point $x_{2n-1}$ in $X_{2n-1} = Sx_{2n-2}$ and $x_{2n}$ in $X_{2n} = Tx_{2n-1}$, for $n = 1,2,3\ldots$

Now,

$$\delta(X_{2n+1},X_{2n+2},X_{2n+2}) = \delta(Sx_{2n},Tx_{2n+1},Tx_{2n+1}).$$

Using (1), we get

$$\begin{aligned}\delta(X_{2n+1},X_{2n+2},X_{2n+2}) \leq\ & \alpha(\delta(x_{2n},x_{2n+1},x_{2n+1}),\delta(x_{2n},Sx_{2n},Sx_{2n})\\ & ,\delta(x_{2n+1},Tx_{2n+1},Tx_{2n+1})),\\ \leq\ & \alpha(\delta(x_{2n},x_{2n+1},x_{2n+1}),\delta(x_{2n},X_{2n+1},X_{2n+1})\\ & ,\delta(x_{2n+1},X_{2n+2},X_{2n+2})),\\ \leq\ & \alpha(\delta(X_{2n},X_{2n+1},X_{2n+1}),\delta(X_{2n},X_{2n+1},X_{2n+1})\\ & ,\delta(X_{2n+1},X_{2n+2},X_{2n+2})).\end{aligned}$$

Using definition of $\alpha$, we get

$$\delta(X_{2n+1},X_{2n+2},X_{2n+2}) \leq k(\delta(X_{2n},X_{2n+1},X_{2n+1})),$$

for some k$\in [0,1)$. Proceeding on the same line, we get

$$\begin{aligned}\delta(X_{2n+1}, X_{2n+2}, X_{2n+2}) &\leq k(k\delta(X_{2n-1}, X_{2n}, X_{2n})),\\ &\leq k^2\delta(X_{2n-1}, X_{2n}, X_{2n}).\end{aligned}$$

Continuing in this way, we have

$$\delta(X_{2n+1}, X_{2n+2}, X_{2n+2}) \leq k^{2n}\delta(X_0, X_1, X_1),$$

for all $n \in N$.

Which in fact gives

$$\delta(X_n, X_{n+1}, X_{n+1}) \leq k^n\delta(X_0, X_1, X_1)...(2).$$

Using the relation between $G$ and $\delta$ and applying (2), we get

$$\begin{aligned}G(x_n, x_{n+1}, x_{n+1}) &\leq \delta(X_n, X_{n+1}, X_{n+1}).\\ &\leq k^n\delta(X_0, X_1, X_1).\end{aligned}$$

Now, for $n > m \geq 0$, using rectangular property of $G$-metric space, we get

$$\begin{aligned}G(x_m, x_n, x_n) \leq& G(x_m, x_{m+1}, x_{m+1}) + G(x_{m+1}, x_{m+2}, x_{m+2})\\ &+ G(x_{m+2}, x_{m+3}, x_{m+3})\\ &+ ... + G(x_{n-1}, x_n, x_n),\\ \leq& k^m\delta(X_0, X_1, X_1) + k^{m+1}\delta(X_0, X_1, X_1) + k^{m+2}\delta(X_0, X_1, X_1)\\ &+ ... + k^{n-1}\delta(X_0, X_1, X_1),\\ \leq& \delta(X_0, X_1, X_1)\{k^m + k^{m+1} + k^{m+2} + ... + k^{n-1}\},\\ &\leq \delta(X_0, X_1, X_1)k^m\{1 + k + k^2 + ... + k^{n-m-1}\},\\ \leq& \delta(X_0, X_1, X_1)k^m\left(\frac{1 - k^{n-m}}{1 - k}\right).\end{aligned}$$

As $k < 1$, $G\left(x_m, x_n, x_n\right) \to 0$ as $m \to \infty$.

Thus $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, $x_n \to x$ in $X$.

Next $D\left(Sx, x_{2n}, x_{2n}\right) \leq \delta(Sx, Tx_{2n-1}, Tx_{2n-1})$, using inequality (1), we get

$$\begin{aligned}D(Sx, x_{2n}, x_{2n}) &\leq \alpha(\delta(x, x_{2n-1}, x_{2n-1}), \delta(x, Sx, Sx), \delta(x_{2n-1}, Tx_{2n-1}, Tx_{2n-1})),\\ &\leq \alpha(\delta(x, x_{2n-1}, x_{2n-1}), \delta(x, Sx, Sx), \delta(x_{2n-1}, X_{2n}, X_{2n})),\\ &\leq \alpha(\delta(x, x_{2n-1}, x_{2n-1}), \delta(x, Sx, Sx), \delta(X_{2n-1}, X_{2n}, X_{2n})),\\ &\leq \alpha(\delta(x, x_{2n-1}, x_{2n-1}), \delta(x, Sx, Sx), k^{2n-1}\delta(X_0, X_1, X_1)),\end{aligned}$$

Taking limit as $n \to \infty$ and using the fact that $X$ is symmetric $G$-metric space, we have

$$\begin{aligned} D(Sx, x, x) &\leq \alpha(\delta 0, (Sx, x, x), 0), \\ &\leq k0, \\ &= 0. \end{aligned}$$

This implies, $D\,(Sx, x, x) = 0$, which finally gives that $x \in Sx$.

Similarly, we can show that $x \in Tx$.

Thus $x$ is common fixed point of $S$ and $T$.

**Corollary 3.2.** *Let $T : X \to B(X)$ is mapping such that*

$$\delta(Tx, Ty, Ty) \leq \alpha(\delta(x, y, y), \delta(x, Tx, Tx), \delta(y, Ty, Ty)),$$

*where $\alpha \in A$, then $T$ have a fixed point.*

*Proof.* The proof follows by taking $S = T$ in above theorem.

**Corollary 3.3.** *Let $S, T : X \to B(X)$ are mapping satisfying any one of the following contractive conditions,*

*(i) There exist a number $a \in [0, \frac{1}{2})$ such that for all $x, y$ in $X$,*

$$\delta(Sx, Ty, Ty) \leq a(\delta(x, Sx, Sx) + \delta(y, Ty, Ty)).$$

*(ii) There exist a number $h \in [0, 1)$ such that for all $x, y$ in $X$,*

$$\delta(Sx, Ty, Ty) \leq h\sqrt{\delta(x, Sx, Sx)\delta(y, Ty, Ty)}.$$

*(iii) There exist a number $h \in [0, 1)$ such that for all $x, y$ in $X$,*

$$\delta(Sx, Ty, Ty) \leq hmax\{\delta(x, Sx, Sx), \delta(y, Ty, Ty)\}.$$

*(iv) There exist numbers $a, b, c \in [0, 1)$ such that $a + b + c < 1$ and for all $x, y$ in $X$,*

$$\delta(Sx, Ty, Ty) \leq a\delta(x, y, y) + b\delta(x, Sx, Sx) + c\delta(y, Ty, Ty).$$

*(v) There exist a number $h \in [0, 1)$ such that for all $x, y, z$ in $X$,*

$$G(Sx, Ty, Tz) \leq kmax\{G(x, Sx, Sx), G(y, Ty, Ty), G(z, Tz, Tz)\}.$$

*Then $S$ and $T$ have a common fixed point in $X$.*

*Proof.* By defining an appropriate $\alpha$, it is not difficult to show that above contractions are $A^{\square}$-type contraction. Also, we have proved that $A^{\square}$-type contraction has fixed point in complete symmetric $G$-metric space, hence it follows that the above contractions have fixed point.

**Theorem 3.4.** *Let $f : X \to X$ and $S, T : X \to B(X)$ be mappings such that*

*(i)* $f, S$ *or* $T$ *are continuous.*

*(ii)* $Sx \subseteq f(X)$ *and* $Tx \subseteq f(X)$.

*(iii)* $f$ *weakly commutes with* $S$ *and* $T$.

*(iv)* $\delta(Sx, Ty, Ty) \leq \alpha(\delta(x, y, y), \delta(x, Sx, Sx), \delta(y, Ty, Ty))$ *for all* $x, y$ *in* $X$ *and for some* $\alpha \in A$.

*Then $f, S$ and $T$ have a common fixed point.*

*Proof.* Let $x_0 \in X$, define a sequence $\{x_n\}$ in $X$ as $x_{2n+1}$=fx$_{2n} \in X_{2n+1} = Sx_{2n}$ and $x_{2n+2}$fx$_{2n+1} \in X_{2n+2} = Tx_{2n+1}$.

Now, using the relation between $G$ and $\delta$ and applying the definition of $\alpha$, we have

$$G(fx_{2n}, fx_{2n+1}, fx_{2n+1}) \leq \delta(X_{2n+1}, X_{2n+2}, X_{2n+2}) = \delta(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}),$$

$$\begin{aligned}
\delta(X_{2n+1}, X_{2n+2}, X_{2n+2}) \leq & \alpha(\delta(x_{2n}, x_{2n+1}, x_{2n+1}), \delta(x_{2n}, Sx_{2n}, Sx_{2n}) \\
& , \delta(x_{2n+1}, Tx_{2n+1}, Tx_{2n+1})), \\
\leq & \alpha(\delta(x_{2n}, x_{2n+1}, x_{2n+1}), \delta(x_{2n}, X_{2n+1}, X_{2n+1}) \\
& , \delta(x_{2n+1}, X_{2n+2}, X_{2n+2})), \\
\leq & \alpha(\delta(X_{2n}, X_{2n+1}, X_{2n+1}), \delta(X_{2n}, X_{2n+1}, X_{2n+1}) \\
& , \delta(X_{2n+1}, X_{2n+2}, X_{2n+2})).
\end{aligned}$$

Using definition of $\alpha$, we get $\delta(X_{2n+1}, X_{2n+2}, X_{2n+2}) \leq k\delta(X_{2n}, X_{2n+1}, X_{2n+1})$, for some k$\in [0, 1)$.

Continuing in the same way, we get

$$\delta(X_{2n+1}, X_{2n+2}, X_{2n+2}) \leq k^{2n}\delta(X_0, X_1, X_1).$$

In general

$$G(fx_{n-1}, fx_n, fx_n) \leq \delta(X_n, X_{n+1}, X_{n+1}) = k^n \delta(X_0, X_1, X_1) ...(1).$$

Using the relation between $G$ and $\delta$ and applying (1), we get

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq \delta(X_n, X_{n+1}, X_{n+1}), \\ &\leq k^n \delta(X_0, X_1, X_1). \end{aligned}$$

Now, for $n > m \geq 0$, using rectangular property of $G$-metric space, we get

$$\begin{aligned} G(x_m, x_n, x_n) \leq & G(x_m, x_{m+1}, x_{m+1}) \\ & + G(x_{m+1}, x_{m+2}, x_{m+2}) + G(x_{m+2}, x_{m+3}, x_{m+3}) \\ & + ... + G(x_{n-1}, x_n, x_n), \\ \leq & k^m \delta(X_0, X_1, X_1) + k^{m+1} \delta(X_0, X_1, X_1) + k^{m+2} \delta(X_0, X_1, X_1) \\ & + ... + k^{n-1} \delta(X_0, X_1, X_1), \\ \leq & \delta(X_0, X_1, X_1) \{k^m + k^{m+1} + k^{m+2} + ... + k^{n-1}\}, \\ \leq & \delta(X_0, X_1, X_1) k^m \{1 + k + k^2 + ... + k^{n-m-1}\}, \\ \leq & \delta(X_0, X_1, X_1) k^m \left( \frac{1 - k^{n-m}}{1-k} \right). \end{aligned}$$

As $k < 1$, so when $m \to \infty$ we get $G\left(x_m, x_n, x_n\right) = G(fx_{m-1}, fx_{n-1}, fx_{n-1}) = 0$.

Thus $\{fx_n\}$ or $\{x_n\}$ is a Cauchy sequence in $X$.

Since $X$ is complete $fx_n \to x$ in $X$.

Next, using the relation between $D$ and $\delta$ and applying the definition of $\alpha$, we have

$$\begin{aligned} D\left(Sx, fx_{2n-1}, fx_{2n-1}\right) = D(Sx, x_{2n}, x_{2n}) \leq \delta(Sx, x_{2n}, x_{2n}) \\ = \delta(Sx, Tx_{2n-1}, Tx_{2n-1}). \end{aligned}$$

$$\begin{aligned} D(Sx, x_{2n}, x_{2n}) &\leq \alpha(\delta(x, x_{2n-1}, x_{2n-1}), \delta(x, Sx, Sx), \delta(x_{2n-1}, Tx_{2n-1}, Tx_{2n-1}), \\ &= \alpha(\delta(x, x_{2n-1}, x_{2n-1}), \delta(x, Sx, Sx), \delta(x_{2n-1}, X_{2n}, X_{2n})), \\ &\leq \alpha(\delta(x, x_{2n-1}, x_{2n-1}), \delta(x, Sx, Sx), \delta(X_{2n-1}, X_{2n}, X_{2n})). \end{aligned}$$

Since $X$ is symmetric $G$-metric space, we can write

$$\begin{aligned} D(Sx, x_{2n}, x_{2n}) &\leq \alpha(\delta(x, x_{2n-1}, x_{2n-1}), \delta(Sx, x, x), \delta(X_{2n-1}, X_{2n}, X_{2n})), \\ &\leq \alpha(\delta(x, x_{2n-1}, x_{2n-1}), \delta(Sx, x, x), k^{2n-1} \delta(X_0, X_1, X_1)). \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$\begin{aligned} D(Sx, x, x) &\le \alpha(0, \delta(Sx, x, x), 0), \\ &\le k0, \\ &= 0. \end{aligned}$$

This implies that $x \in Sx$.

Similarly we can show that $x \in Tx$.

Now suppose $f$ and $S$ is continuous and since $f$ weakly commutes with $S$, we have

$$\begin{aligned} G(x, fx, fx) &\le G(x, ffx_{2n+1}, ffx_{2n+1}) + G(ffx_{2n+1}, fx, fx), \\ &\le qH(Sx, fSx_{2n}, fSx_{2n}) + G(ffx_{2n+1}, fx, fx), \text{ where q} > 1, \\ &\le q\{H(Sx, Sfx_{2n}, Sfx_{2n}) + H(Sfx_{2n}, fSx_{2n}, fSx_{2n})\} \\ &\quad + G(ffx_{2n+1}, fx, fx). \end{aligned}$$

Now by using definition of weakly commuting, we can write

$$\begin{aligned} G(x, fx, fx) \le q\{H(Sx, Sfx_{2n}, Sfx_{2n}) + D(fx_{2n}, Sx_{2n}, Sx_{2n})\} \\ + G(ffx_{2n+1}, fx, fx) ...(2). \end{aligned}$$

Taking limit as $n \to \infty$, we have $H(Sx, Sfx_{2n}, Sfx_{2n}) = H(x, x, x) = 0$, $D(Sx_{2n}, fx_{2n}, fx_{2n})) = G(x, x, x) = 0$, and $G(ffx_{2n+1}, fx, fx) = G(x, x, x) = 0$.

So equation (2) becomes, $G(x, fx, fx) \le 0$.

That is, $G(x, fx, fx) = 0$.

This gives that $x = f(x) \in Sx \bigcap Tx$

**Corollary 3.5.** *Let $f : X \to X$ and $S, T : X \to B(X)$ be mappings such that it satisfies conditions $(i)$, $(ii)$ and $(iii)$ of Theorem 3.4 and condition $(iv)$ is replaced by any of the following contractive conditions:*

*(i) There exist a number $a \in [0, \frac{1}{2})$ such that for all $x, y$ in $X$,*

$$\delta(Sx, Ty, Ty) \le a(\delta(x, Sx, Sx) + \delta(y, Ty, Ty)).$$

*(ii) There exist a number $h \in [0, 1)$ such that for all $x, y$ in $X$,*

$$\delta(Sx, Ty, Ty) \le h\sqrt{\delta(x, Sx, Sx)\delta(y, Ty, Ty)}.$$

*(iii)* *There exist a number* $h \in [0, 1)$ *such that for all* $x, y$ *in* $X$,

$$\delta(Sx, Ty, Ty) \leq hmax\{\delta(x, Sx, Sx), \delta(y, Ty, Ty)\}.$$

*(iv)* *There exist numbers* $a, b, c \in [0, 1)$ *such that* $a + b + c < 1$ *and for all* $x, y$ *in* $X$,

$$\delta(Sx, Ty, Ty) \leq a\delta(x, y, y) + b\delta(x, Sx, Sx) + c\delta(y, Ty, Ty).$$

*(v)* *There exist a number* $h \in [0, 1)$ *such that for all* $x, y, z$ *in* $X$,

$$G(Sx, Ty, Tz) \leq kmax\{G(x, Sx, Sx), G(y, Ty, Ty), G(z, Tz, Tz)\}.$$

*Then* $f, S$ *and* $T$ *have a common fixed point.*

*Proof.* By defining an appropriate $\alpha$, its not difficult to show that above contractions are $A^{\square}$-type contraction. Also, we have proved that the mapping $f, S$ and $T$ satisfying the $A^{\square}$-type contraction condition along with condition $(i)$, $(ii)$ and $(iii)$ have common fixed point, hence it follows that $f, S$ and $T$ have common fixed point.

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