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ON THE LATTICE OF GENERALIZED TOPOLOGIES

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Abstract: In this paper we discuss some properties of the lattice $LGT(X)$ of generalized topologies on a fixed set X and determine the automorphism group of $LGT(X)$. We define simple expansion of a generalized topological space and prove that any cover of a generalized topology μ on a set X is a simple expansion of μ. Further if X is finite, cardinality of any cover of μ is exactly one element more than that of μ.

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1. Introduction

Császár introduced the notion of Generalized Topology in [2]. A collection μ of subsets of a set X is said to form a generalized topology on X if $\emptyset \in \mu$ and arbitrary union of elements in μ is again in μ. Let $LGT(X)$ denotes the collection of all generalized topologies on X. Then $LGT(X)$ is a poset under the order of set inclusion. Baskaran, Murugalingam and Sivaraj proved that $LGT(X)$ forms a complete lattice and studied the properties of $LGT(X)$ in their paper[6]. The lattice $(L, \leq)$ is denoted by L, when there is no confusion.

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We use the following definitions and theorems in this paper. For more details see[3, 4].

Let L be a lattice. For $a, b \in L$, by a covers b we mean $b < a$ and for any $c \in L$ with $b \leq c \leq a$, we have either $c = b$ or $c = a$. An atom of a lattice is an element which covers the smallest element 0 if it exists. A lattice is atomic if every element other than the least element can be written as the join of atoms. An anti-atom is an element which is covered by the largest element 1 (if it exists) in the lattice. An anti-atom is also called dual atom. A lattice is anti-atomic if every element other than the largest element can be written as the meet of anti-atoms.

A lattice is called distributive if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in L$. The lattice L called modular if for any $a, b, c \in L$, $a \leq c$ implies $a \vee (b \wedge c) = (a \vee b) \wedge c$. A lattice L is modular if and only if it has no sublattice isomorphic to a pentagon[4]. The lattice L is called semi-modular if for any $a, b \in L$ with $a \neq b$, and if a and b cover $a \wedge b$, then $a \vee b$ covers a and b.

A bijective map from a lattice L to a lattice K is called a lattice isomorphism if it preserves finite meets and joins. Equivalently an isomorphism between two lattices L and M is a bijection $f : L \to M$ such that for $x, y \in L$, $x \leq y$ if and only if $f(x) \leq f(y)$ for $x, y \in L$. Two lattices are said to be isomorphic if there exist an isomorphism between them and an isomorphism of a lattice with itself is called an automorphism. A lattice is called self dual if it is isomorphic to its dual lattice.

A poset P is called graded if we can define an integer-valued function h on P such that for $x, y \in P$, $x \leq y$ and $h(x) + 1 = h(y)$ if and only if x covers y.

2. Some Properties of *LGT*(*X*)

Lattice $LGT(X)$ of topologies on a set X is both atomic and anti-atomic [7]. Here we prove that $LGT(X)$ is atomic but not anti atomic.

Theorem 1. *$LGT(X)$ is an atomic lattice. The atoms are generalized topologies of the form $\{\emptyset, A\}$, where $\emptyset \subsetneq A \subseteq X$. If X is finite and if $|X| = n$, then $LGT(X)$ contain $2^n - 1$ atoms. If X is infinite and $|X| = \alpha$, then $LGT(X)$ contain 2^α atoms.*

Proof. It can be seen that the atoms in $LGT(X)$ are precisely the generalized topologies of the form $\{\emptyset, A\}$, where A is a nonempty subset of X. Also given any generalized topology μ on X, we have $\mu = \bigvee_{\substack{A \in \mu \\ A \neq \phi}} \{\emptyset, A\}$. Thus $LGT(X)$

is an atomic lattice. Since we are considering every non empty subset A of X here, the total number of atoms in $LGT(X)$ is $2^n - 1$ where $n = |X|$ and number of atoms in $LGT(X)$ is 2^α if X is an infinite set of cardinality α. □

Note 2.1. $LGT(X)$ possess anti-atoms. The anti-atoms are precisely the generalized topologies of the form $P(X) \setminus \{\{x\}\}$, $x \in X$. Because given any generalized topology μ on X with $\mu \neq P(X)$, there exist an $x \in X$ such that $\{x\}$ does not belong to μ. This implies $\mu \subset P(X)\setminus\{\{x\}\}$ and also $P(X)\setminus\{\{x\}\}$ is a generalized topology since $\emptyset \in P(X) \setminus \{\{x\}\}$ and any arbitrary union of elements in $P(X) \setminus \{\{x\}\}$ is again in the same collection. Also it is obvious that there is no proper subset of $P(X)$ between $P(X)$ and $P(X) \setminus \{\{x\}\}$ and hence no generalized topology exists between them for any $x \in X$. If $|X| = \alpha$, $LGT(X)$ contain α anti-atoms. But $LGT(X)$ is not anti-atomic since $\{\emptyset, \{x\}\}$ cannot be written as the meet of any collection of anti-atoms because every anti-atom contains X so is their intersection.

It is known that every distributive lattice is modular[4]. But it is proved that[6] $LGT(X)$ is not distributive. In the next theorem we enquires when $LGT(X)$ become modular.

Theorem 2. *$LGT(X)$ is modular if $|X| \leq 2$ and not modular if $|X| \geq 3$.*

Proof. If $|X| \leq 2$, we can see from the lattice diagrams below that it has no sublattice isomorphic to a pentagon. Hence $LGT(X)$ is modular if $|X| \leq 2$.

Let $|X| = n$ with $n \geq 3$. Then there exist $a, b, c \in X$ with $a \neq b \neq c$. Consider the generalized topologies G_i, $i = 1, 2, \dots 5$ on X, where $G_1 = \{\emptyset\}$, $G_2 = \{\emptyset, \{a, b\}\}$, $G_3 = \{\emptyset, \{a, c\}\}$, $G_4 = \{\emptyset, \{a, b\}, \{a, b, c\}\}$ and $G_5 = \{\emptyset, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. It can be observed that in the lattice diagram of $LGT(X)$, the above five generalized topologies constitute a pentagon as below and hence is not modular. □

Definition 3. [4] Let L be a lattice with least element 0. We define the height function as follows: for $a \in L$, let $h(a)$ denotes the length of a longest maximal chain in $[0, a]$, if it exists; otherwise, put $h(a) = \infty$.

Theorem 4. *[4] Let L be a finite lattice. L is semimodular if and only if $h(a) + h(b) \geq h(a \wedge b) + h(a \vee b)$ for all a, b and c in L, where h is the height function.*

We use above theorem to prove that $LGT(X)$ is not semimodular when X is finite with $|X| \geq 3$.

Theorem 5. *Let X be a finite set. Then $LGT(X)$ is semimodular if and only if $|X| \leq 2$.*

Proof. When $|X| \leq 2$, we proved that it is modular and hence semimodular. Let $|X| \geq 3$ and let a, b and c be three distinct elements in X, and consider the generalized topologies $\mu_1 = \{\emptyset, \{a\}$
$, \{b\}, \{a, b\}\}$, and $\mu_2 = \{\emptyset, \{c\}, \{b, c\}\}$. Then $\mu_1 \vee \mu_2 = P(\{a, b, c\})$ and $\mu_1 \wedge \mu_2 = \{\emptyset\}$. Also $h(\mu_1) = 3, h(\mu_2) = 2$ and $h(\mu_1 \vee \mu_2) = 7$, $h(\mu_1 \wedge \mu_2) = 0$. We can see that h does not satisfy the condition in previous theorem and there by concluding that lattice of generalized topologies on any finite set of cardinality greater than 3 is not semimodular. □

Theorem 6. *$LGT(X)$ is self dual if and only if $|X| = 1$.*

Proof. If $|X| = 1$, $LGT(X)$ contain only two elements, namely $\{\emptyset\}$ and $\{\emptyset, X\} = P(X)$. Hence $LGT(X)$ is obviously self dual. When $|X| > 2$, as the number of atoms and dual atoms are different, $LGT(X)$ is not self dual . □

Set of all topological spaces of X is not a sublattice of $LGT(X)$ if $|X| \geq 3$. Let a, b and c be three distinct elements in X and let $\tau_1 = \{\emptyset, X, \{a, b\}\}$ and $\tau_2 = \{\emptyset, X, \{a, c\}\}$, then $\tau_1 \vee \tau_2 = \{\emptyset, X, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ is not a topology on X.

3. Simple Expansion of a Generalized Topological Spaces

Simple expansion of lattice of topologies has been studied previously by many mathematicians and the similar concept can be carried over to generalized topologies.

Definition 7. Let X be a set, $\mu \in LGT(X)$ and A be a subset of X which does not belong to μ. Then the simple expansion of μ by A, denoted by $\mu(A)$, is defined as

$$\mu(A) = \mu \cup \{G \cup A : G \in \mu\}$$

Remark 3.1. *We can prove that a simple expansion of μ forms a generalized topological space. Also it is obvious from the definition that $\mu(A)$ is the smallest generalized topology containing μ and A. Hence we have the following theorem.*

Theorem 8. *Let $\mu \in LGT(X)$ and suppose that $\mu(A)$ is a simple expansion of μ. Then*

$$\mu(A) \in LGT(X) \text{ and } \mu(A) = \mu \vee \{\emptyset, A\}.$$

We can see that $\mu(A)$ is obviously an upper neighbor of μ. The following example shows that it need not always be a cover.

Example 3.1. Consider the set $X = \{a, b, c, d\}$ and the generalized topology $\mu = \{\emptyset, X, \{a, b, c\}, \{d\}\}$ on X. Let $A = \{a\}$. Then

$$\mu(A) = \{\emptyset, X, \{a, b, c\}, \{d\}, \{a\}, \{a, d\}\}.$$

We can see that $\mu(A)$ is not a cover of μ since if we take

$$\tau = \{\emptyset, X, \{a, b, c\}, \{d\}, \{a, d\}\}, \quad \tau \in LGT(X)$$

and $\mu < \tau < \mu(A)$.

But we can prove that every cover is a simple expansion of some subset of X.

Theorem 9. *Let $\mu \in LGT(X)$. If τ is a cover of μ in $LGT(X)$, then there exist $A \subsetneqq X$ such that $\tau = \mu(A)$.*

Proof. Let $A \in \tau \setminus \mu$. Then $\mu(A)$ is the smallest generalized topology containing μ and A. Thus $\mu < \mu(A) \leq \tau$. But τ is a cover of μ, hence $\mu(A) = \tau$. ☐

Remark 3.2. *From the above proof it also implies that every cover τ of μ is of the form $\mu(A)$, for every $A \in \tau \setminus \mu$.*

We now show the existence of a cover of μ which has cardinality exactly one more than that of μ when X is finite.

Lemma 3.1. *Let μ be a generalized topology on a finite set X and $\mu \neq \mathcal{P}(X)$. Then there exists a subset A of X that does not belong to μ such that $\mu(A) = \mu \cup \{A\}$.*

Proof. Let $|X| = n$. It is enough to show the existence of a set $A \subset X$ such that $\mu \cup \{A\}$ is a generalized topology on X. If X does not belong to μ, take $A = X$, so that $\mu \cup \{X\} \in LGT(X)$. If $X \in \mu$, then consider the collection of all subsets of X with cardinality $n - 1$. Let us denote the collection by F_{n-1}. If $F_{n-1} \nsubseteq \mu$ then choose $A \in F_{n-1}$ such that $A \notin \mu$.

Claim. *$\mu \cup \{A\}$ is a generalized topology on X.*

Let $U, V \in \mu \cup \{A\}$. If $U, V \in \mu$, then obviously $U \cup V \in \mu$. If $U \in \mu$ and $V = A$, then either $U \cup V = X$ or $U \cup V = A$. In either case $U \cup V \in \mu \cup \{A\}$. Also $\emptyset \in \mu$ implies that $\mu \cup \{A\} \in LGT(X)$.

If $F_{n-1} \subset \mu$, then consider F_{n-2}, which is the collection of all subsets of X with cardinality $n-2$. If $F_{n-2} \not\subseteq \mu$ then choose $A \in F_{n-2}$ such that $A \notin \mu$ and we can prove that $\mu \cup \{A\}$ is a generalized topology on X. Proceeding similarly if μ contain all 2 element sets and its supersets, since $\mu \neq P(X)$, there exist an $x \in X$ such that $\{x\}$ does not belong to μ, then take $A = \{x\}$. Then A will satisfy the required property. Hence the theorem. □

Theorem 10. *If X is finite, every cover of μ is of the form $\mu \cup \{A\}$ for some set $A \subset X$.*

Proof. Let τ be a cover of μ with $|\tau| = |\mu| + k$ where $k \geq 2$. Let us write τ as $\mu \cup \{A_1, A_2 \ldots A_k\}$ where $A_1, A_2 \ldots A_k$ are distinct subsets of X which do not belong to μ. Let $\mathcal{F} = \{A_1, A_2 \ldots A_k\}$ and $I = \{1, 2, 3, \ldots k\}$.

Case 1. Let $A_i \not\subseteq A_j$ for $i, j \in I$ with $i \neq j$.

Consider $\mu \cup \{A_1\}$. $\emptyset \in \mu \cup \{A_1\}$ since μ is a generalized topology. Let $U, V \in \mu \cup \{A_1\}$. If $U, V \in \mu$, then $U \cup V \in \mu$ since μ is a generalized topological space. If $U \in \mu$ and $V = A_1$ then also $U \cup V \in \mu \cup \{A_1\}$, for otherwise if $U \cup V = A_j$ for some $j \neq 1$ then $A_1 \subset A_j$ which is not possible. Therefore $\mu \cup \{A_1\}$ is a generalized topological space. Then $\mu < \mu \cup \{A_1\} < \tau$ which is a contradiction since τ is a cover of $\{\mu\}$.

Case 2. There exist $i, j \in I$ with $A_i \subset A_j$ where $i \neq j$.

Let $\mathcal{F}_j = \{A_i : i \in I, i \neq j, A_i \subset A_j\}$. Consider $\mathcal{F} \setminus \mathcal{F}_j$. Rename the elements in $\mathcal{F} \setminus \mathcal{F}_j$ such that $\mathcal{F} \setminus \mathcal{F}_j = \{A_j, A_{j+1}, \ldots A_k\}$. Let us denote $\mathcal{F} \setminus \mathcal{F}_j$ by $\mathcal{F}_j^c$. $\mathcal{F}_j^c \neq \emptyset$ as $A_j \in \mathcal{F}_j^c$.

Claim. *$\mu \cup \mathcal{F}_j^c$ is a generalized topology.*

$\emptyset \in \mu \cup \mathcal{F}_j^c$ since $\emptyset \in \mu$. Let $U, V \in \mu \cup \mathcal{F}_j^c$. If $U, V \in \mu$, then obviously $U \cup V \in \mu$. If $U \in \mu$ and $V \in \mathcal{F}_j^c$, then also $U \cup V \in \mu \cup \mathcal{F}_j^c$, otherwise $U \cup V \in \mathcal{F}_j$. Then $U \cup V = A_h$ for some $h \in \{1, 2, \ldots, j-1\}$ so that $V \subset A_h \subset A_j$. Thus $V \in \mathcal{F}_j$ which is not possible since $V \in \mathcal{F}_j^c$. Similarly if $U, V \in \mathcal{F}_j^c$ then $U \cup V \in \mu \cup \mathcal{F}_j^c$ by the same argument. Hence the claim.

Thus we get $\mu < \mu \cup \mathcal{F}_j^c < \tau$ a contradiction since τ is a cover. Thus in both cases we proved that any cover of μ can not have a cover with cardinality $|\mu| + k$

where $k \geq 2$. By Lemma[3.1] we have that there exist a cover with cardinality $|\mu| + 1$. Thus every cover of μ is of the form $\mu \cup \{A\}$ for some set $A \subset X$ and $A \notin \mu$. ☐

Corollary 11. *$LGT(X)$ is a graded poset when X is finite.*

Proof. Define a function

$$h : LGT(X) \to \mathbb{Z}^+$$

by $h(\mu) = |\mu|$. That is, h maps each generalized topology into its cardinal number. Then by the above theorem, if $\tau \in LGT(X)$ is a cover of $\mu \in LGT(X)$, then $|\tau| = |\mu| + 1$. Thus $LGT(X)$ is a graded poset. ☐

Remark 3.3. *The following example shows that there exists generalized topology whose simple expansion by any subset contains an infinite number of elements other than that in generalized topology. Consider $\mathbb{R}$, the set of real numbers and the following generalized topology on $\mathbb{R}$,*

$$\mu = \{\emptyset, \mathbb{R}\} \cup P(\mathbb{Q}) \cup \{X \cup Y : X = \mathbb{Q} \setminus A \text{ where A is a finite subset of } \mathbb{Q} \text{ and } Y \subset \mathbb{R} \setminus \mathbb{Q}\}.$$

If A is any subset of $\mathbb{R}$ which does not belongs to μ then A is of the form $G \cup H$ where G is a subset of $\mathbb{Q}$ from which an infinite subset of $\mathbb{Q}$ is removed and H is any subset of $\mathbb{R} \setminus \mathbb{Q}$. In this case $\mu(A) \setminus \mu$ contains infinite number of elements.

4. Automorphisms of Lattice of Generalized Topologies

In[5] it is proved that when X is infinite or X consists of atmost two elements, the lattice $LT(X)$ of topologies on X is isomorphic to the symmetric group on X. From this it can be seen that if X is an infinite set and P is any topological property, then the set of all topologies in $LT(X)$ possessing the property P may be identified exclusively from the lattice structure of $LT(X)$ and hence the topological properties of elements of $LT(X)$ must be determined by the position of the topologies in $LT(X)$ [7]. Here we prove that automorphism group of lattice of generalized topologies on any set X is isomorphic to the symmetric group on X.

Lemma 4.1. *Let $S(X)$ be the group of all bijections of X with operation as composition of functions. For $p \in S(X)$ and $\mu \in LGT(X)$ let $p(\mu) = \{p(G) : G \in \mu\}$ where $p(G) = \{p(x) : x \in G\}$. Then $p(\mu)$ is a generalized topology on X.*

Proof. $p(\emptyset) = \emptyset$ implies $\emptyset \in p(\mu)$. Now consider an arbitrary collection of sets $\{G_i\}_{i \in I}$ in $p(\mu)$. Then for every $i \in I$, $G_i = p(U_i)$ for some $U_i \in \mu$. Also $\bigcup_{i \in I} G_i = \bigcup_{i \in I} p(U_i) = p(\bigcup_{i \in I} U_i) \in p(\mu)$ since $\bigcup_{i \in I} U_i \in \mu$. Thus $p(\mu)$ is a generalized topology on X. □

Now we prove that each bijection in X naturally induces an automorphism in $LGT(X)$.

Theorem 12. *Let $p \in S(X)$, define a map A_p on $LGT(X)$ by $A_p(\mu) = p(\mu)$ for $\mu \in LGT(X)$. Then A_p is an automorphism of $LGT(X)$.*

Proof. Let $\mu, \tau \in LGT(X)$. $A_p(\mu) = A_p(\tau)$ implies $p(\mu) = p(\tau)$. Now

$$\begin{aligned} G \in \mu &\Leftrightarrow p(G) \in p(\mu) \\ &\Leftrightarrow p(G) \in p(\tau) \\ &\Leftrightarrow G \in \tau. \end{aligned}$$

This proves that $\mu = \tau$. Hence A_p is one-one. Let $\mu \in LGT(X)$ and take $\tau = \{p^{-1}(G) : G \in \mu\}$ where $p^{-1}(G) = \{x \in X : p(x) \in G\}$. Then $\tau = p^{-1}(\mu)$ and $A_p(\tau) = p(p^{-1}(\mu)) = \mu$ proving that A_p is onto.

Let $\mu \subseteqq \tau$. That is

$$\begin{aligned} (G \in \mu \Rightarrow G \in \tau) &\Leftrightarrow (p(G) \in p(\mu) \Rightarrow p(G) \in p(T)) \\ &\Leftrightarrow p(\mu) \subseteqq p(\tau) \\ &\Leftrightarrow A_p(\mu) \subseteqq A_p(\tau). \end{aligned}$$

Hence A_p is an automorphism of $LGT(X)$. □

Note 4.1. *An automorphism of $LGT(X)$ maps atoms of a lattice to atoms and dual atoms to dual atoms[3].*

Lemma 4.2. *An automorphism of $LGT(X)$ maps a generalized topology consisting of n elements to a generalized topology consisting of same number of elements.*

Proof. Let μ be a generalized topology consisting of n elements. and A be an automorphism of $LGT(X)$. Then μ is larger than precisely $n-1$ atoms. Therefore $A(\mu)$ must be larger than precisely $n-1$ atoms. Hence $A(\mu)$ consists of n elements. □

Lemma 4.3. *Let A be an automorphism of $LGT(X)$ and $\mu, \tau \in LGT(X)$ are complements to each other. Then $A(\mu)$ and $A(\tau)$ are compliments to each other.*

Proof. We have $\mu \vee \tau = P(X)$ and $\mu \wedge \tau = \{\emptyset\}$ where $P(X)$ and $\{\emptyset\}$ being the largest and smallest elements of $LGT(X)$. Also $A(\mu) \vee A(\tau) = A(\mu \vee \tau) = A(P(X)) = P(X)$ and $A(\mu) \wedge A(\tau) = A(\mu \wedge \tau) = A(\{\emptyset\}) = \{\emptyset\}$, since an automorphism always preserves the largest and smallest element of a lattice. Hence $A(\mu)$ and $A(\tau)$ are complements to each other. $\square$

For $p \in S(X)$ consider the map A_p on $LGT(X)$ where $A_p(\mu) = p(\mu)$ for $\mu \in LGT(X)$ and $p(\mu) = \{p(G) : G \in \mu\}$ where $p(G) = \{p(x) : x \in G\}$. Then we have the following theorem.

Theorem 13. *The set of automorphisms of $LGT(X)$ is precisely $\{A_p : p \in S(X)\}$.*

Proof. In theorem [4.1], it is proved that A_p is an automorphism of $LGT(X)$ for every $p \in S(X)$. Now let A be an automorphism of $LGT(X)$. Let $\mathcal{N}$ denotes the collection of all atoms $I_x = \{\emptyset, \{x\}\}$ where $x \in X$.

Claim: A maps $\mathcal{N}$ onto itself.

Let $I_x \in \mathcal{N}$. Consider the dual atom $\mu = \mathcal{P}(X) \setminus \{\{x\}\}$. As μ and I_x are complements to each other, $A(\mu)$ and $A(I_x)$ are complements to each other. Since $A(\mu)$ is also a dual atom there exists a $y \in X$ such that $A(\mu) = \mathcal{P}((X) \setminus \{\{y\}\}$. Then $A(I_x)$ must contain $\{y\}$ since $A(\mu) \vee A(I_x) = \mathcal{P}(X)$ and therefore $\{\emptyset, \{y\}\} \subseteqq A(I_x)$. But $A(I_x)$ is an atom implying that $A(I_x) = \{\emptyset, \{y\}\}$. Thus A maps $\mathcal{N}$ to itself. Now take $I_y \in \mathcal{N}$. Consider $\tau = \mathcal{P}(X) \setminus \{\{y\}\}$. Since A is onto, there exists a dual atom, say $\mu \in LGT(X)$ such that $A(\mu) = \tau$. Let $\mu = \mathcal{P}(X) \setminus \{\{x\}\}$ and $I_x = \{\emptyset, \{x\}\}$. As $A(I_x) \vee \tau = A(I_x) \vee A(\mu) = A(I_x \vee \mu) = A(\mathcal{P}(X)) = \mathcal{P}(X)$, $\{y\}$ must belong to $A(I_x)$. Since $A(I_x)$ is an atom $A(I_x) = I_y$. Hence the claim.

Now define a map $p : X \to X$ such that $p(x) = y$ whenever $A(I_x) = I_y$. Since y is unique for a fixed x implying that p is well defined. Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Since A is one-one $A(I_x) \neq A(I_y)$. Hence $p(x_1) \neq p(x_2)$ implies that p is one-one. Let $y \in X$ and consider $I_y = \{\emptyset, \{y\}\}$. Since A is onto there exists an $x \in X$ such that $A(I_x) = I_y$ implies $p(x) = y$. Hence p is onto.

Now consider the automorphism A_p on $LGT(X)$ induced by the bijection p.

Claim. *$A = A_p$ on $\mathcal{N}$.*

Let $I_x \in \mathcal{N}$. Then $A(I_x) = I_{p(x)} = \{\emptyset, \{p(x)\}\} = p(\{\emptyset, \{x\}\}) = p(I_x) = A_p(I_x)$. Hence $A = A_p$ on $\mathcal{N}$.

Let $\alpha = \{\emptyset, G\}$, $G \subsetneqq X$, be an atom which does not belong to $\mathcal{N}$. Let $A(\alpha) = \{\emptyset, H\}$ where $H \subsetneqq X$. Consider $A_p(\alpha) = \{\emptyset, p(G)\}$. We have to prove that $A_p(\alpha) = A(\alpha)$.

Let $x \in G$ and $A(I_x) = I_y$. Then $y = p(x) \in p(G)$. Now $A(\{\emptyset, \{x\}, G\}) = A(\{\emptyset, \{x\}\} \vee \{\emptyset, G\}) = A(\{\emptyset, \{x\}\}) \vee A(\{\emptyset, G\}) = \{\emptyset, \{y\}\} \vee \{\emptyset, H\} = \{\emptyset, \{y\}, H, H \cup \{y\}\}$. But A maps an n element set to n element set only, hence $H \cup \{y\} = H$. This implies $y \in H$. Since $x \in G$ is arbitrary $p(G) \subsetneqq H$.

To prove the reverse inclusion, let $y \notin p(G)$ and $A(I_x) = I_y$ implies $x \notin G$. $I_x \vee \alpha = \{\emptyset, H\}$. As $I_x \vee \alpha = \{\emptyset, \{x\}\} \vee \{\emptyset, G\} = \{\emptyset, \{x\}, G, G \cup \{x\}\}$. Since $x \notin G$, $G \neq G \cup \{x\}$, hence $A(I_x \vee \alpha)$ is a four element set. That is $A(I_x \vee \alpha) = A(I_x) \vee A(\alpha) = I_y \vee \{\emptyset, H\} = \{\emptyset, \{y\}, H, H \cup \{y\}\}$ is a four element set, which implies $y \notin H$ proving that $H \subsetneqq p(G)$.

Hence $H = p(G)$ and hence $A(\{\emptyset, G\}) = \{\emptyset, H\} = \{\emptyset, p(G)\} = A_p(\{\emptyset, G\})$. Thus we proved that $A = A_p$ on all atoms of $LGT(X)$. Since $LGT(X)$ is an atomic lattice, $A = A_p$ on $LGT(X)$. $\square$

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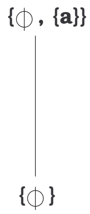
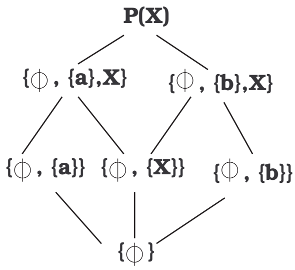


LGT(X) when X={a}



LGT(X) when X={a,b}