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LOCALLY FUNCTION CHAINABLE SETS IN TOPOLOGICAL SPACES

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Abstract: The concept of function chainable sets has been introduced in [7]. In this paper, the function chainability between sets in topological spaces have been defined in locally terms. It is shown, that under some restrictions, that concept of function chainability between sets is partially hereditary and partially topological. We use the terminology and preliminary definitions from [4], [7], and [8].

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Key Words: function-chainable sets, function-chainable space, self function chainable sets, strongly function chainable sets, strongly self function chainable sets

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Throughout this paper X denotes the topological space with topology τ, $f : X \to [0,\infty)$ is real-valued, non-constant continuous function, unless stated otherwise. Also, for $A \subset X$, A°_{τ}, will stand for $\tau-$interior of A. Let $\varepsilon > 0$. Then (referring to Definition 1 in [7]):

$$V_{f\varepsilon}(a) = \{\ x \in X\ : |f\,(x) - f\,(a)| < \varepsilon\}.$$

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Definition 1. Let $A, B \subset X$. Then $\langle A, B\rangle$ is locally function$-f-$chainable at $(a,b) \in A \times B$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left\langle V_{f\delta}(a) \cap \overset{\circ}{A_\tau},\ V_{f\delta}(b) \cap \overset{\circ}{B_\tau} \right\rangle$$

is function$-f-\varepsilon-$chainable, where $V_{f\delta}(a) \cap \overset{\circ}{A_\tau}$ and $V_{f\delta}(b) \cap \overset{\circ}{B_\tau}$ are non-empty sets with $V_{f\delta}(a) \cap \overset{\circ}{A_\tau} \subset V_{f\varepsilon}(a)$ and $V_{f\delta}(b) \cap \overset{\circ}{B_\tau} \subset V_{f\varepsilon}(b)$. $\langle A, B\rangle$ is locally function$-f-$chainable if $\langle A, B\rangle$ is locally function$-f-$chainable at each point of $A \times B$.

Definition 2. A subset A of X is said to be function$-f-$open if for every $a \in A$ there exists an $\varepsilon > 0$ such that $V_{f\varepsilon}(a) \subset A$.

Every function$-f-$open set is open in X.

The collection of all function$-f-$open sets in X form a topology τ_f on X coarser than τ.

Theorem 1. *Let $A, B \subset X$ such that $\langle A, B\rangle$ is locally function$-f-$chainable. Let C and D be function$-f-$open sets in X such that $C \subset A$ and $D \subset B$. Then $\langle C, D\rangle$ is locally function$-f-$chainable.*

Proof. Let $(a,b) \in C \times D$ and $\varepsilon > 0$. Then there exists an $\varepsilon_1 < \varepsilon$ such that $V_{f\varepsilon_1}(a) \subset C \subset A$ and $V_{f\varepsilon_1}(b) \subset D \subset B$. For $(a,b) \in A \times B$ and $\varepsilon_1 > 0$ there exists a $\delta > 0$ such that $\langle V_{f\delta}(a) \cap \overset{\circ}{A_\tau},\ V_{f\delta}(b) \cap \overset{\circ}{B_\tau}\rangle$ is function$-f-\varepsilon_1-$chainable and $V_{f\delta}(a) \cap \overset{\circ}{A_\tau} \neq \emptyset$ and $V_{f\delta}(b) \cap \overset{\circ}{B_\tau} \neq \emptyset$ and $V_{f\delta}(a) \cap \overset{\circ}{A_\tau} \subset V_{f\varepsilon_1}(a) \subset C$ and $V_{f\delta}(b) \cap \overset{\circ}{B_\tau} \subset V_{f\varepsilon_1}(b) \subset D$. Since C and D are also $\tau-$open sets in X, $V_{f\delta}(a) \cap C = V_{f\delta}(a) \cap \overset{\circ}{A_\tau}$ and $V_{f\delta}(b) \cap D = V_{f\delta}(b) \cap \overset{\circ}{B_\tau}$. Since $\varepsilon_1 < \varepsilon$ it follows that $\langle C, D\rangle$ is locally function$-f-$chainable at (a,b). Hence the result.

Corollary 2. *Let $A, B \subset X$ are sets such that $\langle A, B\rangle$ is locally function$-f-$chainable then $\left\langle \overset{\circ}{A_{\tau_f}}, \overset{\circ}{B_{\tau_f}} \right\rangle$ is locally$-f-$chainable.*

Corollary 3. *Let A, B, C and D be function$-f-$open subsets of X such that $\langle A, B\rangle$ and $\langle C, D\rangle$ are locally function$-f-$chainable then $\langle A \cap C\ , B \cap D\rangle$ is locally function$-f-$chainable.*

Theorem 4. *Let g be a homeomorphism from topological space (X, τ) to topological space (Y, τ'). Let $A, B \subset X$ such that $\langle A, B\rangle$ is be locally* *function−f−chainable. Then $\langle g(A),\ g(B)\rangle$ is locally function−$f\circ g^{-1}$−chainable.*

Proof. Let $(g(a), g(b)) \in g(A)\times g(B)$ and $\varepsilon > 0$. Then there exists a $\delta > 0$ such that
$$\left\langle V_{f\delta,}(a)\cap \overset{\circ}{A_\tau},\ V_{f\delta}(b)\cap \overset{\circ}{B_\tau}\right\rangle$$
is function−$f-\varepsilon-$ chainable where $V_{f\delta}(a)\cap \overset{\circ}{A_\tau}\neq\emptyset$ and $V_{f\delta}(b)\cap \overset{\circ}{B_\tau}\neq\emptyset$ and $V_{f\delta}(a)\cap \overset{\circ}{A_\tau}\subset V_{f\varepsilon}(a)$; $V_{f\delta}(b)\cap \overset{\circ}{B_\tau}\subset V_{f\varepsilon}(b)$.

We have $g\left(\overset{\circ}{A_\tau}\right) = \overset{\circ}{(g(A))}_{\tau'}$; $g\left(\overset{\circ}{B_\tau}\right) = \overset{\circ}{(g(B))}_{\tau'}$ and
$$V_{(f\circ g^{-1})\delta}(g(a))\cap \overset{\circ}{(g(A))}{}_{\tau'} = \left\{g(x) : x\in V_{f\delta}(a)\cap \overset{\circ}{A_\tau}\right\};$$
$$V_{(f\circ g^{-1})\delta}(g(b))\cap \overset{\circ}{(g(B))}_{\tau'} = \left\{g(x) : x\in V_{f\delta}(b)\cap \overset{\circ}{B_\tau}\right\}.$$
Hence
$$V_{(f\circ g^{-1})\delta}(g(a))\cap \overset{\circ}{(g(A))}_{\tau'}\neq\emptyset;$$
$$V_{(f\circ g^{-1})\delta}(g(b))\cap \overset{\circ}{(g(B))}_{\tau'}\neq\emptyset.$$

Next, let us consider $y\in V_{(f\circ g^{-1})\delta}(g(a))\cap \overset{\circ}{(g(A))}_{\tau'}$ or
$$\left|f\circ g^{-1}(y) - f\circ g^{-1}(g(a))\right| = \ |f(x) - f(a)| <\ \delta,$$
where $x = g^{-1}(y)$.

Therefore $x\in V_{f\delta}(a)$. Again, $y\in \overset{\circ}{(g(A))}_{\tau'}$ yields $g^{-1}(y)\in\ g^{-1}(W)\subset A$ for some τ'−open set W in Y or $x\in \overset{\circ}{A_\tau}$. Hence x is function−$f-\varepsilon$−chainable to some $z\in V_{f\delta}(b)\cap \overset{\circ}{B_\tau}$ or there exist function−$f-\varepsilon$−chain $x = x_0, x_1, \ldots, x_{n-1}$, $x_n = z$ in X such that $|f(x_i) - f(x_{i-1})| < \varepsilon$; $1\le i\ \le n$. Since
$$g(z)\in V_{(f\circ g^{-1})\ \delta}\ (g(b))\cap \overset{\circ}{(g(B))}_{\tau'}$$
and
$$\left|\left(f\circ g^{-1}\right)(g(x_i)) - \left(f\circ g^{-1}\right)(g(x_{i-1}))\right| < \varepsilon;$$
$$y = g(x_0), g(x_1), \ldots, g(x_{n-1}),\quad g(x_n) = g(z)$$
is function−$f\circ g^{-1}-\varepsilon$−chain between $y\in V_{(f\circ g^{-1})\delta}(g(a))\cap \overset{\circ}{(g(A))}_{\tau'}$ and $g(z)\in V_{(f\circ g^{-1})\delta}(g(b))\cap \overset{\circ}{(g(B))}_{\tau'}$, or
$$\left\langle V_{(f\circ g^{-1})\ \delta}\ (g(a))\cap \overset{\circ}{(g(A))}{}_{\tau'}, V_{(f\circ g^{-1})\ \delta}\ (g(b))\cap \overset{\circ}{(g(B))}_{\tau'}\right\rangle.$$

Also $V_{(f\circ g^{-1})\ \delta}(g(a))\cap (g(A))^{\circ}_{\tau'}\subset V_{(f\circ g^{-1})\ \varepsilon}(g(a))$ and

$$V_{(f\circ g^{-1})\delta}(g(b))\cap (g(B))^{\circ}{}_{\tau'}\subset V_{(f\circ g^{-1})\ \varepsilon}(g(b)).$$

Hence $\langle g(A), g(B)\rangle$ is locally function $-f\circ g^{-1}-$chainable.

Theorem 5. *Let A, B, C, D be open subsets of (X,τ) such that $\langle A, B\rangle$ and $\langle C, D\rangle$ are locally function$-f-$chainable at (a,b) Then $\langle A\cup C\ \ B\cup C\rangle$ is locally function$-f-$chainable at (a,b).*

Proof. Let $\varepsilon > 0$. Now $a\in A\cap C$ and $b\in B\cap D$ Then there exists $\delta_1 > 0$, $\delta_2 > 0$ such that $\langle V_{f\delta_1}(a)\cap A,\ V_{f\delta_1}(b)\cap B\rangle$ and $\langle V_{f\delta_2}(a)\cap C, V_{f\delta_2}(b)\cap D\rangle$ are function$-f-\varepsilon-$ chainable. Moreover each of these sets is non-empty and

$$V_{f\delta_1}(a)\cap A\subset\ V_{f\varepsilon}(a);\quad V_{f\delta_2}(a)\cap C\subset V_{f\varepsilon}(a)$$

and

$$V_{f\delta_1}(b)\cap B\subset\ V_{f\varepsilon}(b);\quad V_{f\delta_2}(b)\cap D\subset V_{f\varepsilon}(b).$$

Choose $\delta=\min(\delta_1,\delta_2)$. Then any point $x\in\ V_{f\delta}(a)\cap(A\cup C)$ is function$-f-\varepsilon-$chainable to either a point y of $V_{f\delta_1}(b)\cap B$ or to a point z of $V_{f\delta_2}(b)\cap D$. Since both y and z are in $V_{f\varepsilon}(b)$, it follows that x is function$-f-\varepsilon-$chainable to $b\in V_{f\delta}(b)\cap(B\cup D)$.

Likewise any point of $V_{f\delta}(b)\cap(B\cup D)$ is function$-f-\varepsilon-$chainable to $a\in V_{f\delta}(a)\cap(A\cup C)$.

Hence $\langle V_{f\delta}(a)\cap(A\cup C),\ V_{f\delta}(b)\cap(B\cup D)\rangle$ is function$-f-\varepsilon-$chainable. That the sets $V_{f\delta}(a)\cap(A\cup C)$ and $V_{f\delta}(b)\cap(B\cup D)$ are all non-empty and are respectively contained in $V_{f\varepsilon}(a)$ and $V_{f\varepsilon}(b)$ are obvious conclusions.

Theorem 6. *Let $A,\ B\subset X$ and let $C,\ D$ be subsets of X such that $C\subset A,\ D\subset B$. Let $\langle A, B\rangle$ be locally function$-f-$chainable at (a,b) where $a\in C^{\circ}_{\tau_f};\ b\in D^{\circ}_{\tau_f}$. Then $\langle C, D\rangle$ is locally function$-f-$chainable at (a,b).*

Proof. Let $\varepsilon > 0$. Now there exists $0 < \varepsilon_1 < \varepsilon$ such that $V_{f\varepsilon_1}(a)\subset C\subset A$ and $V_{f\varepsilon_1}(a)\subset D\subset B$ Again there exists a $\delta > 0$ such that $\left\langle V_{f\delta}(a)\cap A^{\circ}_{\tau},\right.$ $\left.V_{f\delta}(b)\cap B^{\circ}_{\tau}\right\rangle$ is function$-f-\varepsilon_1-$chainable where $V_{f\delta}(a)\cap A^{\circ}_{\tau}\subset V_{f\varepsilon_1}(a)\subset V_{f\varepsilon}(a)$ and $V_{f\delta}(b)\cap B^{\circ}_{\tau}\subset V_{f\varepsilon_1}(b)\subset\ V_{f\varepsilon}(b)$. Now $C^{\circ}_{\tau_f}\subset C^{\circ}_{\tau}\subset A^{\circ}_{\tau}$ and $D^{\circ}_{\tau_f}\subset D^{\circ}_{\tau}\subset B^{\circ}_{\tau}$ and as $V_{f\delta}(a)\cap C^{\circ}_{\tau_f}\neq\emptyset$, $V_{f\delta}(b)\cap D^{\circ}_{\tau_f}\neq\emptyset$ it follows that every point of $V_{f\delta}(a)\cap C^{\circ}_{\tau_f}$ is function$-f-\varepsilon_1-$chainable or a priori function$-f-\varepsilon-$chainable to $b\in V_{f\delta}(b)\cap D^{\circ}_{\tau_f}$ and every point of $V_{f\delta}(b)\cap D^{\circ}_{\tau_f}$ is function$-f-\varepsilon-$chainable to $a\in\ V_{f\delta}(a)\cap C^{\circ}_{\tau_f}$.

Hence $\left\langle V_{f\delta}(a) \cap C^{\circ}_{\tau_f},\ \ V_{f\delta}(b) \cap D^{\circ}_{\tau_f} \right\rangle$ is function$-f-\varepsilon-$chainable. Clearly $V_{f\delta}(a) \cap C^{\circ}_{\tau_f} \subset V_{f\varepsilon}(a)$ and $V_{f\delta}(b) \cap D^{\circ}_{\tau_f} \subset V_{f\varepsilon}(b)$.

Therefore $\langle C, D\rangle$ is locally function$-f-$chainable at (a,b).

Corollary 7. *Let $A, B \subset X$ and $a \in A^{\circ}_{\tau_f}$, $b \in B^{\circ}_{\tau_f}$. If $\langle \overline{A}, \overline{B}\rangle$ is locally function$-f-$chainable at (a,b) then $\langle A, B\rangle$ is locally function$-f-$chainable at (a,b).*

Theorem 8. *Let $A,\ B, C,\ D$ be subsets of X and let A, B be function$-f-$open sets. If $\langle A, B\rangle$ is locally function$-f-$chainable at (a,b) where $a \in C^{\circ}_{\tau_f}, b \in D^{\circ}_{\tau_f}$. Then $\langle A\cap C, B\cap D\rangle$ is locally function$-f-$chainable at (a,b).*

Proof. Now $a \in (A\cap C\ )^{\circ}_{\tau_f}$; $b \in (B\cap D\ )^{\circ}_{\tau_f}$. Hence by the theorem 4, $\langle A\cap C, B\cap D\rangle$ is locally function$-f-$chainable at (a,b).

Theorem 9. *Let (X,τ) be a topological space and $A,\ B \subset X$. Let $\langle A, B\rangle$ be locally function$-f-$chainable. Then for each $\varepsilon > 0$, there exists open sets C and D; $C \subset\ V_{f\varepsilon}(A),\ D \subset V_{f\varepsilon}(B)$ such that $\langle C, D\rangle$ is function$-f-\varepsilon-$chainable.*

Proof. Let $\varepsilon > 0$ and $(a,b) \in A\times B$ be arbitrary. Then there exists a $\delta > 0$ such that $\langle V_{f\delta}(a) \cap A^{\circ}_{\tau},\ V_{f\delta}(b) \cap B^{\circ}_{\tau}\rangle$ is function$-f-\varepsilon-$chainable and $V_{f\delta}(a) \cap A^{\circ}_{\tau} \subset V_{f\varepsilon}(a)$ and $V_{f\delta}(b) \cap B^{\circ}_{\tau} \subset V_{f\varepsilon}(b)$. Let

$$C = \bigcup_{\substack{a\in A\\ \delta>0}} V_{f\delta}(a) \cap\ A^{\circ}_{\tau} \text{ and } D = \bigcup_{\substack{b\in B\\ \delta>0}} V_{f\delta}(b) \cap\ B^{\circ}_{\tau}.$$

Then, $C,\ D$ are open sets of X, $\langle C, D\rangle$ is function$-f-\varepsilon-$chainable and

$$C \subset V_{f\varepsilon}(A), \quad D \subset V_{f\varepsilon}(B).$$

Definition 3. Let X be a topological space and $f : X \to [0,\infty)$ be a non-constant continuous map. For $\varepsilon > 0$ and for each subset A of X, $C_{f\varepsilon}(A)$ is defined to be the set of all points of X which can be joined to points of A by a function$-f-\varepsilon-$chain in X. Equivalently:

$$C_{f\varepsilon}(A) = \cup\{U_n(A) : n \in N\},$$

where $U_0(A) = A$, $U_1(A) = \{x \in X / |f(x) - f(A)| < \varepsilon\}$ and inductively $U_{n+1}(A) = U_1(U_n(A))$. Let $C_f(A) = \cap\{C_{f\varepsilon}(A) : \varepsilon > 0\}$.

Theorem 10. *Let A, B be open subsets of X such that $\langle A, B\rangle$ is locally function$-f-$chainable then:*

1. $C_f(A) = C_f(B)$,
2. $A, B, C_f(A)$ *or* $C_f(B)$ *are self function$-f-$chainable,*
3. $\langle A, B\rangle$ *is strongly function$-f-$chainable,*
4. $C_f(A)$ *or* $C_f(B)$ *are open sets of X whenever A or B are function$-f-$ open sets of X.*

Proof.

1. Let $x \in C_f(A)$, *then* $x \in C_{f\varepsilon}(A)$ for every $\varepsilon > 0$. Or for each $\varepsilon > 0$ there exists a point $a \in A$ such that a is function$-f-\varepsilon-$chainable to x. Let $b \in B$. Then for $\varepsilon > 0$, there is a $\delta > 0$ such that $\langle V_{f\delta}(a) \cap A$, $V_{f\delta}(b) \cap B\rangle$ is function$-f-\varepsilon-$chainable and $V_{f\delta}(a) \cap A \subset V_{f\varepsilon}(a)$; $V_{f\delta}(b) \cap B \subset V_{f\varepsilon}(b)$. Now a is function$-f-\varepsilon-$chainable to some $b_1 \in V_{f\varepsilon}(b) \cap B$ and hence to $b \in B$ or x is function$-f-\varepsilon-$chainable to $b \in B$ for every $\varepsilon > 0$ or $x \in C_f(B)$ or $C_f(A) \subset C_f(B)$. Likewise $C_f(B) \subset C_f(A)$ Hence $C_f(A) = C_f(B)$.

2. Let $x, y \in A$. Then $x,\ y \in C_f(A)$ by (1) both x and y are function$-f-\varepsilon-$chainable to some $b \in B$ for every $\varepsilon > 0$.

   Hence A is self function$-f-$chainable.

   Similarly B is self function$-f-$chainable. Again

$$C_f(A) = \bigcap_{\varepsilon>0} C_{f\varepsilon}(A),$$

   where $C_{f\varepsilon}(A)$ is the set of all points which can be joined to points of A by a function$-f-\varepsilon-$chain. Since A is self function$-f-$ chainable $C_f(A)$ is self function$-f-$chainable. Similarly $C_f(B)$ is self function$-f-$chainable.

3. Now $A \subset C_f(A) = C_f(B)$. By (1) any $a \in A$ is function$-f-\varepsilon-$chainable to every $b \in B$ for every $\varepsilon > 0$. Hence $\langle A, B\rangle$ is strongly function$-f-$chainable.

4. Let $x \in C_f\ (A)$ Then by (1) x is function$-f-\varepsilon-$chainable to every $b \in B$ for every $\varepsilon > 0$. Now choose $0 < \varepsilon_1 < \varepsilon$ such that $V_{f\varepsilon_1}(b) \subset B$. Let $y \in V_{f\varepsilon_1}(x)$ or $|f(y) - f(x)| < \varepsilon_1 < \varepsilon$. Hence y is function$-f-$ $\varepsilon-$chainable to b. Since $\varepsilon > 0$ is arbitrary, $y \in C_f(B) = C_f(A)$ or $V_{f\varepsilon_1}(x) \subset C_f(B) = C_f(A)$ is open in X. Similarly $C_f(B)$ is open in X.

Theorem 11. *Let $A, B \subset X$ be open sets such that $\langle A, B\rangle$ is strongly function$-f-$chainable Then $\langle A, B\rangle$ is locally function$-f-$chainable.*

Proof. Let $(a, b) \in A \times B$ and $\varepsilon > 0$. Consider $x \in V_{f\varepsilon}(a) \cap A$ .Then x is function$-f-\varepsilon-$chainable to point $b \in B$ and hence of $V_{f\varepsilon}(b) \cap B$ or $\langle V_{f\varepsilon}(a) \cap A, V_{f\varepsilon}(b) \cap B\rangle$ is function$-f-\varepsilon-$chainable. Since $\varepsilon > 0$ is arbitrary $\langle A, B\rangle$ is locally function$-f-$chainable.

Theorem 12. *Let A, B be self function$-f-$chainable and function$-f-$open subsets of X such that $C_f(A) = C_f(B)$. Then $\langle A, B\rangle$ is locally function$-f-$chainable.*

Proof. Let $(a, b) \in A \times B$ and $\varepsilon > 0$. Then for some $0 < \delta < \varepsilon$, $V_{f\varepsilon}(a) \subset A$ and $V_{f\varepsilon}(b) \subset B$. Consider $x \in V_{f\varepsilon}(a) \cap A = V_{f\delta}(a)$ Since $C_f(A) = C_f(B)$; x is function$-f-\varepsilon-$chainable to some point $b_1 \in B$ and hence to b or $\langle V_{f\delta}(a) \cap A, V_{f\delta}(b) \cap B\rangle$ is function$-f-$chainable or $\langle A, B\rangle$ is locally function $-f-$chainable.

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