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ON THE SEGRE UPPER BOUND OF THE REGULARITY FOR FAT POINTS IN $\mathbb{P}^4$, I

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Abstract: We study the generalized Segre bound in $\mathbb{P}^4$ for fat points schemes. In this first part we only prove the initial case, i.e. with respect to homogeneous degree 3 polynomials.

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1. Introduction

Fix a projective space $\mathbb{P}^n$, $n \ge 2$, defined over an algebraically closed field $\mathbb{K}$. Fix a finite set $S \subset \mathbb{P}^n$. For each $p \in S$ fix an integer $m_p > 0$ and set $Z := \cup_{p\in S} m_p p$. Set $s := \sharp(S)$ and let $m_1 \ge \cdots \ge m_s$ denote the multiplicities m_p, $p \in S$, listed in non-increasing order. For each linear space $L \subseteq \mathbb{P}^n$ we write $w(L) := \sum_{p\in S\cap L} m_p$. The generalized Segre conjecture introduced in [7] and [5] says that $h^1(\mathcal{I}_Z(d)) > 0$ if $m_1 \le d$ and $w(L) \le \dim(L)d+1$ for all linear subspaces $L \subseteq \mathbb{P}^n$. It is known to be true if S is in linearly general position ([4, Theorem 1.4]), if $n = 2$ ([3], [6]), if $n = 3$ ([5], [7]), if $s = n+2$ and S spans $\mathbb{P}^n$) and if $s = n+3$ and S spans $\mathbb{P}^n$ ([9] in the quasi-equimultiple case, [1] in the

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general case, but only in characteristic zero). In this paper we prove it when $n = 4$ and $d \leq 3$ and introduced some tools useful for the case $n = 4$, $d > 3$, on which we plan to go soon. We assume characteristic zero, because we use a case in [1] ($n = 4$, $d = 3$, $s = 7$, $m_1 = \cdots = m_6 = 2$, $m_7 = 1$, done in [1] in characteristic zero).

From now on we assume $n = 4$.

Proposition 1. *The generalized Segre conjecture is true if* $n = 4$ *and* $d \leq 3$.

See Lemma 2 for the case $d = 2$ and Proposition 2 for the case $d = 3$.

2. The Proofs and Preliminary Lemmas for Parts II and III

We fix an integer $d > 0$ and assume that the generalized Segre conjecture is true in $\mathbb{P}^4$ for all positive integers $< d$. We fix $Z = \sum_{p \in S} m_p p$ and assume that Z satisfies the assumptions of the generalized Segre conjecture in degree d, i.e. we assume $d \geq m_1$, $w(\mathbb{P}^4) \leq 4d + 1$ and $w(E) \leq id + i - 2$ for $i = 1, 2, 3$ and each i-dimensional linear subspace $E \subset \mathbb{P}^4$. We use induction on the integer $w(\mathbb{P}^4) = \sum_{p \in S} m_p$. In particular we assume $h^1(\mathcal{I}_W(d)) = 0$ if $W = \sum_{p \in S'} m_p p$, where $S' \subsetneq S$. For each $p \in \mathbb{P}^4$ let m_p denote the multiplicity of p in Z. Hence $m_p = 0$ if and only if $p \notin S$.

Notation 1. For any set $A \subseteq \mathbb{P}^4$ set $w(A) := \sum_{P \in S \cap A} m_P$. Write $w_Z(A) := w(A)$. If $U \subset \mathbb{P}^4$ is any finite subset and for each $P \in U$ we take a non-negative integer n_P, then set $w_W(A) := \sum_{P \in U \cap A} n_P$, where $W := \sum_{P \in U} n_P P$. Let B_1 (resp. B_1') be the set of all lines $L \subset \mathbb{P}^4$ such that $w(L) = d + 1$ (resp. $w(L) = d$). Let B_2 (resp. B_2') be the set of all planes $L \subset \mathbb{P}^4$ such that $2d \leq w(L) \leq 2d + 1$ (resp. $2d - 2 \leq w(L) \leq 2d - 1$). Let B_3 (resp. B_3') be the set of all hyperplanes $L \subset \mathbb{P}^4$ such that $3d - 1 \leq w(L) \leq 2d + 1$ (resp. $2d - 4 \leq w(L) \leq 3d - 2$).

Remark 1. Fix an integer $t > 0$. Let $A \subset \mathbb{P}^4$ be a zero-dimensional scheme such that $h^1(\mathcal{I}_A(t)) = 0$. Let B be a subscheme of A. Since the restriction map $H^0(\mathcal{O}_A(t)) \to H^0(\mathcal{O}_B(t))$ is surjective we have $h^1(\mathcal{I}_B(t)) = 0$. In particular for every hypersurface $T \subset \mathbb{P}^4$ we have $h^1(\mathcal{I}_{A \cap T}(t)) = 0$. Hence $h^1(T, \mathcal{I}_{T \cap A}(t)) = 0$.

For any hyperplane $H \subset \mathbb{P}^4$ the scheme $Z \cap H$ is a fat point scheme in $H =$

$\mathbb{P}^3$ and it satisfies the Segre conditions in $H = \mathbb{P}^3$. Hence $h^1(H, \mathcal{I}_{Z\cap H,H}(d)) = 0$ ([5], [7]). For any hypersurface $T \subset \mathbb{P}^4$ and any scheme $A \subset \mathbb{P}^4$ the residual scheme $\mathrm{Res}_T(A)$ of A with respect to T is the closed subscheme of $\mathbb{P}^4$ with $\mathcal{I}_A : \mathcal{I}_T$ as its ideal scheme. Set $a := \deg(T)$. We have an exact sequence (called the residual exact sequence of T):

$$0 \to \mathcal{I}_{\mathrm{Res}_T(A)}(t-a) \to \mathcal{I}_A(t) \to \mathcal{I}_{T\cap A,T}(t) \to 0 \qquad (1)$$

Remark 2. By (1) to prove that $h^1(\mathcal{I}_Z(d)) = 0$ it is sufficient to find T such that $h^1(T, \mathcal{I}_{T\cap Z,T}(d)) = 0$ and $h^1(\mathcal{I}_{\mathrm{Res}_T(Z)}(t-a)) = 0$. We will use the cases $a = 1, 2$. In the case of hyperplanes we always have $h^1(T, \mathcal{I}_{Z\cap T,T}(d)) = 0$. Now assume $\deg(T) > 1$. We have $h^1(T, \mathcal{I}_{Z\cap T,T}(d)) = 0$ if $T \cap S \neq S$, because in this case $Z \cap T = E \cap T$, where $E := \sum_{p\in S\cap T} m_p p$, and $h^1(\mathcal{I}_E(d)) = 0$, because $w(E) < w(Z)$ and E satisfies the Segre conditions in degree d; we also use Remark 1. We have $w_{\mathrm{Res}_T(Z)}(A) = w(A) - \sharp(A \cap T \cap S)$.

For any set $E \subseteq \mathbb{P}^4$ let $\langle E\rangle$ denote the linear span of E, i.e. the minimal linear subspace of $\mathbb{P}^4$ containing E.

Lemma 1. *If $m_1 = d$, then $h^1(\mathcal{I}_Z(d)) = 0$.*

Proof. Since Z satisfies the Segre conditions in degree d for lines, we have $m_i = 1$ for all $i > 1$, there is a unique $p \in S$ with $m_p = d$ and $\sharp(S \cap L) \leq 2$ for each line L through p. Set $B := S \setminus \{p\}$. Let $\ell : \mathbb{P}^4 \setminus \{p\} \to \mathbb{P}^3$ be the linear projection from p. Set $E := \ell(B)$. For each linear space $A \subseteq \mathbb{P}^3$ set $A_1 := \ell^{-1}(A) \cup \{p\}$. A_1 is a linear space and $\dim(A_1) = \dim(A) + 1$. Since $\sharp(S \cap L) \leq 2$ for each line L through p, $\ell|B$ is injective. Hence for any linear space $A \subseteq \mathbb{P}^4$ we have $w(A_1) = d + w_E(A)$. If A has dimension c, then $w_E(A) = w(A_1) - m \leq cd + 1$ and so E satisfies the Segre conditions in degree d. Hence $h^1(\mathbb{P}^3, \mathcal{I}_E(d) = 0$, i.e. $h^0(\mathbb{P}^3, \mathcal{I}_E(d)) = \binom{d+3}{3} - \sharp(S) + 1$. Since $|\mathcal{I}_{mP}(d)|$ is the set of all degree m cones with vertex containing p, we get $h^0(\mathbb{P}^4, \mathcal{I}_Z(d)) = h^0(\mathbb{P}^3, \mathcal{I}_E(d)) = \binom{d+3}{3} - \sharp(S) + 1$, i.e. $h^1(\mathcal{I}_Z(d)) = 0$. □

By Lemma 1 we may assume $m_1 \leq d - 1$.

Remark 3. If $A \in B_i \cup B_i'$, then $A \cap S$ spans A (use the Segre conditions for the proper linear subspaces of A, for B_1' we also need the assumption $m_1 \leq d - 1$).

Remark 4. Tale $L, R \in B_1$ such that $L \cap R \neq \emptyset$. Since $w(E) \leq 2d+1$ and $w(L) = w(R) = d+1$, we have $L \cap R \cap S \neq \emptyset$.

Remark 5. Assume the existence of $L_i \in B_1$, $i = 1, 2, 3$, such that $L_i \cap L_j = \emptyset$ for all $i \neq j$. Let $L \subset \mathbb{P}^4$ be a line with $w(L) \geq d-1$. Since $w(\mathbb{P}^4) \leq 4d+1$, then $S \cap L \cap (L_1 \cup L_2 \cup L_3) \neq \emptyset$.

Lemma 2. *If $d \leq 2$, then $h^1(\mathcal{I}_Z(d)) = 0$.*

Proof. The cases $d = 1$ and $d = 2$, $m_1 = 2$, are covered by Lemma 1. Assume $d = 2$ and $m_1 = 1$, i.e. $Z = S$. The Segre conditions means that S is in linearly general position. Apply [4, Theorem 1.4]. □

Lemma 3. *If there is a hyperplane $H \subset \mathbb{P}^4$ such that $\sharp(S \cap H) \geq \sharp(S) - 1$, then $h^1(\mathcal{I}_Z(d)) = 0$.*

Proof. Since the case $\sharp(S) \leq 3$ is trivial, we may assume $\sharp(S) \geq 4$. Since $Z \cap H$ satisfies the Segre conditions in degree d in $H = \mathbb{P}^3$, we have $h^1(H, \mathcal{I}_{Z \cap H}(d)) = 0$ ([5], [7]). By the residual exact sequence (1) of H and the inductive assumption it is sufficient to check that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree $d-1$. For any linear space $A \subseteq \mathbb{P}^4$ we have $w_{\mathrm{Res}_H(Z)}(A) = w(A) - \sharp(A \cap H \cap S)$. Since Z satisfies the Segre conditions in degree d and $w_{\mathrm{Res}_H(Z)}(A) \leq w(A)$, it is sufficient to test $\mathbb{P}^4$ and the elements of B_c, $c = 1, 2, 3$. If $A \in B_c$ use that $A \cap S$ spans A (Remark 3) and that $\sharp(A \cap H \cap S) \geq \sharp(A \cap S) - 1 \geq c$. If $A = \mathbb{P}^4$ use that either S spans $\mathbb{P}^4$ (and hence $\sharp(S) \geq 5$) or we may take $H \supset S$ (changing if necessary H) and hence $w_{\mathrm{Res}_H(Z)(A)}(\mathbb{P}^4) \leq 4d + 1 - \sharp(S)$. □

Lemma 4. *Let $E, A \subset \mathbb{P}^4$ be planes such that $E \cap A$ is a single point, o. Then $h^1(\mathcal{I}_{A \cup E}(t)) = 0$ for all $t \geq 2$ and $\mathcal{I}_{A \cup E}(2)$ is spanned.*

Proof. Fix lines L, D of $\mathbb{P}^3$ such that $L \cap D = \emptyset$. We have $h^1(\mathcal{I}_{L \cup D}(2)) = 0$, $h^0(\mathcal{I}_{A \cup D}(2)) = 4$ and $\mathcal{I}_{L \cup D}(2)$ is spanned. Since $|\mathcal{I}_{A \cup D}(2)|$ is the linear system of all quadric cones with vertex containing o, we get $h^1(\mathcal{I}_{A \cup D}(2)) = 0$ and that $\mathcal{I}_{A \cup E}(2)$ is spanned outside o. Take affine coordinates x_1, x_2, x_3, x_4 centered at o such that the affine part of A (resp. E) has equations $x_1 = x_2 = 0$ (resp. $x_3 = x_4 = 0$). Since the quadratic equations $x_i x_j = 0$, $i = 1, 2$, $j = 3, 4$, defines $A \cup E$ scheme-theoretically in a neighborhood of o, we get that $\mathcal{I}_{A \cup E}(2)$ is spanned at o. The case $t > 2$ follows from Castelnuovo-Mumford's lemma. □

Proposition 2. *If $d = 3$, then $h^1(\mathcal{I}_Z(3)) = 0$.*

Proof. By Lemma 1 we may assume $m_1 \le 2$. Let h be the cardinality of the set of all $p \in S$ with $m_p = 2$. Since $w(\mathbb{P}^4) \le 13$, we have $0 \le h \le 6$, $s \ge h$, and $s + h \le 13$. By Lemma 3 we may assume $s \ge 5$. Set $S_2 := \{p \in S; m_p = 2\}$ and $S_1 := S \setminus S_2$. The Segre conditions imply $\sharp(A \cap S_2) \le \dim(A) + 1$ if A is either a line or a plane and $\dim(A \cap S_2) \le 5$ if A is a hyperplane.

(a) Assume $s = h$, i.e. $S_2 = S$. If S is in linearly general position, then we may use [4, Theorem 1.4]. If S is not in linearly general position, then there is a hyperplane $H \subset \mathbb{P}^4$ such $\sharp(S) \setminus S \cap H \le 1$. Apply Lemma 3. Alternatively, apply either [2] or [8].

(b) Assume $s > h$.

(b1) Assume $h \le 1$. If $B_1 \cup B_2 \cup B_3 = \emptyset$, we take any hyperplane H with $\sharp(H \cap S) \ge 4$ and see that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 2 and hence $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(2)) = 0$ (Lemma 2). Since $h \le 1$, for all $B \in B_1$ we have $\sharp(B \cap S) \ge 3$ for all $B \in B_1$ and $\sharp(S \cap B) = 3$ if and only if $h = 1$ and $S_2 \subset B$.

(b1.1) Assume the existence of $M \in B_3$ and $L \in B_1$ such that $M \cap L \cap S = \emptyset$. Since $w(\mathbb{P}^4) \le 13$ and $w(L) = 4$, we have $8 \le w(M) \le 9$. Until step (b1.1.5) we assume $S \subset M \cup L$. Since $\sharp(B \cap S) \ge 3$ for all $B \in B_1$, each element of $B_1 \setminus \{L\}$ is contained in M.

(b1.1.1) Assume the existence of a plane $E \subset M$ with $w(E) = 7$. We have $\sharp(E \cap S) \ge 5$ and equality holds only if $h = 1$ and E contains the point with maximal multiplicity. Fix $q \in L \cap S$ with maximal multiplicity and set $H := \langle E \cup \{q\} \rangle$. We check that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 2. We have $w_{\mathrm{Res}_H(Z)}(\mathbb{P}^4) \le 13 - \sharp(S \cap L \cap H) - \sharp(E \cap S)$ and hence to check the Segre condition in degree 2 for $\mathbb{P}^4$ and H with respect to $\mathrm{Res}_H(Z)$ it is sufficient to use that $\sharp(E \cap S) \ge 3$. M satisfies the Segre condition in degree 2 for $\mathrm{Res}_H(Z)$, because $w(M) \le 9$ and $\sharp(M \cap H \cap S) = \sharp(E \cap S) \ge 3$. Since $L \cap H \cap S \ne \emptyset$, L satisfies the Segre condition in degree 2 for $\mathrm{Res}_H(Z)$. Take $R \in B_1$ with $R \ne L$. Then $R \subset M$, because $\sharp(B \cap S) \ge 3$ and hence R contains two points of M. Since $w(M) \le 9$, we have $w(E) + w(R) > w(M)$ and hence $E \cap R \cap S \ne \emptyset$. Thus R satisfies the Segre condition in degree 2. Fix $A \in B_2$ with $A \supset L$. Since $S \cap H \cap L \ne \emptyset$, to check the Segre condition for A we may assume $w(A) = 7$; in this case it is sufficient to check that $A \cap E \cap S \ne \emptyset$; we have $w(A \setminus L) = 3$; since $w(M \setminus E) \le 2$, we get $A \cap E \cap S \ne \emptyset$. Fix $A \in B_2$ with $A \subset M$. Since $w(A) + w(E) > w(M)$, we get $E \cap A \cap S \ne \emptyset$; to check the Segre condition for A we may assume $w(A) = 7$ and in this case we have

$\sharp(A\cap E\cap S)\ge 2$, because $m_1\le 2$ and $w(A)+w(E)-w(M)\ge 4$. Fix $A\in B_2$ such that $A\not\supseteq L$ and $A\not\subseteq M$. Hence $A\cap M$ is a line. Since $m_1\le 2$ and $w(A\cap M)\le 4\le w(A)-2$, we get $w(A)=6$, $w(A\cap M)=4$ and that $A\cap L$ is a point of S_2; since $w(E)=7\ge w(M)-2$, we have $E\cap A\cap S\ne\emptyset$ and hence A satisfies the Segre condition in degree 2.

Since $q\in L$, L satisfies the Segre condition for $\mathrm{Res}_H(Z)$ in degree 2. Fix $D\in B_1\setminus\{L\}$. We saw that $D\subset M$. Since $w(M\setminus D)>w(E)$, we have $D\cap E\cap S\ne\emptyset$, concluding the proof that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 2.

(b1.1.2) Assume the existence of $R,D\in B_1$ such that $R\cap D=\emptyset$ and $R\cup D\subset M$. Since $w(M)\le 9$, either $S\cap M\subset R\cup D$ or $S\cap M\setminus S\cap(R\cup D)$ is a point e with multiplicity 1 in Z; in the former case we write $e:=\emptyset$ and $m_e=0$.

(b1.1.2.1) Assume that either $e=\emptyset$ or $h=0$ or $h=1$ and the point of S with multiplicity 1 is contained in L. Fix $p,q\in L\cap S$ with $p\ne q$; if $h=1$ and $S_2\subset L$, then assume that p is the point with multiplicity 2. Set $Q:=\langle D\cup\{p\}\rangle\cup\langle R\cup\{q\}\rangle$. Since $\sharp(S\cap L)\ge 3$, we have $Q\cap S\ne S$. Therefore the inductive assumption gives $h^1(Q,\mathcal{I}_{Z\cap Q}(3))=0$. Our assumptions give $\sharp(Q\cap S)\ge 10$ (case $m_e=1$) and $\sharp(Q\cap S)\ge 9$ (case $m_e=0$) and hence $w_{\mathrm{Res}_Q(Z)}(\mathbb{P}^4)\le 3$. Since $\mathrm{Res}_Q(Z)$ is reduced and not formed by 3 collinear points (since $M\cap L\cap S=\emptyset$) we get $h^1(\mathcal{I}_{\mathrm{Res}_Q(Z)}(1))=0$.

(b1.1.2.2) Assume $e\ne\emptyset$, $h=1$ and that the point $o\in S$ with multiplicity 2 is not contained in L, say it is contained in D. We fix another point $o'\in D\cap S$ with $o\ne o'$ and take a general $Q'\in|\mathcal{I}_{\{o,o'\}\cup R\cup L}(2)|$. We use that $e\notin D$ and hence $\mathrm{Res}_{Q'}(Z)$ is formed by 3 non-collinear points.

(b1.1.3) Assume that M contains at least one element of B_1, say R, and that any two elements of B_1 contained in M meets. Set $H:=\langle L\cup R\rangle$. It is sufficient to prove that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 2. Since $\sharp(B\cap S)\ge 3$ for all $B\in B_1$, the Segre condition is satisfied by $\mathbb{P}^4$, M, all lines, and all linear spaces containing either L or R.

Fix $A\in B_2$ such that $A\not\supseteq L$ and $A\not\supseteq L$. First assume $A\subset M$. By step (b1.1.1) we may assume $w(A)=6$ and hence to check the Segre condition for $\mathrm{Res}_H(Z)$ it is sufficient to check that $A\cap H\cap S\ne\emptyset$; this is true, because $w(A)+w(R)>9\ge w(M)$. Now assume $A\not\subseteq M$. Since $S\subset M\cup L$, we have $w(A)=w(A\cap M)+w(A\cap L)\le w(A\cap M)+2$; hence $w(A\cap M)=4$ and $w(A)=6$; hence $A\cap M\in B_1$; hence $A\cap R\ne\emptyset$; Remark 4 gives $A\cap R\cap S\ne\emptyset$ and hence $w_{\mathrm{Res}_H(Z)}(A)\le 5$.

Fix $U\in B_3$ such that $U\ne M$ and $U\not\supseteq L$. Since $S\subset M\cup L$ and $w(U\cap L)\le 2$, we get $w(U\cap M)\ge 6>w(M)-4$. Hence $U\cap M\cap S\ne\emptyset$. Therefore to check the Segre condition for U we may assume $w(U) = 10$ and hence $w(U \cap M) \geq 8$, contradicting the Segre condition for planes of Z.

(b1.1.4) Assume $B_1 = \{L\}$. Let H be a hyperplane containing L and at least two points p, q of $M \cap S$. $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 2 with respect to $\mathbb{P}^4$ and all lines. It satisfies the Segre condition with respect to M, because $w(M) \leq 9$ and $\sharp(M \cap H \cap S) \geq 2$. Fix $U \in B_3$ with $U \neq M$. If $U \supset L$, then we get $\sharp(U \cap H \cap S) \geq 3$. Now assume $U \not\supseteq L$ and call u the point $U \cap L$. Since $7 \geq w(U \cap M) = w(U) - m_u$, we get $m_u > 0$ and hence $S \cap U \cap L \neq \emptyset$; to check the Segre condition for U we may assume $w(U) \geq 9$ and hence we may assume $m_u = 2$ and $w(U \cap M) = 7$; step (b1.1.1) gives $h^1(\mathcal{I}_Z(3)) = 0$.

If $B_2 = \emptyset$, then $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 2 for planes. If there is $A \in B_2$ with $A \supset L$, then $\sharp(A \cap H \cap S) \geq 3$. Now assume the existence of $A \in B_2$ not contained in M. Since $w(A \cap M) \leq 3$ and $m_1 \leq 2$, we have $A \supset L$ and hence $\sharp(A \cap H \cap S) \geq 2$ and hence $\mathrm{Res}_H(Z)$ satisfies the Segre conditions for A.

Now assume the existence of $E \in B_2$ with $E \subset M$. By step (b1.1.1) we may assume $w(E) = 6$. To get the Segre condition for $\mathrm{Res}_H(Z)$ it is sufficient to take p or q in E. We may do that for all $E \in B_2$ with $E \subset M$ if they are at most two. Fix $F, G \in B_2$ with $w(F) = w(G) = 6$, $F \cup G \subset M$. and E, F, G distinct. Since $w(M) \leq 9$ and $B_1 = \{L\}$, we have $w(M) = 9$, $w(E \cap F) = 3$, $E \cap F \cap S$ spans $E \cap F$ (Remark 3) and $S \cap M = (S \cap E) \cup (S \cap F)$ (and the same is true with F, G). Hence $S \cap G = (S \cap G \cap E) \cup (S \cap G \cap F)$. Since $h \leq 1$, we have $\sharp(S \cap G) \geq 5$. Since $B_1 = \{L\}$, we get $\sharp(S \cap G \cap E) \geq 2$ and $\sharp(S \cap G \cap E) \geq 2$. Since any two points of M spans a line, we get $E \cap F = E \cap G = F \cap G$. Hence it is sufficient to take $p \in E \cap F \cap S$ to get the Segre condition for $\mathrm{Res}_H(Z)$.

(b1.1.5) Now assume $S \nsubseteq M \cup L$. Since $w(\mathbb{P}^4) \leq 13$, $w(M) \geq 8$ and $w(L) = 4$, we get that $S \setminus S \cap (L \cup M)$ is a single point, o, with $m - o = 1$. Take a hyperplane $N \supset L$ such that $o \notin N$. Since $\mathrm{Res}_{M \cup N}(Z) = \{o\} \cup S_2$ and $\sharp(S_2) = h \leq 1$, we have $h^1(\mathcal{I}_{\mathrm{Res}_{M \cup N}(Z)}(1)) = 0$. Since $o \notin M \cup N$, the inductive assumption gives $h^1(\mathcal{I}_{Z \cap (M \cup N)}(3)) = 0$. Use the residual sequence of $M \cup N$.

(b1.2) Assume $B_3 \neq \emptyset$ and $L \cap M \cap S \neq \emptyset$ for all $M \in B_3$ and all $L \in B_1$. Fix $H \in B_3$ with maximal weight among all elements of B_3. We have $8 \leq w(H) \leq 10$. By Lemma 2 it is sufficient to prove that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 2. The conditions for $\mathbb{P}^4$ and H are satisfied, because $\sharp(H \cap S) \geq 4$ (Remark 3), while the conditions for the lines are satisfied by our assumption on B_1. Fix $U \in B_3 \backslash \{H\}$. We have $w(H \cap U) \geq w(H) + w(U) - 13 > 0$. Hence $H \cap U \cap S \neq \emptyset$ and to check the Segre condition for $\mathrm{Res}_H(Z)$ and U we may assume $w(U) \geq 9$. Since $w(H) \geq w(U)$, we get $w(H \cap U) \geq 5$. Since

$h \le 1$, we get $\sharp(H \cap U) \ge 4$ and so U satisfies the Segre condition for $\mathrm{Res}_H(Z)$ in degree 2. Now take $A \in B_2$. We have $6 \le w(A) \le 7$. Since $w(A) + w(H) > 13$, we have $A \cap H \cap S \ne \emptyset$. Hence to check the Segre condition for $\mathrm{Res}_H(Z)$ and A in degree 2 we may assume $w(A) = 7$. In this case $w(A \cap H) \ge 2$. We get that A satisfies the Segre condition, unless $h = 1$, $w(H) = 8$, $w(\mathbb{P}^4) = 13$, $S \subset A \cup H$ and $A \cap H \cap S$ is the only point, o, with multiplicity 2. For each $q \in H \cap S \setminus \{o\}$ set $H_q := \langle A \cup \{q\}\rangle$. Since $w(H_q) \ge 8$, we conclude taking H_q instead of H, unless in H_q there is a plane $A' \subset H$ with $\sharp(A' \cap S_1) = 5$ and $o \in A'$. Assume for the moment $A \cap A' = \{o\}$. Let H_1 be a general plane containing A and H_2 a general plane containing A'. We have $(H_1 \cup H_2) \cap S = (A \cup A') \cup S$ and hence $w(H_1 \cup H_2) = 12$. Hence $(H_1 \cup H_2) \cap S \ne S$. The inductive assumption gives $h^1(H_1 \cup H_2, \mathcal{I}_{H_1 \cup H_2) \cap Z}(3)) = 0$. Since $\mathrm{Res}_{H_1 \cup H_2}(Z)$ is the union of o and another point, we have $h^1(\mathcal{I}_{\mathrm{Res}_{H_1 \cup H_2}(Z)}(1)) = 0$ and hence it is sufficient to apply the residual sequence of $H_1 \cup H_2$. Noe assume $A \cap A' \ne \{o\}$. Since $A \ne A'$, then $A \cap A'$ is a line. We get $w(\langle A \cup A'\rangle) \ge 10$, contradicting the assumption (in this case) that $w(U) = 8$ for all $U \in B_3$.

(b1.3) From now on we assume $B_3 = \emptyset$.

(b1.3.1) Assume the existence of $L_1, L_2, L_3 \in B_1$ such that $L_i \cap L_j = \emptyset$ for all $i \ne j$. Since $w(\mathbb{P}^4) \le 13$, either $S \subset L_1 \cup L_2 \cup L_3$ or $S \setminus (L_1 \cup L_2 \cup L_3)$ is a unique point, e, and $m_e = 1$. If $S \subset L_1 \cup L_2 \cup L_3$, then set $e = \emptyset$. In particular if $h = 1$ the only point, p, with $m_p = 2$ is contained in one of these lines, say $p \in L_1$. If $h = 0$ fix $p \in L_1$. Fix $o \in L_2 \cap S$ and set $H := \langle \{p, q\} \cup L_3\rangle$ Since $\sharp(L_3 \cap S) = 4$ and $p \in H$, the scheme $\mathrm{Res}_H(Z)$ is reduced and it has cardinality ≤ 7. Hence $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(2)) = 0$, unless either $\mathrm{Res}_H(Z)$ has 4 collinear points or it has 6 points contained in a conic or it has cardinality 7 and it is contained in a plane. The first possibility is excluded by Remark 5. The third possibility does not occur, because it would implies that it contains 3 points of L_1 and 3 points of L_2 and hence it spans the 3-dimensional linear space $\langle L_1 \cup L_2\rangle$. In the second possibility 3 of these 6 points must be in a line L_1 or L_2 and at least two in the other one, so the conic is $L_1 \cup L_2$, contradicting the assumption $L_1 \cap L_2 = \emptyset$.

(b1.3.2) Assume the existence of $L, R \in B_1$ such that $L \cap R = \emptyset$ and that each element of B_1 meets $L \cup R$. Set $H := \langle L \cup R\rangle$. Since $\sharp(S \cap (L \cup R)) \ge 7$ (with strict inequality unless $h = 1$ and the only point p_1 of multiplicity 2 is contained in $L \cup R$), we have $w_{\mathrm{Res}_H(Z)}(\mathbb{P}^4) \le 6$ (with strict inequality if either $h = 0$ or $\mathrm{Res}_H(Z)$ is reduced). By Remark 3 every element of B_1 meets $S \cap (L \cup R)$. Hence $\deg(T \cap \mathrm{Res}_H(Z)) \le 3$ for each line T.

If $h = 0$, then $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(2)) = 0$.

Now assume $h = 1$ and $p_1 \in L \cup R$. We get $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(2)) = 0$, unless

$\mathrm{Res}_H(Z)$ is formed by 6 point in a conic T. Since $p_1 \in \mathrm{Res}_H(Z)$ the plane $\langle \mathrm{Res}_H(Z)\rangle$ satisfies $w(\langle \mathrm{Res}_H(Z)\rangle) \ge 7$. Fix $u \in S \setminus S \cap \langle \mathrm{Res}_H(Z)\rangle$. We have $\langle \{u\} \cup \mathrm{Res}_H(Z)\rangle \in B_3$, a contradiction.

Now assume $h = 1$ and $p_1 \notin (L \cup R)$. The scheme $\mathrm{Res}_H(Z)$ is the union of $2p_1$ and $S' \subset S_1$ with $\sharp(S') \le 3$. Since each line $T \in B_1$ meets $L \cup R$, no 2 of the points of S' are contained in a line through p_1. Hence $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(2)) = 0$.

(b1.3.3) Assume the existence of $L \in B_1$ such that each element of B_1 meets L. Fix $p, q \in S \setminus S \cap L$ with $p \ne q$ and let H be a hyperplane containing $L \cup \{p, q\}$. $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 2 with respect to $\mathbb{P}^4$, lines and hyperplanes. Hence we may assume $B_2 \ne \emptyset$. Fix $A \in B_2$ with maximal weight. First assume $w(A) = 7$ and $A \cap L \cap S = \emptyset$. To use $\mathrm{Res}_H(Z)$ we need to take $\{p, q\} \subset A$. Assume the existence of another $E \in B_2$ with $E \cap L \cap S = \emptyset$. Since $w(\mathbb{P}^4 \setminus (A \cup L)) \le 2$, we have $w(E \cap A) \ge 4$ and hence $E \cap A \in B_1$. Hence $E \cap A \cap L \cap S \ne \emptyset$, a contradiction. Now take $E \in B_2$ with $E \cap L \cap S \ne \emptyset$. To check the Segre condition for E and $\mathrm{Res}_H(Z)$ we may assume $w(E) = 7$. Since $\sharp(L \cap S) \ge 2$, we may also assume that $E \cap L$ is a single point o. Since $w(\mathbb{P}^4 \setminus (A \cup L)) \le 2$ and $m_o \le 2$, we get $w(A \cap E) \ge 3$ and hence $A \cap E$ is a line spanned by $A \cap E \cap S$. We get $w(\langle A \cup E\rangle) \ge w(A) + w(E) - w(A \cap E)$ and hence $A \cap E \in B_1$. Hence $A \cap E \cap L \cap S \ne \emptyset$, contradicting the assumption $A \cap L \cap S = \emptyset$.

Now assume $w(A) = 7$ and $\sharp(A \cap L \cap S) = 1$, say $\{u\} = A \cap L$. To use $\mathrm{Res}_H(Z)$ we need $\{p, q\} \cap A \ne \emptyset$, say $p \in A$. Take another $E \in B_2$; we saw that we may assume that either $w(E) = 6$ or $E \cap L \cap S \ne \emptyset$. First assume $w(E) = 6$; to check that E satisfies the Segre condition we may assume $L \cap E \cap S = \emptyset$; since $w(\mathbb{P}^4 \setminus (A \cup L)) \le 2 + m_u$, then $w(A \cap E) \ge 4 - m_u$; since $h \le 1$, we get $\sharp(A \cap E \cap S) \ge 2$ and hence $A \cap E$ is a line; we have $w(\langle A \cup E\rangle) \ge w(A) + w(E) - w(A \cap E) \ge 7 + 6 - 4$, contradicting the assumption $B_3 = \emptyset$. Now assume $w(E) = 7$ and $E \cap L \cap S \ne \emptyset$; since $\sharp(L \cap S) \ge 2$, to check the Segre condition for E we may assume that $E \cap L$ is a single point o. We get $w(A \cap E) \ge 5 - m_u - m_o$; since $h \ge 1$, we get $\sharp(A \cap E \cap S) \ge 2$ and hence $A \cap E$ is a line. We get $w(\langle A \cup E\rangle) \ge 7 + 7 - 4$, contradicting the assumption $B_3 = \emptyset$.

Now assume $w(A) = 6$. In this case to check the Segre condition for A we may assume $A \cap L \cap S = \emptyset$ and it is sufficient to take $p \in A \cap S$. Assume the existence of $E \in B_2$ such that $E \ne A$. Since A has maximal weight, we have $w(E) = 6$. Since $w(\mathbb{P}^4 \setminus (A \cup L)) \le 3$, we have $w(A \cap E) \ge 4$. Hence $A \cap E \in B_1$ and so $A \cap E \cap L \cap S \ne \emptyset$ and hence $w_{\mathrm{Res}_H(Z)}(E) \le 5$.

(b1.3.4) Assume $B_1 = B_3 = \emptyset$. We may assume $B_2 \ne \emptyset$ and take $E \in B_2$ with maximal weight. Fix $p \in S \setminus S \cap E$ and set $H := \langle \{p\} \cup E\rangle$. Since

$w(H) \ge 1 + w(E)$ and $B_3 = \emptyset$, we get $w(E) = 6$ and that if $h = 1$, then E contains the only point with multiplicity 2 of Z. $\mathrm{Res}_H(Z)$ satisfies the Segre condition in degree 2 with respect to $\mathbb{P}^4$, lines, hyperplanes and E. Assume the existence of another $A \in B_2$. Since $w(A) \le w(E)$, we have $w(A) = 6$. To check the Segre condition for $\mathrm{Res}_H(Z)$ we may assume $A \cap E \cap S = \emptyset$. In this case we are forced to take $p \in A \cap S$ and we need to check that we may take the same p for all $B \in B_2$ with $B \cap E \cap S = \emptyset$.

First assume $S \subset A \cup E$. Take any $B \in B_2$. Since $B \cap S = (B \cap A \cap S) \sqcup (B \cap E \cap S) = B \cap A \cap$ and $B \cap S$ spans B (Remark 3), we get $B = A$. Hence our choice of p works for all $B \in B_2$.

Now assume $S \nsubseteq A \cup E$. Fix general hyperplanes $M \supset A$, $N \supset E$. Since $S \cap (M \cup N) = S \cap (A \cup E) \subsetneq S$, the inductive assumption gives $h^1(\mathcal{I}_{Z \cap (H \cup M)}(3)) = 0$. Since $\mathrm{Res}_{H \cup M}(Z)$ is a point, the residual sequence of the quadric $M \cup N$ gives $h^1(\mathcal{I}_Z(3)) = 0$.

(b2) Assume $h = 2$. We have $s \le 11$. The Segre condition for Z gives $\langle S_2 \rangle \cap S_1 = \emptyset$. $\langle S_2 \rangle$ is the only element B of B_1 with $\sharp(B \cap S) = 2$. The Segre condition for Z gives $S_1 \cap \langle S_2 \rangle = \emptyset$.

(b2.1) Assume the existence of L in B_1 and M in B_3 such that $M \cap L \cap S = \emptyset$. We have $8 \le w(M) \le 9$. The proof that the case $S \subset M \cup L$ implies the general case made in step (b1.1.5) works, because $S_1 \cap \langle S_2 \rangle = \emptyset$ and so $h^1(\mathcal{I}_{S_2 \cup \{o\}}(1)) = 0$. Hence we assume $S \subset M \cup L$.

(b2.1.1) Assume the existence of a plane $E \subset M$ with $w(E) = 7$. We have $\sharp(E \cap S) \ge 5$ and equality holds only if $S_2 \subset E$. Fix $q \in L \cap S$ with maximal multiplicity and set $H := \langle E \cup \{q\} \rangle$. We check that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 2. We have $w_{\mathrm{Res}_H(Z)}(\mathbb{P}^4) \le 13 - \sharp(S \cap L \cap H) - \sharp(E \cap S)$ and hence to check the Segre condition in degree 3 for $\mathbb{P}^4$ and H with respect to $\mathrm{Res}_H(Z)$ is is sufficient to use that $\sharp(E \cap S) \ge 3$. M satisfies the Segre condition in degree 2 for $\mathrm{Res}_H(Z)$, because $w(M) \le 9$ and $\sharp(M \cap H \cap S) = \sharp(E \cap S) \ge 3$. Since $L \cap H \cap S \ne \emptyset$, L satisfies the Segre condition in degree 2 for $\mathrm{Res}_H(Z)$. Take $R \in B_1$ with $R \ne L$. If $R \notin M$, then it contains a point of $S_2 \cap M$ and a point of $S_2 \cap L$; since in this case $\sharp(S_2 \cap L) = 1$, this point is q and hence $w_{\mathrm{Res}_H(Z)}(R) \le 2$. Now assume $R \subset M$. Since $w(M) \le 9$, we have $w(E) + w(R) > w(M)$ and hence $E \cap R \cap S \ne \emptyset$. Thus R satisfies the Segre condition in degree 2. Fix $A \in B_2$ with $A \supset L$; since $S \cap H \cap L \ne \emptyset$, to check the Segre condition for A we may assume $w(A) = 7$; in this case it is sufficient to check that $A \cap E \cap S \ne \emptyset$; we have $w(A \setminus L) = 3$; since $w(M \setminus E) \le 2$, we get $A \cap E \cap S \ne \emptyset$. Fix $A \in B_2$ with $A \subset M$; since $w(M \setminus E) \le 2$, we have $w(A \cap E) \ge 4$ and hence $\sharp(A \cap E \cap S) \ge 2$ and so A satisfies the Segre condition.

Take $A \in B_2$ such that $A \nsupseteq L$ and $A \nsubseteq M$; since $\sharp(S \cap A \cap L) \le 1$ and

$m_1 \le 2$, we have $w(A \cap M) \ge w(A) - 2$ with equality only if $A \cap L$ is a point of S_2; since $A \cap M$ is a line, we get $w(A) = 6$, $A \cap L \in S_2$ and $w(A \cap M) = 4$; since $w(M \setminus E) \le 2$, we get $A \cap M \cap S \ne \emptyset$ and hence A satisfies the Segre condition for $\mathrm{Res}_H(Z)$.

Since $q \in L$, L satisfies the Segre condition for $\mathrm{Res}_H(Z)$. Fix $D \in B_1$ such that $D \ne L$ and $D \nsubseteq M$; we saw that $D = \langle S_2 \rangle$, that one of the points of S_2 is in M and that the other one is in L; since L has a unique point with multiplicity 2, this point is q and hence $w_{\mathrm{Res}_H(Z)}(D) \le 3$. Fix $D \in B_1$ such that $D \subset M$; since $w(M \setminus D) > w(E)$, we have $D \cap E \cap S \ne \emptyset$.

(b2.1.2) Assume the existence of $R, D \in B_1$ such that $R \cap D = \emptyset$ and $R \cup D \subset M$. Since $w(M) \le 9$, either $S \cap M \subset R \cup D$ or $S \cap M \setminus S \cap (R \cup D)$ is a point e with multiplicity 1 in Z; in the former case we write $e := \emptyset$ and $m_e = 0$. Fix $p, q \in L \cap S$ with $\{p, q\} \supseteq S_2 \cap L$. Set $Q := \langle D \cup \{p\} \rangle \cup \langle R \cup \{q\} \rangle$. If $e \ne \emptyset$, then $e \notin Q$, because $M \cap L \notin S$ and so neither p nor q are contained in the plane spanned by e and one of the lines R, D. We write $\{L, D, R\} = \{L_1, L_2, L_3\}$, because in our proof the role of L, D, R is symmetric.

(b2.1.2.1) Assume $e \ne \emptyset$ and $S_2 \subset L_i$ for some i, say $S_2 \subset L_1$. Since $\mathcal{I}_{S_2 \cup L_2 \cup L_3}(2)$ is spanned, there is $Q \in |\mathcal{I}_{S_2 \cup L_1 \cup L_2}(2)|$ such that $e \notin Q$. The inductive assumption gives $h^1(\mathcal{I}_{Z \cap Q}(3)) = 0$. We have $\mathrm{Res}_Q(Z) = \{e\} \cup S_2$ with $e \notin \langle S_2 \rangle$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_Q(Z)}(1)) = 0$.

(b2.1.2.2) Assume $e = \emptyset$ and $S_2 \subset L_i$ for some i, say $S_2 \subset L_1$. Fix $p \in L_2 \cap S$ and $q \in L_3 \cup S$. Set $H := \langle S_2 \cup \{p, q\} \rangle$. Since $\mathrm{Res}_H(Z)$ is the union of S_2, 3 points of L_2 and 3 points of L_3 and $h^1(\mathcal{I}_{L_1 \cup L_2 \cup L_3}(2)) = 0$, we get $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(2)) = 0$.

(b2.1.2.3) Assume $S_2 \nsubseteq L_i$ for all i, say $S_2 \cap L_1 \ne \emptyset$ and $S_2 \cap L_2 \ne \emptyset$. Set $H := \langle S_2 \cup L_3 \rangle$. The scheme $\mathrm{Res}_H(Z)$ is the union of e, 3 points of L_1 and 3 points of L_2. Since $h^1(\mathcal{I}_{L_1 \cup L_2}(2)) = 0$ and $\mathcal{I}_{L_1 \cup L_2}(2)$ is spanned, we get $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(2)) = 0$.

(b2.1.3) Assume that M contains at least one element of B_1, say R, and that any two elements of B_1 contained in M meets. If $S_2 \subset M$, we get $\langle S_2 \rangle \cap R \ne \emptyset$ and hence $S_2 \cap R \ne \emptyset$. Set $H := \langle L \cup R \rangle$. It is sufficient to prove that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 2.

(b2.1.3.1) Assume that either $S_2 \subset M$ or $S_2 \subset L$. Since $\sharp(B \cap S) \ge 3$ for all $B \in B_1 \setminus \{\langle S_2 \rangle\}$, the Segre conditions is satisfied by $\mathbb{P}^4$, M, and all lines, and all linear spaces containing L, except (if $S_2 \subset U$) the hyperplane with $w(U) = 10$. Fix $U \in B_3$ such that $U \ne M$ and $U \nsupseteq L$. Since $S \subset M \cup L$ and $w(U \cap L) \le 2$, we get $w(U \cap M) \ge 6 > w(M) - 4$. Hence $U \cap M \cap S \ne \emptyset$. Therefore to check the Segre condition for $\mathrm{Res}_H(Z)$ we may assume $w(U) = 10$ and hence $w(U \cap M) \ge 8$, contradicting the Segre condition for planes of Z. Now assume

$S_2 \subset L \subset U$ and $w(U) = 10$; we have $w(M \cap U) = 6 < w(M) - w(R)$ and hence $U \cap R \cap S \neq \emptyset$, which implies $w_{\mathrm{Res}_H(Z)}(U) \leq 7$. Fix $A \in B_2$ such that $A \nsubseteq L$. First assume $A \subset M$. By step (b1.1.1) we may assume $w(A) = 6$ and hence to check the Segre condition for A it is sufficient to check that $A \cap H \cap S \neq \emptyset$; this is true, because $w(A) + w(R) > 9 \geq w(M)$. Now assume $A \nsubseteq M$; since $S \subset M \cup L$, we get $w(A) = w(A \cap M) + w(A \cap L)$; since $w(A \cap L) \leq 2$, we get $w(A \cap M) = 4$ and $w(A) = 6$; since $w(A \cap L) > 0$, we get $A \cap L \cap S \neq \emptyset$ and hence $w_{\mathrm{Res}_H(Z)}(A) \leq 5$.

(b2.1.3.2) Assume $S_2 \cap M \neq \emptyset$ and $S_2 \cap L \neq \emptyset$. Since $S_2 \cap L \neq \emptyset$, $\mathrm{Res}_H(Z)$ is the union of S_2 and at most 3 points, not on a line meeting S_2. Thus $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(2)) = 0$.

(b2.1.4) Assume $B_1 = \{L\}$. Since $\langle S_2 \rangle \in B_1$, we have $L = \langle S_2 \rangle$, $S_2 \cap M = \emptyset$ and $S_1 \subset M$. Hence $w(B) = \sharp(B \cap S)$ for any set $B \subseteq M$. Let H be a hyperplane containing L and at least two points p, q of $M \cap S$. $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 2 with respect to $\mathbb{P}^4$, H, and all lines. It satisfies it with respect to M, because $w(M) \leq 9$ and $\sharp(M \cap H \cap S) \geq 2$. Fix $U \in B_3$ with $U \neq M$. Since $w(U \cap M) \leq 7$, we have $U \cap L \cap S \neq \emptyset$. First assume $U \supset L$; to check the Segre condition we may assume $w(U) = 10$; in this case we have $w(U \cap M) = 6$; we get $U \cap M \in B_2$ and to win it is sufficient that $A \cap M$ satisfies the Segre condition for $\mathrm{Res}_H(Z)$ (see below). Now assume $U \not\supset L$; since $w(A \cap M) < 8 \leq w(U)$, we have $U \cap L \cap S \neq \emptyset$; to check the Segre condition we may assume $w(U) \geq 9$; we get $w(U \cap M) \geq w(U) - m_{U \cap L} \geq 7$; in this case step (b2.1.1) gives $h^1(\mathcal{I}_Z(3)) = 0$.

If there is $A \in B_2$ with $A \supset L$, then $\sharp(A \cap H \cap S) \geq 2$. Fix $A \in B_2$ with $A \nsubseteq M$. Since $w(A \cap M) \leq 3$ and $m_1 \leq 2$, we have $A \supset L$ and hence $\sharp(A \cap H \cap S) \geq 2$ and $\mathrm{Res}_H(Z)$ satisfies the Segre conditions for As.

Now assume the existence of $E \in B_2$ with $E \subset M$. By step (b1.1.1) we may assume $w(E) = 6$. To get the Segre condition for $\mathrm{Res}_H(Z)$ it is sufficient to take p or q in E. We may do that for all $E \in B_2$ with $E \subset M$ if they are at most two. Fix $F, G \in B_2 \setminus \{E\}$ with $w(F) = w(G) = 6$, $F \cup G \subset M$ and $F \neq G$. Since $w(M) \leq 9$ and $B_1 = \{L\}$, we have $w(M) = 9$, $w(E \cap F) = 3$, $E \cap F \cap S$ spans $E \cap F$ and $S \cap M = (S \cap E) \cup (S \cap F)$ (and the same is true with F, G). Hence $S \cap G = S \cap G \cap E \cup S \cap G \cap F$. Since $h \leq 1$, we have $\sharp(S \cap G) \geq 5$. Since $B_1 = \{L\}$, we get $\sharp(S \cap G \cap E) \geq 2$ and $\sharp(S \cap G \cap E) \geq 2$. Since any two points of M spans a line, we get $E \cap F = E \cap G = F \cap G$. Hence it is sufficient to take $p \in E \cap F \cap S$ to get the Segre condition for $\mathrm{Res}_H(Z)$.

(b2.2) Assume $B_3 \neq \emptyset$ and $M \cap L \cap S \neq \emptyset$ for all $M \in B_3$ and all $L \in B_1$. Fix $H \in B_3$ with maximal weight among all elements of B_3. We have $8 \leq w(H) \leq 10$ and $H \cap S_2 \neq \emptyset$. It is sufficient to prove that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 2. The conditions for $\mathbb{P}^4$ and for H are satisfied, because $\sharp(H \cap S) \geq 4$ (Remark 3). The conditions for lines are satisfied by our assumption on B_1. Fix $U \in B_3 \backslash \{H\}$. We have $w(H \cap U) \geq w(H) + w(U) - 13 > 0$. Hence $H \cap U \cap S \neq \emptyset$ and to check the Segre condition for $\mathrm{Res}_H(Z)$ we may assume $w(U) \geq 9$. Since $w(H) \geq w(U)$, we get $w(H \cap U) \geq 5$. Since $h = 2$, we get $\sharp(H \cap U \cap S) \geq 3$ and so U satisfies the Segre condition for $\mathrm{Res}_H(Z)$ in degree 2. Now take $A \in B_2$. We have $6 \leq w(A) \leq 7$. Since $w(A) + w(H) > 13$, we have $A \cap H \cap S \neq \emptyset$. Hence to check the Segre condition for $\mathrm{Res}_H(Z)$ in degree 2 we may assume $w(A) = 7$. In this case $w(A \cap H) \geq 2$ with strict inequality if $w(H) \geq 9$. We have $\sharp(A \cap H \cap S_1) \geq 2$ if $A \cap H \cap S_2 = \emptyset$. A satisfies the Segre condition in degree 2 if $S_2 \subset A$. Hence we may assume $\sharp(S_2 \cap A) = 1$. In this case the hyperplane $\langle A \cup S_2 \rangle$ satisfies $w(\langle A \cup S_2 \rangle) \geq 9$. Hence $w(H) \geq 9$ and so $w(A \cap H) \geq 3$. Therefore $\sharp(A \cap H \cap S) \geq 2$ and so A satisfies the Segre condition in degree 2.

(b2.3) By Steps (b2.1) and (b2.2) we may assume $B_3 = \emptyset$.

(b2.3.1) Assume the existence of $L_1, L_2, L_3 \in B_1$ such that $L_i \cap L_j = \emptyset$ for all $i \neq j$. See the proof of step (b2.1.2).

(b2.3.2) Assume the existence of $L, R \in B_1$ such that $L \cap R = \emptyset$ and that each element of B_1 meets $L \cup R$. Set $H := \langle L \cup R \rangle$. Since $\sharp(S \cap (L \cup R)) \geq 6$, we have $w_{\mathrm{Res}_H(Z)}(\mathbb{P}^4) \leq 5$.

(b2.3.2.1) Assume $S_2 \subset L \cup R$. In this case the scheme $\mathrm{Res}_H(Z)$ is reduced and it has at most 5 points. We have $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(2)) > 0$ if and only if it has 4 collinear points. This is not the case because every element of B_1 contains a point of $S \cap (L \cup R)$ by Remark 4.

(b2.3.2.2) Assume $S_2 \nsubseteq L \cup R$. Since $\langle S_2 \rangle$, we have $\sharp(S_2 \cap (L \cup R)) = 1$, say $S_2 = \{p, q\}$ with $p \in L$. In this case $\mathrm{Res}_H(Z)$ is the union of $2q$, p, and the union G of $s - 6 \leq 3$ points of S_1. Since each element of B_1 meets $L \cup R$, we have $\langle S_2 \rangle \cap G = \emptyset$ and no line through q contains two points of G. Taking the linear projection from q we see that $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(2)) > 0$ if and only if $\sharp(G) = 3$ and $\{p\} \cup G$ are collinear. In this case $\langle \{q\} \cup L \cup G \rangle$ is a hyperplane with $w(\langle \{q\} \cup L \cup G \rangle) \geq 8$, contradicting the assumption $B_3 = \emptyset$.

(b2.3.3) Assume the existence of $L \in B_1$ such that each element of B_1 meets L (this is the last case needed to conclude the proof of the case $h = 2$, because $\langle S_2 \rangle \in B_1$). Since $\langle S_2 \rangle \in B_1$, we have $S_2 \cap L \neq \emptyset$. Fix $p, q \in S \setminus S \cap L$ with $p \neq q$ and let H be a hyperplane containing $L \cup \{p, q\}$. $\mathrm{Res}_H(Z)$ satisfies the Segre condition in degree 2 with respect to $\mathbb{P}^4$, lines and hyperplanes. Hence we may assume $B_2 \neq \emptyset$. Fix $A \in B_2$ with maximal weight. First assume $w(A) = 7$ and $A \cap L \cap S = \emptyset$. To use $\mathrm{Res}_H(Z)$ we need to take $\{p, q\} \subset A$. Assume the existence of another $E \in B_2$ with $E \cap L \cap S = \emptyset$. Since $w(\mathbb{P}^4 \setminus (A \cup L)) \leq 2$, we have $w(E \cap A) \geq 4$ and hence $E \cap A \in B_1$. Hence $E \cap A \cap L \cap S \neq \emptyset$, a contradiction. Now take $E \in B_2$ with $E \cap L \cap S \neq \emptyset$. To check the Segre condition for $\mathrm{Res}_H(Z)$ we may assume $w(E) = 7$. Since $\sharp(L \cap S) \geq 2$, we may also assume that $E \cap L$ is a single point o. Since $w(\mathbb{P}^4 \setminus (A \cup L)) \leq 2$ and $m_o \leq 2$, we get $w(A \cap E) \geq 3$ and hence $A \cap E$ is a line spanned by $A \cap E \cap S$. We have $w(\langle A \cup E \rangle) \geq w(A) + w(E) - w(A \cap E) \geq 10$, contradicting the assumption $B_3 = \emptyset$.

Now assume $w(A) = 7$ and $\sharp(A \cap L \cap S) = 1$, say $\{u\} = A \cap L$. To use $\mathrm{Res}_H(Z)$ we need $\{p, q\} \cap A \neq \emptyset$. Take another $E \in B_2$; we saw that we may assume that either $w(E) = 6$ or $E \cap L \cap S \neq \emptyset$. First assume $w(E) = 6$; to check that E satisfies the Segre condition we may assume $L \cap E \cap S = \emptyset$; we get $w(\mathbb{P}^4 \setminus (A \cup L))) \leq 2 + m_u$ and hence $w(A \cap E) \geq 4 - m_u$; since $h = 2$, we get $\sharp(A \cap E \cap S) \geq 2$ and hence $A \cap E$ is a line; the inequality $w(\langle A \cup \rangle) \geq 7 + 6 - 4$ contradicts the assumption $B_3 \neq \emptyset$. Now assume $w(A) = 6$. In this case we may assume $A \cap L \cap S = \emptyset$ and it is sufficient to take $p \in A \cap S$. Assume the existence of $E \in B_2$ such that $E \neq A$. Since A has maximal weight, we have $w(E) = 6$ and hence to check the Segre condition in degree 2 for E we may assume $E \cap L \cap S = \emptyset$. Since $w(\mathbb{P}^4 \setminus (A \cup L)) \leq 3$, we have $w(A \cap E) \geq 3$ and hence (since $m_1 \leq 2$), $A \cap E$ is a line; since $\langle A \cup E \rangle \geq 6 + 6 - 4$, we have $\langle A \cup E \rangle \in B_3$, a contradiction.

(b3) Assume $h = 6$ and hence $s = 7$. This case is covered in [1].

(b4) Assume $h = 5$ and hence $1 \leq s - h \leq 3$. First assume that $H := \langle S_2 \rangle$ is a hyperplane. It is sufficient to prove that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 2. Set $S' := S \setminus (S_1 \cap H)$. We have $S' = \mathrm{Res}_H(Z)$. The Segre condition for lines and the inequality $s \leq 3$, implies that no 4 of the points of S' are in a plane and no 6 are in a hyperplane. Hence $h^1(\mathcal{I}_{S'}(2)) = 0$. Now assume that S_2 is in linearly general position. We take as H a hyperplane spanned by 4 points of S_2 (later we will add other restrictions on H). $\mathrm{Res}_H(Z)$ has one point p with multiplicity 2, all points of $S_2 \setminus \{p\}$ with multiplicity 1 and all points of $S_1 \setminus S_1 \cap H$ with multiplicity 1. We have $w_{\mathrm{Res}_H(Z)}(\mathbb{P}^4) \leq w(\mathbb{P}^4) - \sharp(S \cap H) \leq 13 - 4$. The same computation works for H. Fix $A \in B_3$ with $A \neq H$. Since $w(\mathbb{P}^4 \setminus H) \leq 2 + (s - h) \leq 5$, we have $w(A \cap H) \geq w(A) - 5$. Hence $\sharp(S \cap A \cap H) > 0$. Therefore to check the Segre condition in degree 2 for $\mathrm{Res}_H(Z)$ we may assume $w(A) \geq 9$. We get $\sharp(A \cap H \cap S) \geq 2$ and hence it is sufficient to check the case $w(A) = 10$, in which we get $w(A \cap H) \geq 5$ and hence $\sharp(A \cap H \cap S) \geq 3$. Fix $A \in B_2$. Since $w(A \cap H) \geq w(A) - 5$, we have $A \cap S \neq \emptyset$ and $\sharp(A \cap S) \geq 2$ if $w(A) = 7$. Take $A \in B_1$ and assume $A \cap H \cap S = \emptyset$. Since $w(A) = 4$, A contains p and at least 2 points of S_1. Since $s - h \leq 3$, S_1 spans at most 3 lines, all of them contained in a plane $U \supset S_2$. Therefore we may take as H a hyperplane spanned by $U \cap S_2$ and $4 - \sharp(U \cap S_2)$ other points of S_2.

(b5) Assume $h = 4$ and hence $1 \le \sharp(S_1) \le 5$. The Segre conditions for Z give that $H := \langle S_2 \rangle$ is a hyperplane. If S is in linearly general position, then we use [4, Theorem 1.4]. If there is no $B \in B_1$ with $\sharp(B \cap S) = 4$ (i.e. with $B \cap S \subseteq S_1$) and with $B \cap H \cap S = \emptyset$, then we may take $H := \langle S_2 \rangle$ as in the case $h = 5$. Now assume the existence of $B \in B_1$ with $\sharp(B \cap S) = 4$ and $B \cap H \cap S = \emptyset$. In this case we have $h^1(\mathcal{I}_S(2)) = 1$ and we may find $S' \subset S$ such that $S_2 \subset S'$, $\sharp(S \setminus S') = 2$, $\sharp(S' \cap B) = 2$ and the line B spanned by $S \setminus S'$ does not meet $\langle S_2 \rangle$. Adapt the proof of Claim 2 below.

(b6) Assume $h = 3$ and hence $s \le 10$. Write $S_2 = \{p_1, p_2, p_3\}$. Set $E := \langle S_2 \rangle$. The Segre condition for planes implies $\sharp(S_1 \cap E) \le 1$. Let τ be the set of all $B \in B_1$ such that $B \cap S_2 = \emptyset$. In steps (b6.1) and (b6.2) we will prove that $h^1(\mathcal{I}_Z(3)) = 0$ if $\tau \neq \emptyset$. From step (b6.3) on we will assume $\tau = \emptyset$. Let Δ be the set of all lines $L \subset \mathbb{P}^4$ such that $\sharp(L \cap S_1) = 2$ and $L \cap S_2 \neq \emptyset$. Since Z satisfies the Segre condition for lines, then $L \cap S_2 = \emptyset$ for all $L \in \Delta$.

(b6.1) In this step we prove that $h^1(\mathcal{I}_Z(3)) = 0$ if $\sharp(\tau) \ge 2$. Since $s - h \le 7$, we have $\sharp(\tau) = 2$, $s = 10$, the two elements L, R, of τ meet at a point of S_1 and $S_1 \subset L \cup R$.

(b6.1.1) Assume $L \cap E = L \cap E = \emptyset$. Set $H := \langle \{p_1, p_2\} \cup L \rangle$ and $M := \langle \{p_1, p_2\} \cup R \rangle$. Since $L \nsubseteq E$ and $R \nsubseteq E$, we have $p_3 \notin H \cup M$. Hence the inductive assumption gives $h^1(H \cup M, \mathcal{I}_{Z \cap (H \cup M)}(3)) = 0$. Since $\mathrm{Res}_{H \cup M}(Z) = 2p_3$, we have $h^1(\mathcal{I}_{\mathrm{Res}_{H \cup M}(Z)}(1)) = 0$.

(b6.1.2) Assume that one of the elements of τ, say L, meets E. Set $H := \langle E \cup L \rangle$. The scheme $\mathrm{Res}_H(Z)$ is the union of S_2 and the points of $R \cap S_1 \setminus R \cap S_1 \cap (E \cup L)$. Since S_2 spans E, $R \nsubseteq E$ and $R \cap S_2 = \emptyset$ (by the Segre condition for Z), we have $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(2)) = 0$.

(b6.2) In this step we prove that $h^1(\mathcal{I}_Z(3)) = 0$ if $\tau \neq \emptyset$. By step (b6.1) we may assume $\sharp(\tau) = 1$, say $\tau = \{L\}$.

(b6.2.1) Assume $L \cap E \neq \emptyset$. Set $H := \langle E \cup L \rangle$. The scheme $\mathrm{Res}_H(Z)$ is the union of S_2 and at most 3 points, none of them in E, and if $\sharp(\mathrm{Res}_H(Z) \setminus S_2) = 3$, then these 3 points do not not span a line intersecting S_2 (by the Segre condition for Z). Since $\langle S_2 \rangle = E$, then $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(2)) = 0$.

(b6.2.2) Assume $E \cap L = \emptyset$ and that either $s \le 9$ or the 3 points of $S_1 \setminus S_1 \cap L$ are not contained in a line intersecting E. These assumption imply the existence of a quadric $Q \supset S_2 \cup L$, such that $Q \cap S_1 \neq Q \cap S_1$, $Q \cap E \cap (S_1 \setminus L)) = \emptyset$ and either $\sharp(S_1 \setminus S_1 \cap Q) = 1$ or $\sharp(S_1 \setminus S_1 \cap Q) = 2$ and the two points of $S_1 \setminus S_1 \cap Q$ span a line disjoint from E. Since none of the points of $S_1 \setminus S_1 \cap Q$ is contained in $E = \langle S_2 \rangle$ and $\mathrm{Res}_Q(Z) = S_2 \cup (S_1 \setminus S_1 \cap Q)$, we have $h^1(\mathcal{I}_{\mathrm{Res}_Q(Z)}(1)) = 0$. Since $Q \cap S \neq S$, the inductive assumption gives $h^1(Q, \mathcal{I}_{Z \cap Q}(3)) = 0$.

(b6.2.3) Assume $E\cap L=\emptyset$, $s=10$ and that the 3 points of $S_1\setminus S_1\cap L$ are contained in a line R with $E\cap R\neq\emptyset$. The Segre conditions for lines implies $R\cap S_2=\emptyset$. We have $|\mathcal{I}_S(2)|=|\mathcal{I}_{S_2\cup L\cup R}(2)|$. Let Q be a general element of $|\mathcal{I}_S(2)|$. Since $\mathrm{Res}_Q(S)=S_2$ and $h^1(\mathcal{I}_{S_2}(1))=0$, the residual sequence of Q shows that it is sufficient to prove that $h^1(Q,\mathcal{I}_{Z\cap Q}(3))=0$.

(b6.2.4.1) Assume $R\cap\langle\{p_i,p_j\}\rangle\neq\emptyset$ for some i,j, say $R\cap\langle\{p_1,p_2\}\rangle\neq\emptyset$. Set $M:=\langle\{p_1,p_2\}\cup L\rangle$ and $N:=\langle\{p_1,p_2\}\cup R\rangle$. The Segre conditions for Z give $L\nsubseteq E$ and $R\nsubseteq E$. Hence $p_3\notin M\cup N$. The inductive assumption gives $h^1(M\cup N,\mathcal{I}_{Z\cap(M\cup N)}(3))=0$. Since $\mathrm{Res}_{M\cup N}(Z)=2p_3$, we have $h^1(\mathcal{I}_{\mathrm{Res}_{M\cup N}(Z)}(1))=0$.

(b6.2.4.2) Assume $R\cap\langle\{p_i,p_j\}\rangle=\emptyset$ for all $i\neq j$ and $S_2\cap\langle L\cup R\rangle=\emptyset$. This assumption implies that $|\mathcal{I}_{S_2\cup L\cup R}(2)|$ has $L\cup R\cup S_2$ as its base locus (scheme-theoretically). Fix general $Q',Q''\in|\mathcal{I}_S(2)|$. Since $\dim(Q\cap Q')=2$, Q' is an effective Cartier divisor of Q and we may take the residual scheme $\mathrm{Res}_{Q\cap Q'}(Q\cap Z)$. Since $\mathrm{Res}_{Q\cap Q'}(Z\cap Q)\subseteq S_2$, we have $h^1(Q,\mathcal{I}_{\mathrm{Res}_{Q\cap Q'}(Z\cap Q)}(1))=0$. Hence it is sufficient to prove that $h^1(Q\cap Q',\mathcal{I}_{Q\cap Q'\cap Z}(3))=0$. Since $\dim(Q\cap Q'\cap Q'')=1$, $Q\cap Q'\cap Q''$ is an effective Cartier divisor of $Q\cap Q'$. Taking the residual scheme we see that it is sufficient to prove that $h^1(Q\cap Q'\cap Q'',\mathcal{I}_{Q\cap Q'\cap Q''\cap Z}(3))=0$. The scheme $Q\cap Q'\cap Q''$ is locally Cohen-Macaulay with pure dimension 1. We have $\deg(Q\cap Q'\cap Q'')=8$ and $L\cup R\subset Q\cap Q'\cap Q''$. Since $L\cup R\cup S_2$ is the scheme-theoretic base locus of $|\mathcal{I}_{S_2\cup L\cup R}(2)|$, L and R occurs with multiplicity one in $Q\cap Q'\cap Q''$ and each p_i is a smooth point of $Q\cap Q'\cap Q''$ and hence $Q\cap Q'\cap Q''=L\cup R\cup C$ with C a degree 6 curve containing S_2 and smooth at each point of S_2. Hence $Q\cap Q'\cap Q''\cap Z$ is the union of S_1 and 3 degree two connected schemes v_1,v_2,v_3 with $(v_i)_{\mathrm{red}}=\{p_i\}$. Therefore it is sufficient to prove that $h^1(\mathcal{I}_{S_1\cup v_1\cup v_2\cup v_3}(2))=0$. Hence it is sufficient to prove that $h^1(\mathcal{I}_{S_1\cup 2p_1\cup 2p_2\cup v_3}(3))=0$. By the case $h=2$ (or a quick easy proof) we have $h^1(\mathcal{I}_{S_1\cup 2p_1\cup 2p_2\cup\{p_3\}}(3))=0$ Since p_3 is a connected component of the scheme-theoretic base locus of $|\mathcal{I}_S(2)|$, v_3 may be taken as a general degree 2 connected subscheme of $\mathbb{P}^4$ with p_3 as its reduction. Since $h^1(\mathcal{I}_{S_1\cup 2p_1\cup 2p_2\cup\{p_3\}}(3))=0$, it is sufficient to prove that $h^1(\mathcal{I}_Z(3))\leq 3$. The proof of step (b6.2.3) gives $h^1(\mathcal{I}_Z(3))\leq 1$.

(b6.3) Assume the existence of a hyperplane $H\supset E$ such that $\sharp(S_1\cap H)\geq 3$. By steps (b6.1) and (b6.2) we may assume $\tau=0$. Hence every element of B_1 meets E and so $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree two with respect to all lines. Since $S_2\subset E$, satisfies the Segre conditions with respect to $\mathbb{P}^4$, H and E. Fix $A\in B_2$ with $A\neq E$. Since $\sharp(S_1\setminus S_1\cap H)\leq 4$, we have $A\cap H\cap S\neq\emptyset$. Hence to check that A satisfies the Segre condition for $\mathrm{Res}_H(Z)$ we may assume $w(A)=7$. Since $w(A\setminus A\cap(L\cup E))\geq 3$, we get

$\sharp(A\cap H\cap S)\ge 2$ and hence $w_{\mathrm{Res}_H(Z)}(A)\le 5$. Fix $U\in B_3$ with $U\ne H$. Since $w(U\setminus U\cap(L\cup E))\ge 4$ and $m_1=2$ we have $\sharp(U\cap H\cap S)\ge 2$. Hence to test the Segre condition for U we may assume $w(U)=10$. We get $w(U\setminus U\cap(L\cup E))\ge 6$ and hence $\sharp(U\cap H\cap S)\ge 3$.

(b6.4) Assume the existence of a hyperplane $H\supset E$ such that $\sharp(H\cap S_1)=2$ and set $L:=\langle S_1\cap H\rangle$. Since $\tau=\emptyset$, every element of B_1 meets E and so $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree two with respect all lines. Since $S_2\subset E$, $\mathrm{Res}_H(Z)$ satisfies the Segre conditions with respect to $\mathbb{P}^4$, H and E. Fix $U\in B_3$ with $U\ne H$. Since $w(U\setminus U\cap(L\cup E))\ge 3$ and $m_1=2$ we have $\sharp(U\cap H\cap S)\ge 2$. Hence to test the Segre condition for U we may assume $w(U)=10$. We get $w(U\setminus U\cap H)\ge 5$ and hence $\sharp(U\cap H\cap S)\ge 3$. Fix $A\in B_2$ with $A\ne E$. Since $\sharp(S_1\setminus S_1\cap H)\le 5$, we have $A\cap H\cap S\ne\emptyset$. Hence to check that A satisfies the Segre condition for $\mathrm{Res}_H(Z)$ we may assume $w(A)=7$. The Segre condition is not satisfied by A only if $s=10$, A contains a single point of S_2 and the 5 points of $S_1\setminus S_1\cap L$.

(b6.4.1) Assume $L\cap S_2\ne\emptyset$, say $p_1\in L$. First assume $p_1\notin A$, say $p_2\in A$. Set $M:=\langle\{p_1\}\cup A\rangle$. Let $\ell:\mathbb{P}^4\setminus\{p_3\}\to\mathbb{P}^3$. We have $\mathrm{Res}_M(Z)=2p_3\cup\{p_1\}\cup(L\cap S)$ with $\sharp(L\cap S)=3$. No line through p_3 contains two points of $\{p_2\}\cup(L\cap S)$. Hence $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(2))=h^1(\mathbb{P}^3,\mathcal{I}_{\ell(\{p_2\}\cup(L\cap S))}(2))=0$. We have $h^1(\mathbb{P}^3,\mathcal{I}_{\ell(\{p_2\}\cup L\cap S)}(2))=0$, because $\sharp(\ell(\{p_1\}\cup(L\cap S)))=4$ and $\ell(\{p_1\}\cup(L\cap S))$ is not contained in a line (since $\dim(\langle\{p_2,p_3\}\cup L\rangle)=3$).

The same proof works if $p_1\in A$, just taking $\langle\{p_2\}\cup A\rangle$.

(b6.4.2) Assume $L\cap\langle\{p_i,p_j\}\rangle=\emptyset$ for all $i\ne j$. In this case the proof of step (b6.4.1) works verbatim.

(b6.4.3) Assume $L\cap S_2=\emptyset$ and $L\cap\langle\{p_i,p_j\}\rangle\ne\emptyset$ for some i,j, say $L\cap\langle\{p_1,p_2\}\rangle$. First assume that A contains one of the points p_1,p_2, say p_2. In this case the proof of step (b6.4.1) works verbatim, because $\ell(\{p_1,p_2\}\cup(L\cup S_1))$ is formed by 4 non-collinear points. Now assume $p_3\in A$. Set $N:=\langle A\cup\{p_2\}\rangle$. We have $\mathrm{Res}_N(Z)=2p_1\cup\{p_2,p_3\}\cup(L\cap S_1)$. Let $\mu:\mathbb{P}^4\setminus\{p_1\}\to\mathbb{P}^3$ be the linear projection from p_1. The Segre conditions for Z gives the injectivity of the restriction of μ to the set $\{p_2,p_3\}\cup(L\cap S_1)$. Since $\langle S_2\cup L\rangle=H$, the 4 points of $\mu(\{p_2,p_3\}\cup(L\cap S_1))$ are not collinear and hence $h^1(\mathcal{I}_{\mathrm{Res}_N(Z)}(2))=0$.

Claim 1: $h^1(\mathcal{I}_Z(3))=0$ if $h^1(\mathcal{I}_S(2))=0$.

Proof of Claim 1: Fix any $p\in S_1\setminus S_1\cap E$. Since $h^1(\mathcal{I}_S(2))=0$, there is a quadric Q such that $Q\cap S=S\setminus\{p\}$. The inductive assumption gives $h^1(\mathcal{I}_{Q\cap Z}(3))=0$. Since $\mathrm{Res}_Q(Z)=S_2\cup\{p\}$ and $p\notin E$, we have $h^1(\mathcal{I}_{\mathrm{Res}_Q(Z)}(1))=0$.

Claim 2: $h^1(\mathcal{I}_Z(3))=0$ if $h^1(\mathcal{I}_S(2))\le 1$.

Proof of Claim 2: By Claim 1 we may assume $h^1(\mathcal{I}_S(2))=1$. Since

$\sharp(S_1 \cap E) \le 1$, we have $h^1(\mathcal{I}_{S\cap E}(2)) = 0$. Let $S' \subset S$ be a maximal subset of S such that $S' \supseteq S \cap E$ and $h^1(\mathcal{I}_{S'}(2)) = 0$. Since $h^1(\mathcal{I}_S(2)) = 1$ and S' is maximal, we have $h^0(\mathcal{I}_{S'}(2)) = h^0(\mathcal{I}_S(2))$ and $\sharp(S') = s-1$. Set $\{q\} := S \setminus S'$. Fix $p \in S' \cap S_1 \setminus S' \cap S_1 \cap E$ and set $S'' = S' \setminus \{q\}$. Since $h^1(\mathcal{I}_{S'}(2)) = 0$, we have $h^0(\mathcal{I}_{S''}(2)) = h^0(\mathcal{I}_{S'}(2)) - 1$ and p is not in the base locus of $\mathcal{I}_{S''}(2)$. Fix a general $Q \in |\mathcal{I}_{S''}(2)|$. Since Q is general, we have $p \notin Q$ and hence $Q \cap S \ne S$. The inductive assumption gives $h^1(\mathcal{I}_{Q\cap Z}(3)) = 0$. We have $\mathrm{Res}_Q(Z) \subseteq S_2 \cup \{p,q\}$ with $p, q \notin E$. Step (b6.4) gives that the line spanned by p and q does nor meet E, i.e. $h^1(\mathcal{I}_{S_2\cup\{p,q\}}(1)) = 0$.

(b6.5) Assume the existence of a plane A with $S_1 \subset A$. Since $\langle S\rangle = \mathbb{P}^4$, the set $A \cap E$ is a single point, o. By step (b6.4) we have $o \notin S_1$. If $o \in S_2$, then $\sharp(S_1) \le 5$, because $o \in S_2$ implies $w(A) = 2 + \sharp(S_1)$; in this case $h^1(\mathcal{I}_S(2)) \le 1$, because $\tau = 0$. Hence we may assume $o \notin S$. We have $h^1(E, \mathcal{I}_{S_2\cup\{o\}}(2)) = 0$. By Lemma 4 and Claim 2 we may assume $s = 10$ and that S_1 is contained in a conic D of A. Fix general hyperplanes H, H' containing E and M, M' containing A. Set $Q := H \cup M$ and $T := Q \cap (H' \cup M')$.

Since $\mathrm{Res}_Q(Z) = S_2$ and $h^1(\mathcal{I}_{S_2}(1)) = 0$, the residual sequence of $H \cup M$ shows that it is sufficient to prove that $h^1(Q, \mathcal{I}_{Q\cap Z}(3)) = 0$. T is an effective Cartier divisor of Q and $\mathrm{Res}_T(Z \cap Q) = S_2$. Hence the residual sequence of T in Q shows that it is sufficient to prove $h^1(T, \mathcal{I}_{Z\cap T}(3)) = 0$. We have $A \cup E \subset T$ (T has embedded point at o). Since $o \notin S$, we have $Z \cap (A \cup E) = Z \cap T$. Since the restriction map $H^0(\mathcal{O}_{\mathbb{P}^4}, \mathcal{O}_{\mathbb{P}^4}(3)) \to H^0(\mathcal{O}_{A\cup E}(3))$ is surjective (Lemma 4) and it factors through the restriction map $H^0(\mathcal{O}_{\mathbb{P}^4}, \mathcal{O}_{\mathbb{P}^4}(3)) \to H^0(\mathcal{O}_T(3))$, it is sufficient to prove that $h^1(A \cup E, \mathcal{I}_{Z\cap(A\cup E}(3)) = 0$. This is true, because $o \notin S$, $h^1(E, \mathcal{I}_{2S_2\cup\{o\}}(3)) = 0$ and $h^1(A, \mathcal{I}_{S_1}(3)) = 0$, e.g. because $\tau = \emptyset$.

(b6.6) Assume the existence of a plane A with $\sharp(S_1 \setminus S_1 \cap A) \le 1$. By Step (b6.6) we may assume $S_1 \nsubseteq A$. Set $\{p\} = S_1 \setminus S_1 \cap A$ and $S' := S \setminus \{p\}$. Since S has no 5-secant line and $\sharp(S_1 \cap A) \le 6$, we have $h^1(A, \mathcal{I}_{S_2\cap A}(2)) \le 1$. By Step (b6.4) we may assume $p \notin E$. Lemma 4 gives $h^1(\mathcal{I}_S(2)) = h^1(\mathcal{I}_{S'}(2))$. By Claim 2 it is sufficient to note that $h^1(\mathcal{I}_{S'}(2)) \le 1$ (use a Mayer-Vietoris exact sequence, that $h^1(A, \mathcal{I}_{S_1\cap A}(2)) \le 1$ and $h^1(E, \mathcal{I}_{S_2\cup\{o\}}(2)) = 0$).

(b6.7) By Claim 2 to conclude the proof Lemma 2 it is sufficient to show that if $h^1(\mathcal{I}_Z(3)) > 0$, then $h^1(\mathcal{I}_S(2)) \le 1$. By step (b6.4) we may assume $E \cap S_1 = \emptyset$. First assume the existence of a plane A such that $\sharp(A \cap S) \ge 3$. The Segre condition for Z gives $A \cap S_2 = \emptyset$. Since $s \le 10$, we get $S_1 \subset A$. Step (b6.6) gives $h^1(\mathcal{I}_Z(3)) = 0$.

Now assume the existence of a plane A such that $\sharp(A \cap S) = 6$. Since $s \le 10$, step (b6.7) gives $S_2 \cap A \ne \emptyset$. Since $w(A) \le 7$, we get $\sharp(S_2 \cap A) = 1$, say $S_2 \cap H = \{p_1\}$, and $\sharp(S_1 \setminus S_1 \cap A) = 2$. Set $H := \langle\{p_2\} \cup A\rangle$. We have

$\mathrm{Res}_H(Z) = 2p_3 \cup \{p_1, p_2\} \cup B$ with $B := S_1 \setminus S_1 \cap A$. By step (b6.4) we have $\langle B\rangle \cap E = \emptyset$. Hence as in step (b6.4) we get $h^1(\mathcal{I}_{2p_3\cup\{p_1,p_2\}\cup B}(2)) = 0$. Thus from now on we may assume $\sharp(S \cap A) \leq 5$ for all planes A. Since $\tau = \emptyset$, we have $h^1(A, \mathcal{I}_{A\cap S}(2)) = 0$ for every plane A.

First assume that S has no trisecant line. Let $H \subset \mathbb{P}^4$ be a hyperplane with maximal $a := \sharp(H \cap S)$. By [4, Theorem 1.4] we may assume $a \geq 5$. Since $\sharp(A \cap S) \leq 5$ for every plane A, we have $h^1(H, \mathcal{I}_{S\cap H}(2)) = 0$. If $a \geq 6$, we get $h^1(\mathcal{I}_S(2)) \leq 1$, because S has no trisecant line. If $a = 5$, then each plane contains at most 4 points of S. We get $h^1(\mathcal{I}_{S\setminus S\cap H}(1)) \leq 1$ and hence $h^1(\mathcal{I}_S(2)) \leq 1$.

Now assume the existence of lines L, R such that $L \cap R = \emptyset$ and $\sharp(S \cap L) = \sharp(S\cap R) \geq 3$ and set $H := \langle L\cup R\rangle$. Since $\tau = \emptyset$, we have $\sharp(S\cap L) = \sharp(S\cap R) = 3$ and hence $h^1(\mathcal{I}_{S\cap(L\cup R)}(2)) = 0$ and $L \cup R$ is the base locus of $|(\mathcal{I}_{S\cap(L\cup R)}(2)|$. Since $\Delta = \emptyset$ (step (b6.4)), we have $S_2 \cap (L \cup R) = \emptyset$ and hence $\sharp(S_1 \setminus S_1 \cap (L \cup L)) \leq 1$. Set $e := S_1 \setminus S_1 \cap (L\cup R)$. If $S_2 \cap H = \emptyset$, we get $h^1(\mathcal{I}_S(2)) \leq 1$, because $h^1(\mathcal{I}_{S_2}(1)) = 0$. If $\sharp(S_2 \cap H) = 2$, we use that $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(2)) = 0$. Now assume $\sharp(S_2 \cap H) = 1$. If $e \subset H$, we get $h^1(H, \mathcal{I}_{S\cap H}(2)) \leq 1$ and $h^1(\mathcal{I}_{S\setminus S\cap H}(1)) = 0$. If $e \nsubseteq H$, we get $h^1(H, \mathcal{I}_{S\cap H}(2)) = 0$ and $h^1(\mathcal{I}_{S\setminus S\cap H}(1)) \leq 1$

Thus we may assume that any two trisecant lines of S meet and that S has at least a trisecant line. Let H be a hyperplane containing at least one trisecant line and, among the hyperplanes with this property, maximal $a := \sharp(H \cap S)$. We have $a \geq 5$ and $S \setminus S \cap H$ has no trisecant lines. We conclude as above. □

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