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WO-CONTINUITY AND *WK*-CONTINUITY ON ASSOCIATED *w*-spaces

Young Key Kim[1], Won Keun Min[2 §]

[1]Department of Mathematics
MyongJi University
Youngin, 449-728, KOREA

[2]Department of Mathematics
Kangwon National University
Chuncheon, 200-701, KOREA

Abstract: We introduce the concepts of WO-continuity and WK-continuity on associated w-spaces. We investigate some properties and relationships between WO-continuity, WK-continuity, W-continuity, $W^{\star}$-continuity and continuity on associated w-spaces.

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1. Introduction

In [16], Siwiec introduced the notions of weak neighborhoods and weak base in a topological space. The author introduced the weak neighborhood systems defined by using the notion of weak neighborhoods in [13]. The weak neighborhood system induces a weak neighborhood space (briefly WNS) which is independent of neighborhood spaces [4] and general topological spaces [2]. In [13], the author introduced the notion of new interior operator and closure operator on a WNS. We also introduced the notion of weak structure which is

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§Correspondence author

defined by the properties of new interior operator and closure operator in a WNS. The set of all g-open subsets [5] of a topological space is a kind of weak structure.

In this paper, we introduce the concepts of WO-continuity and WK-continuity on associated w-spaces. In particular, we investigate some properties and relationships between WO-continuity, WK-continuity, W-continuity, W^{*}-continuity and continuity on associated w-spaces.

2. Preliminaries

Let X be a topological space and $S \subseteq X$. The closure (resp. interior) of S will be denoted by clS (resp. $intS$). A subset S of X is called a *preopen* set [11] (resp. α-set [15], *semi-open* [6]) if $S \subseteq int(cl(S))$ (resp. $S \subseteq int(cl(int(S))), S \subseteq cl(int(S))$). The complement of a preopen set (resp. α-set, *semi-open*) is called a *preclosed* set (resp. α-closed set, *semi-closed*). The family of all preopen sets (resp. α-sets, semi-open sets) in X will be denoted by $PO(X)$ (resp. $\alpha(X), SO(X)$). We know the family $\alpha(X)$ is a topology finer than the given topology on X.

A subset A of a topological space (X, τ) is said to be:

(a) g-closed [5] if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X ;

(b) gp-closed [7] if $pCl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X;

(c) gs-closed [3] if $sCl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X;

(d) $g\alpha$-closed [9] if $\tau^{\alpha}Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in X where $\tau^{\alpha} = \alpha(X)$;

(e) $g\alpha^{*}$-closed [8] if $\tau^{\alpha}Cl(A) \subseteq Int(U)$ whenever $A \subseteq U$ and U is α-open in X;

(f) $g\alpha^{**}$-closed [8] if $\tau^{\alpha}Cl(A) \subseteq Int(Cl(U))$ whenever $A \subseteq U$ and U is α-open in X;

(g) αg-closed [9] if $\tau^{\alpha}Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X;

(h) $\alpha^{**}g$-closed [9] if $\tau^{\alpha}Cl(A) \subseteq Int(Cl(U))$ whenever $A \subseteq U$ and U is open in X;

(i) g-open (resp. gp-open, gs-open, $g\alpha$-open, $g\alpha^{*}$-open, $g\alpha^{**}$-open, αg-open, $\alpha^{**}g$-open) if the complement A is g-closed (resp. gp-closed, gs-closed, $g\alpha$-closed, $g\alpha^{*}$-closed, $g\alpha^{**}$-closed, αg-closed, $\alpha^{**}g$-closed).

(j) The family of all g-open (resp. gp-open, gs-open, $g\alpha$-open, $g\alpha^{*}$-open, $g\alpha^{**}$-open, αg-open, $\alpha^{**}g$-open) sets in X will be denoted by $gO(X)$ (resp. $gpO(X), gsO(X), g\alpha O(X),\ g\alpha^{*}O(X), g\alpha^{**}O(X), \alpha gO(X), \alpha^{**}gO(X)$).

Definition 2.1 ([14]). Let X be a nonempty set. A subfamily w_X of the power set $P(X)$ is called a *weak structure* on X if it satisfies the following:

(1) $\emptyset \in w_X$ and $X \in w_X$.

(2) For $U_1, U_2 \in w_X$, $U_1 \cap U_2 \in w_X$.

Then the pair (X, w_X) is called a *w-space* on X. Then $V \in w_X$ is called a *w-open* set and the complement of a w-open set is a *w-closed* set.

Definition 2.2 ([14]). Let (X, w_X) be a w-space. For a subset A of X, the *w-closure* of A and the *w-interior* of A are defined as the following:

(1) $wCl(A) = \cap\{F : A \subseteq F, X - F \in w_X\}$.

(2) $wInt(A) = \cup\{U : U \subseteq A, U \in w_X\}$.

Theorem 2.3 ([14]). *Let (X, w_X) be a w-space and $A \subseteq X$.*

(1) If A is w-open, then $wInt(A) = A$.

(2) If A is w-closed, then $wCl(A) = A$.

A collection $\mathbf{H}$ of subsets of X is called an *m-family* [12] on X if $\cap \mathbf{H} \neq \emptyset$. Let $f : X \to Y$ be a function; then it is obvious $f(\mathbf{H}) = \{f(F) : F \in \mathbf{H}\}$ is an m-family on Y.

3. WO-Continuity; WK-Continuity

Definition 3.1. Let X be a nonempty set and let (X, τ) be a topological space. A subfamily w_τ of the power set $P(X)$ is called an associated *weak structure* on X if $\tau \subseteq w_\tau$. Then the pair (X, w_τ) is called an associated *w-space* with τ.

The collection of all w-open sets [14] (resp. w-closed sets) in a w-space X will be denoted by $WO(X)$ (resp. $WC(X)$). We set $W(x) = \{U \in WO(X) : x \in U\}$. The collection of all open sets (resp. closed sets) in a topological space X will be denoted by $O(X)$ (resp. $C(X)$). We set $O(x) = \{U \in O(X) : x \in U\}$.

Remark 3.2. Let X be a nonempty set and let (X, τ) be a topological space. The family $gO(X), g\alpha O(X)$, $g\alpha^{\square}O(X), g\alpha^{\square\square}O(X), \alpha gO(X)$ and $\alpha^{\square\square}gO(X)$ all associated weak structures on a topological space X.

Definition 3.3. Let $f : (X, w_\tau) \to (Y, \mu)$ be a function on an associated w-space with τ and a topological space (Y, μ). Then f is said to be

(1) *WO-continuous* if for $x \in X$ and $V \in O(f(x))$, there is $U \in W(x)$ such that $f(U) \subseteq V$,

(2) *WK-continuous* if for every open set V in Y, $f^{-1}(V)$ is a w-open set in X.

Theorem 3.4. *Let $f : (X, w_\tau) \to (Y, \mu)$ be a function on an associated w-space with τ and a topological space (Y, μ). Then the following statements are equivalent:*

(1) f is WO-continuous.

(2) $f(wCl(A)) \subseteq cl(f(A))$ for $A \subseteq X$.

(3) $wCl(f^{-1}(V)) \subseteq f^{-1}(cl(V))$ for $V \subseteq Y$.

(4) $f^{-1}(int(V)) \subseteq wInt(f^{-1}(V))$ for $V \subseteq Y$

Proof. (1) $\Rightarrow$ (2) Let $x \in wCl(A)$. If $f(x)$ is not in $cl(f(A))$, then there exists $V \in O(f(x))$ such that $V \cap f(A) = \emptyset$. By WO-continuity, there is $U \in O(x)$ such that $f(U) \subseteq V$ and so $f(U) \cap f(A) = \emptyset$. Hence $U \cap A = \emptyset$ and it is a contradiction.

(2) $\Rightarrow$ (3) Let $A = f^{-1}(B)$ for $B \subseteq Y$; then by (2), $f(wCl(A)) \subseteq cl(f(A)) = cl(f(f^{-1}(B))) \subseteq cl(B)$. Thus $wCl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

(3) $\Rightarrow$ (4) By Theorem 2.3, it is obvious.

(4) $\Rightarrow$ (1) Let $V \in O(f(x))$ for $x \in X$. Then $f(x) \in int(V)$ and by (4), $x \in f^{-1}(int(V)) \subseteq wInt(f^{-1}(V))$. There exists $U \in O(x)$ such that $x \in U \subseteq wInt(f^{-1}(V))$. □

Corollary 3.5. *Let $f : (X, w_\tau) \to (Y, \mu)$ be a function on an associated w-space with τ and a topological space (Y, μ). Then the following statements are equivalent:*

(1) f is WO-continuous.

(2) $f^{-1}(V) = wInt(f^{-1}(V))$ for every open set $V \in Y$.

(3) $f^{-1}(B) = wCl(f^{-1}(B))$ for every closed set $B \subseteq Y$.

Proof. From Theorem 2.3, it is obvious. □

Remark 3.6. Let $f : (X, w_\tau) \to (Y, \mu)$ be a function on an associated w-space and a topological space (Y, μ). If w_τ is $gO(X)$ (resp. $g\alpha O(X)$, $g\alpha^{\square}O(X), g\alpha^{\square\square}O(X)$), then f is $gO(X)$ (resp. $g\alpha O(X)$, $g\alpha^{\square}O(X)$, $g\alpha^{\square\square}O(X)$) continuous.

Theorem 3.7. *Let $f : (X, w_\tau) \to (Y, \mu)$ be a function on an associated w-space and a topological space (Y, μ). Then f is WK-continuous if and only if for every closed set F in Y, $f^{-1}(F)$ is w-closed in X.*

Proof. It is obvious. □

Every WK-continuous function is a WO-continuous function. But the converse may not be true as shown the following.

Example 3.8. For $X = Y = \{a, b, c\}$ let $\tau = \{\emptyset, \{b\}, X\}$ and $\mu = \{\emptyset, \{a, b\}, Y\}$. Consider an associated w-space $w_\tau = \{\emptyset, \{a\}, \{b\}, X\}$ with the topological space (X, τ). Let $f : (X, w_\tau) \to (Y, \mu)$ be a function defined by $f(x) = x$, for $x \in X$. Then f is WO-continuous, but not WK-continuous.

Remark 3.9. Let $f : (X, w_X) \to (Y, w_Y)$ be two w-spaces. Then f is said to be

(1) *W-continuous* [14] if for $x \in X$ and $V \in W(f(x))$, there is $U \in W(x)$ such that $f(U) \subset V$,

(2) *$W^{\square}$-continuous* [14] if for every $A \in W(f(x))$, $f^{-1}(A)$ is in $W(x)$.

Let $f : (X, w_\tau) \to (Y, w_\mu)$ be a function on two associated w-spaces with topological spaces (X, τ) and (Y, μ), respectively. Then we get the following implications but the converses may not be true:

$$\begin{array}{ccccc} \text{continuity} & \Rightarrow & WK\text{-continuity} & \Rightarrow & WO\text{-continuity} \\ & & \Uparrow & & \Uparrow \\ & & W^{\square}\text{-continuity} & \Rightarrow & W\text{-continuity} \end{array}$$

Example 3.10. For $X = \{a, b, c\}$ let $\tau = \{\emptyset, \{b\}, X\}$. Consider an associated w-space $w_\tau = \{\emptyset, \{a\}, \{b\}, X\}$ with the topological space (X, τ).

Let $f : (X, w_\tau) \to (X, w_\tau)$ be a function defined by $f(a) = c$, $f(b) = b$ and $f(c) = a$. Then f is both WK-continuous and WO-continuous, but neither $W^{\square}$-continuous nor W-continuous.

Let (X, w_X) be a w-space and $\mathbf{H}$ an m-family on X. Then we say that an m-family $\mathbf{H}$ *w-converges* [14] to $x \in X$ if $\mathbf{H}$ is finer than $W(x)$ i.e., $W(x) \subseteq \mathbf{H}$. If $\mathbf{F}$ is a filter base, we denote by $< \mathbf{F} >$ the filter generated by $\mathbf{H}$.

Theorem 3.11. *Let $f : (X, w_\tau) \to (Y, \mu)$ be a function on an associated w-space and a topological space (Y, μ). Then if f is WO-continuous, then for an m-family $\mathbf{H}$ w-converging to $x \in X$, a filter $< f(\mathbf{H}) >$ converges to $f(x)$.*

Proof. Suppose f is WO-continuous and $\mathbf{H}$ is an m-family w-converging to $x \in X$. By WO-continuity, for $V \in O(f(x))$, there exists $U \in W(x)$ such that $f(U) \subseteq V$. Since $f(W(x)) \subseteq f(\mathbf{H})$, $V \in < f(\mathbf{H}) >$ i.e., $O(f(x)) \subseteq < f(\mathbf{H}) >$. Hence the filter $< f(\mathbf{H}) >$ converges to $f(x)$. $\square$

From the relationship between WO-continuity and WK-continuity, we get the following.

Corollary 3.12. *Let $f : (X, w_\tau) \to (Y, \mu)$ be a function on an associated w-space and a topological space (Y, μ). Then if f is WK-continuous, then for an m-family $\mathbf{H}$ w-converging to $x \in X$, a filter $< f(\mathbf{H}) >$ converges to $f(x)$.*

Theorem 3.13. *Let $f : (X, w_\tau) \to (Y, w_\mu)$ be a bijective function on two associated w-spaces with topological spaces (X, τ) and (Y, μ), respectively. If $\mu = w_\mu$, then f is WK-continuous iff for an m-family $\mathbf{H}$ w-converging to $x \in X$, $f(\mathbf{H})$ w-converges to $f(x)$.*

Proof. Since $\mu = w_\mu$, every WK-continuous function is $W^{\Box}$-continuous. Hence from Theorem 4.7 in [14], we get the result. $\square$

Let (X, w_X) be a w-space and $Y \subseteq X$. $w \mid Y = \{V \subseteq X : V = U \cap Y \text{ for some } U \in W(x)\}$ is called a weak structure relative [14] to Y. $(Y, w \mid Y)$ is called subspace of the w-space X.

A w-space (X, w_X) is called *W-compact* [14] if every cover of X by w-open sets has finite subcover. A subset A of a w-space (X, w_X) is called *W-compact* [14] relative to A if every collection $\{U_i : i \in J\}$ of w-open subsets of X such that $A \subseteq \cup\{U_i : i \in J\}$, there exists a finite subset J_0 of J such that $A \subseteq \cup\{U_i : i \in J_0\}$. A subset A of a w-space (X, w_X) is said to be *W-compact* if A is W-compact as a subspace of X.

Theorem 3.14. *Let (X, w_τ) be an associated w-space and (Y, μ) a topological space. If $f : (X, w_\tau) \to (Y, \mu)$ is WK-continuous and A is a W-compact subset of X, then $f(A)$ is compact in Y.*

Proof. Let $\{U_i : i \in J\}$ be a cover of $f(A)$ by open subsets of Y. Then $A \subseteq \cup\{f^{-1}(U_i) : i \in J\}$, where $\{f^{-1}(U_i) : i \in J\}$ is a cover of A by w-open subsets of X. Since A is W-compact, there exists a finite subset J_0 of J such that $A \subseteq \cup\{f^{-1}(U_i) : i \in J_0\}$. Hence $f(A) \subseteq \cup\{U_i : i \in J_0\}$. □

Corollary 3.15. *Let $f : (X, w_\tau) \to (Y, w_\mu)$ be a function on two associated w-spaces with topological spaces (X, τ) and (Y, μ), respectively. If f is $WK^{\square}$-continuous and A is a W-compact subset of X, then $f(A)$ is compact in Y.*

Proof. From Remark 3.9, it is obvious. □

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