WO-CONTINUITY AND WK-CONTINUITY
ON ASSOCIATED w-spaces

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Abstract: We introduce the concepts of WO-continuity and WK-continuity on associated w-spaces. We investigate some properties and relationships between WO-continuity, WK-continuity, W-continuity, W*,-continuity and continuity on associated w-spaces.

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1. Introduction

In [16], Siwiec introduced the notions of weak neighborhoods and weak base in a topological space. The author introduced the weak neighborhood systems defined by using the notion of weak neighborhoods in [13]. The weak neighborhood system induces a weak neighborhood space (briefly WNS) which is independent of neighborhood spaces [4] and general topological spaces [2]. In [13], the author introduced the notion of new interior operator and closure operator on a WNS. We also introduced the notion of weak structure which is

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defined by the properties of new interior operator and closure operator in a WNS. The set of all $g$-open subsets [5] of a topological space is a kind of weak structure.

In this paper, we introduce the concepts of $WO$-continuity and $WK$-continuity on associated $w$-spaces. In particular, we investigate some properties and relationships between $WO$-continuity, $WK$-continuity, $W$-continuity, $W^*$-continuity and continuity on associated $w$-spaces.

2. Preliminaries

Let $X$ be a topological space and $S \subseteq X$. The closure (resp. interior) of $S$ will be denoted by $clS$ (resp. $intS$). A subset $S$ of $X$ is called a preopen set [11] (resp. $\alpha$-set [15], semi-open [6]) if $S \subseteq int(cl(S))$ (resp. $S \subseteq int(int(S)))$. The complement of a preopen set (resp. $\alpha$-set, semi-open) is called a preclosed set (resp. $\alpha$-closed set, semi-closed). The family of all preopen sets (resp. $\alpha$-sets, semi-open sets) in $X$ will be denoted by $PO(X)$ (resp. $\alpha(X)$, $SO(X)$). We know the family $\alpha(X)$ is a topology finer than the given topology on $X$.

A subset $A$ of a topological space $(X, \tau)$ is said to be:

(a) $g$-closed [5] if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$
(b) $gp$-closed [7] if $pCl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$
(c) $gs$-closed [3] if $sCl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$
(d) $g\alpha$-closed [9] if $\tau^\alpha Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $X$ where $\tau^\alpha = \alpha(X)$
(e) $g\alpha^*$-closed [8] if $\tau^\alpha Cl(A) \subseteq Int(U)$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $X$
(f) $g\alpha^{**}$-closed [8] if $\tau^\alpha Cl(A) \subseteq Int(Cl(U))$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $X$
(g) $\alpha g$-closed [9] if $\tau^\alpha Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$
(h) $\alpha^{**}g$-closed [9] if $\tau^\alpha Cl(A) \subseteq Int(Cl(U))$ whenever $A \subseteq U$ and $U$ is open in $X$
(i) $g$-open (resp. $gp$-open, $gs$-open, $g\alpha$-open, $g\alpha^*$-open, $g\alpha^{**}$-open, $\alpha g$-open, $\alpha^{**}g$-open) if the complement of $A$ is $g$-closed (resp. $gp$-closed, $gs$-closed, $g\alpha$-closed, $g\alpha^*$-closed, $g\alpha^{**}$-closed, $\alpha g$-closed, $\alpha^{**}g$-closed).
(j) The family of all $g$-open (resp. $gp$-open, $gs$-open, $g\alpha$-open, $g\alpha^*$-open, $g\alpha^{**}$-open, $\alpha g$-open, $\alpha^{**}g$-open) sets in $X$ will be denoted by $gO(X)$ (resp. $gpO(X), gsO(X), g\alpha O(X), g\alpha^*O(X), g\alpha^{**}O(X), \alpha gO(X), \alpha^{**}gO(X)$.
Definition 2.1 ([14]). Let $X$ be a nonempty set. A subfamily $w_X$ of the power set $P(X)$ is called a weak structure on $X$ if it satisfies the following:

1. $\emptyset \in w_X$ and $X \in w_X$.
2. For $U_1, U_2 \in w_X$, $U_1 \cap U_2 \in w_X$.

Then the pair $(X, w_X)$ is called a $w$-space on $X$. Then $V \in w_X$ is called a $w$-open set and the complement of a $w$-open set is a $w$-closed set.

Definition 2.2 ([14]). Let $(X, w_X)$ be a $w$-space. For a subset $A$ of $X$, the $w$-closure of $A$ and the $w$-interior of $A$ are defined as the following:

1. $wCl(A) = \cap \{F : A \subseteq F, X - F \in w_X\}$.
2. $wInt(A) = \cup \{U : U \subseteq A, U \in w_X\}$.

Theorem 2.3 ([14]). Let $(X, w_X)$ be a $w$-space and $A \subseteq X$.

1. If $A$ is $w$-open, then $wInt(A) = A$.
2. If $A$ is $w$-closed, then $wCl(A) = A$.

A collection $H$ of subsets of $X$ is called an $m$-family [12] on $X$ if $\cap H \neq \emptyset$. Let $f : X \to Y$ be a function; then it is obvious $f(H) = \{f(F) : F \in H\}$ is an $m$-family on $Y$.

3. $WO$-Continuity; $WK$-Continuity

Definition 3.1. Let $X$ be a nonempty set and let $(X, \tau)$ be a topological space. A subfamily $w_\tau$ of the power set $P(X)$ is called an associated weak structure on $X$ if $\tau \subseteq w_\tau$. Then the pair $(X, w_\tau)$ is called an associated $w$-space with $\tau$.

The collection of all $w$-open sets [14] (resp. $w$-closed sets) in a $w$-space $X$ will be denoted by $WO(X)$ (resp. $WC(X)$). We set $W(x) = \{U \in WO(X) : x \in U\}$. The collection of all open sets (resp. closed sets) in a topological space $X$ will be denoted by $O(X)$ (resp. $C(X)$). We set $O(x) = \{U \in O(X) : x \in U\}$.

Remark 3.2. Let $X$ be a nonempty set and let $(X, \tau)$ be a topological space. The family $gO(X), g\alpha O(X), g\alpha^\ast O(X), g\alpha^\ast\ast O(X), \alpha O(X)$ and $\alpha^\ast\ast gO(X)$ all associated weak structures on a topological space $X$.

Definition 3.3. Let $f : (X, w_\tau) \to (Y, \mu)$ be a function on an associated $w$-space with $\tau$ and a topological space $(Y, \mu)$. Then $f$ is said to be
(1) **WO-continuous** if for \( x \in X \) and \( V \in O(f(x)) \), there is \( U \in W(x) \) such that \( f(U) \subseteq V \),

(2) **WK-continuous** if for every open set \( V \) in \( Y \), \( f^{-1}(V) \) is a \( w \)-open set in \( X \).

**Theorem 3.4.** Let \( f : (X, w\tau) \rightarrow (Y, \mu) \) be a function on an associated \( w \)-space with \( \tau \) and a topological space \( (Y, \mu) \). Then the following statements are equivalent:

1. \( f \) is WO-continuous.
2. \( f(w\text{Cl}(A)) \subseteq \text{cl}(f(A)) \) for \( A \subseteq X \).
3. \( w\text{Cl}(f^{-1}(V)) \subseteq f^{-1}(\text{cl}(V)) \) for \( V \subseteq Y \).
4. \( f^{-1}(\text{int}(V)) \subseteq w\text{Int}(f^{-1}(V)) \) for \( V \subseteq Y \)

**Proof.** (1) \( \Rightarrow \) (2) Let \( x \in w\text{Cl}(A) \). If \( f(x) \) is not in \( \text{cl}(f(A)) \), then there exists \( V \in O(f(x)) \) such that \( V \cap f(A) = \emptyset \). By WO-continuity, there is \( U \in O(x) \) such that \( f(U) \subseteq V \) and so \( f(U) \cap f(A) = \emptyset \). Hence \( U \cap A = \emptyset \) and it is a contradiction.

(2) \( \Rightarrow \) (3) Let \( A = f^{-1}(B) \) for \( B \subseteq Y \); then by (2), \( f(w\text{Cl}(A)) \subseteq \text{cl}(f(A)) = \text{cl}(f(f^{-1}(B))) \subseteq \text{cl}(B) \). Thus \( w\text{Cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B)) \).

(3) \( \Rightarrow \) (4) By Theorem 2.3, it is obvious.

(4) \( \Rightarrow \) (1) Let \( V \in O(f(x)) \) for \( x \in X \). Then \( f(x) \in \text{int}(V) \) and by (4), \( x \in f^{-1}(\text{int}(V)) \subseteq w\text{Int}(f^{-1}(V)) \). There exists \( U \in O(x) \) such that \( x \in U \subseteq w\text{Int}(f^{-1}(V)) \).

**Corollary 3.5.** Let \( f : (X, w\tau) \rightarrow (Y, \mu) \) be a function on an associated \( w \)-space with \( \tau \) and a topological space \( (Y, \mu) \). Then the following statements are equivalent:

1. \( f \) is WO-continuous.
2. \( f^{-1}(V) = w\text{Int}(f^{-1}(V)) \) for every open set \( V \in Y \).
3. \( f^{-1}(B) = w\text{Cl}(f^{-1}(B)) \) for every closed set \( B \subseteq Y \).

**Proof.** From Theorem 2.3, it is obvious.

**Remark 3.6.** Let \( f : (X, w\tau) \rightarrow (Y, \mu) \) be a function on an associated \( w \)-space and a topological space \( (Y, \mu) \). If \( w\tau \) is \( gO(X) \) (resp. \( g\alpha O(X) \), \( g\alpha^*O(X) \), \( g\alpha^{**}O(X) \)), then \( f \) is \( gO(X) \) (resp. \( g\alpha O(X) \), \( g\alpha^*O(X) \), \( g\alpha^{**}O(X) \)) continuous.
**Theorem 3.7.** Let \( f : (X, w_\tau) \rightarrow (Y, \mu) \) be a function on an associated \( w \)-space and a topological space \((Y, \mu)\). Then \( f \) is \( WK \)-continuous if and only if for every closed set \( F \) in \( Y \), \( f^{-1}(F) \) is \( w \)-closed in \( X \).

**Proof.** It is obvious. \( \square \)

Every \( WK \)-continuous function is a \( WO \)-continuous function. But the converse may not be true as shown the following.

**Example 3.8.** For \( X = Y = \{a, b, c\} \) let \( \tau = \{\emptyset, \{b\}, X\} \) and \( \mu = \{\emptyset, \{a, b\}, Y\} \). Consider an associated \( w \)-space \( w_\tau = \{\emptyset, \{a\}, \{b\}, X\} \) with the topological space \((X, \tau)\). Let \( f : (X, w_\tau) \rightarrow (Y, \mu) \) be a function defined by \( f(x) = x \), for \( x \in X \). Then \( f \) is \( WO \)-continuous, but not \( WK \)-continuous.

**Remark 3.9.** Let \( f : (X, w_X) \rightarrow (Y, w_Y) \) be two \( w \)-spaces. Then \( f \) is said to be

1. \( W \)-continuous [14] if for \( x \in X \) and \( V \in W(f(x)) \), there is \( U \in W(x) \) such that \( f(U) \subseteq V \),

2. \( W^* \)-continuous [14] if for every \( A \in W(f(x)) \), \( f^{-1}(A) \) is in \( W(x) \).

Let \( f : (X, w_\tau) \rightarrow (Y, w_\mu) \) be a function on two associated \( w \)-spaces with topological spaces \((X, \tau)\) and \((Y, \mu)\), respectively. Then we get the following implications but the converses may not be true:

\[
\text{continuity } \Rightarrow WK\text{-continuity } \Rightarrow WO\text{-continuity} \\
\text{\uparrow} \quad \text{\uparrow} \\
W^*\text{-continuity } \Rightarrow W\text{-continuity}
\]

**Example 3.10.** For \( X = \{a, b, c\} \) let \( \tau = \{\emptyset, \{b\}, X\} \). Consider an associated \( w \)-space \( w_\tau = \{\emptyset, \{a\}, \{b\}, X\} \) with the topological space \((X, \tau)\).

Let \( f : (X, w_\tau) \rightarrow (X, w_\tau) \) be a function defined by \( f(a) = c \), \( f(b) = b \) and \( f(c) = a \). Then \( f \) is both \( WK \)-continuous and \( WO \)-continuous, but neither \( W^* \)-continuous nor \( W \)-continuous.

Let \( (X, w_X) \) be a \( w \)-space and \( H \) an \( m \)-family on \( X \). Then we say that an \( m \)-family \( H \) \( w \)-converges [14] to \( x \in X \) if \( H \) is finer than \( W(x) \) i.e., \( W(x) \subseteq H \). If \( F \) is a filter base, we denote by \( < F > \) the filter generated by \( H \).
Theorem 3.11. Let \( f : (X, w_\tau) \to (Y, \mu) \) be a function on an associated \( w \)-space and a topological space \((Y, \mu)\). Then if \( f \) is \( W\text{O} \)-continuous, then for an \( m \)-family \( H \) \( w \)-converging to \( x \in X \), a filter \(< f(H) >\) converges to \( f(x) \).

Proof. Suppose \( f \) is \( W\text{O} \)-continuous and \( H \) is an \( m \)-family \( w \)-converging to \( x \in X \). By \( W\text{O} \)-continuity, for \( V \in O(f(x)) \), there exists \( U \in W(x) \) such that \( f(U) \subseteq V \). Since \( f(W(x)) \subseteq f(H) \), \( V \in < f(H) > \) i.e., \( O(f(x)) \subseteq < f(H) > \). Hence the filter \(< f(H) >\) converges to \( f(x) \).

From the relationship between \( W\text{O} \)-continuity and \( WK \)-continuity, we get the following.

Corollary 3.12. Let \( f : (X, w_\tau) \to (Y, \mu) \) be a function on an associated \( w \)-space and a topological space \((Y, \mu)\). Then if \( f \) is \( WK \)-continuous, then for an \( m \)-family \( H \) \( w \)-converging to \( x \in X \), a filter \(< f(H) >\) converges to \( f(x) \).

Theorem 3.13. Let \( f : (X, w_\tau) \to (Y, w_\mu) \) be a bijective function on two associated \( w \)-spaces with topological spaces \((X, \tau)\) and \((Y, \mu)\), respectively. If \( \mu = w_\mu \), then \( f \) is \( WK \)-continuous iff for an \( m \)-family \( H \) \( w \)-converging to \( x \in X \), \( f(H) \) \( w \)-converges to \( f(x) \).

Proof. Since \( \mu = w_\mu \), every \( WK \)-continuous function is \( W^* \)-continuous. Hence from Theorem 4.7 in [14], we get the result.

Let \((X, w_X)\) be a \( w \)-space and \( Y \subseteq X \). \( w | Y = \{V \subseteq X : V = U \cap Y \text{ for some } U \in W(x)\} \) is called a weak structure relative [14] to \( Y \). \((Y, w | Y)\) is called subspace of the \( w \)-space \( X \).

A \( w \)-space \((X, w_X)\) is called \( W \)-\textit{compact} [14] if every cover of \( X \) by \( w \)-open sets has finite subcover. A subset \( A \) of a \( w \)-space \((X, w_X)\) is called \( W \)-\textit{compact} [14] relative to \( A \) if every collection \( \{U_i : i \in J\} \) of \( w \)-open subsets of \( X \) such that \( A \subseteq \bigcup \{U_i : i \in J\} \), there exists a finite subset \( J_0 \) of \( J \) such that \( A \subseteq \bigcup \{U_i : i \in J_0\} \). A subset \( A \) of a \( w \)-space \((X, w_X)\) is said to be \( W \)-\textit{compact} if \( A \) is \( W \)-\textit{compact} as a subspace of \( X \).

Theorem 3.14. Let \((X, w_\tau)\) be an associated \( w \)-space and \((Y, \mu)\) a topological space. If \( f : (X, w_\tau) \to (Y, \mu) \) is \( WK \)-continuous and \( A \) is a \( W \)-\textit{compact} subset of \( X \), then \( f(A) \) is compact in \( Y \).
Proof. Let \( \{U_i : i \in J\} \) be a cover of \( f(A) \) by open subsets of \( Y \). Then \( A \subseteq \bigcup \{f^{-1}(U_i) : i \in J\} \), where \( \{f^{-1}(U_i) : i \in J\} \) is a cover of \( A \) by \( w \)-open subsets of \( X \). Since \( A \) is \( W \)-compact, there exists a finite subset \( J_0 \) of \( J \) such that \( A \subseteq \bigcup \{f^{-1}(U_i) : i \in J_0\} \). Hence \( f(A) \subseteq \bigcup \{U_i : i \in J_0\} \). \( \square \)

**Corollary 3.15.** Let \( f: (X, w_\tau) \to (Y, w_\mu) \) be a function on two associated \( w \)-spaces with topological spaces \( (X, \tau) \) and \( (Y, \mu) \), respectively. If \( f \) is \( WK^* \)-continuous and \( A \) is a \( W \)-compact subset of \( X \), then \( f(A) \) is compact in \( Y \).

Proof. From Remark 3.9, it is obvious. \( \square \)

**References**


