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SYMMETRIC PROPERTIES FOR THE GENERALIZED TWISTED (h, q)-EULER POLYNOMIALS ASSOCIATED WITH p-ADIC INVARIANT q-INTEGRAL ON $\mathbb{Z}_p$

C.S. Ryoo
Department of Mathematics
Hannam University
Daejeon, 306-791, KOREA

Abstract: In this paper, we study the symmetry for generalized twisted (h, q)-Euler numbers $E^{(h)}_{n,\chi,q,\zeta}$ and polynomials $E^{(h)}_{n,\chi,q,\zeta}(x)$. We obtain some interesting identities of the power sums and generalized twisted (h, q)-Euler polynomials $E^{(h)}_{n,\chi,q,\zeta}(x)$ using the symmetric properties for the p-adic invariant q-integral on $\mathbb{Z}_p$.

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Key Words: Euler numbers and polynomials, generalized twisted (h, q)-Euler numbers and polynomials, symmetric properties, power sums

1. Introduction

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the

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normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$(see [1-9]). Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q}.$$

Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \le 1$ in the present p-adic case. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $g \in UD(\mathbb{Z}_p)$ the fermionic p-adic invariant q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} g(x)(-q)^x, \text{ see [1, 2]} . \tag{1.1}$$

If we take $g_n(x) = g(x+n)$ in (1.1), then we see that

$$q^n I_q(g_n) + (-1)^{n-1} I_q(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l). \tag{1.2}$$

Let $T_p = \cup_{N\ge 1} C_{p^N} = \lim_{N\to\infty} C_{p^N}$, where $C_{p^N} = \{\zeta | \zeta^{p^N} = 1\}$ is the cyclic group of order p^N. For $\zeta \in T_p$, we denote by $\phi_\zeta : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \zeta^x$(see [7, 8]).

Let χ be the primitive Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$ and $\zeta \in T_p$. We assume that $h \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|q-1|_p < 1$. Let $g(y) = \chi(y)\phi_\zeta(y) q^{(h-1)y} e^{(y+x)t}$.

By (1.1), we derive

$$\begin{aligned}\int_X \chi(y)\phi_\zeta(y) q^{(h-1)y} e^{(y+x)t} d\mu_{-q}(y) &= \frac{[2]_q \sum_{a=0}^{d-1} \chi(a)(-1)^a \zeta^a q^{ha} e^{at}}{\zeta^d q^{hd} e^{dt} + 1} e^{xt} \\ &= \sum_{n=0}^{\infty} E_{n,\chi,q,\zeta}^{(h)}(x) \frac{t^n}{n!}.\end{aligned} \tag{1.3}$$

By using Taylor series of $e^{(y+x)t}$ in the above equation (1.3), we obtain

$$\sum_{n=0}^{\infty} \left(\int_X \chi(y)\phi_\zeta(y) q^{(h-1)y} (y+x)^n d\mu_{-q}(y) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} E_{n,\chi,q,\zeta}^{(h)}(x) \frac{t^n}{n!}.$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we have the Witt formula for the generalized twisted (h,q)- Euler polynomials attached to χ as follows:

Theorem 1.1. *For positive integers n and $\zeta \in T_p$, we have*

$$E^{(h)}_{n,\chi,q,\zeta}(x) = \int_X \chi(y)\phi_\zeta(y)(y+x)^n d\mu_{-q}(y). \tag{1.4}$$

Observe that for $x = 0$, the equation (1.4) reduces to (1.5).

Corollary 1.2. *For positive integers n and $\zeta \in T_p$, we have*

$$E^{(h)}_{n,\chi,q,\zeta} = \int_X \chi(y)y^n\phi_\zeta(y)d\mu_{-q}(y). \tag{1.5}$$

By (1.4) and (1.5), we obrian the following theorem.

Theorem 1.3. *For positive integers n and $\zeta \in T_p$, we have*

$$E^{(h)}_{n,\chi,q,\zeta}(x) = \sum_{l=0}^{n}\binom{n}{l}E^{(h)}_{l,\chi,q,\zeta}x^{n-l}.$$

2. Symmetry for the Generalized Twisted (h,q)-Euler Polynomials

In this section, we assume that $q \in \mathbb{C}_p$ and $\zeta \in T_p$. We give some interesting identities of the power sums and generalized twisted (h,q)-Euler polynomials $E^{(h)}_{n,\chi,q,\zeta}(x)$ using the symmetric properties for the p-adic invariant q-integral on $\mathbb{Z}_p$. If n is odd from (1.2), we obtain

$$q^n I_q(g_n) + I_q(g) = [2]_q \sum_{l=0}^{n-1}(-1)^l q^l g(l). \tag{2.1}$$

Substituting $g(x) = \chi(x)\zeta^x q^{(h-1)x}e^{xt}$ into the above, we obtain

$$\begin{aligned}&\zeta^{nd}q^{hnd}\int_X \chi(x)\zeta^x q^{(h-1)x}e^{(x+nd)t}d\mu_{-q}(x) + \int_X \chi(x)\zeta^x q^{(h-1)x}e^{xt}d\mu_{-q}(x)\\ &= [2]_q \sum_{j=0}^{nd-1}(-1)^j\chi(j)\zeta^j q^{hj}e^{jt}.\end{aligned} \tag{2.2}$$

For $k \in \mathbb{N} \cup \{0\}$, let us define the p-adic functional $T^{(h)}_{k,\chi,q,\zeta}(n)$ as follows:

$$T^{(h)}_{k,\chi,q,\zeta}(n) = \sum_{l=0}^{n}(-1)^l \chi(l) q^{hl}\zeta^l l^k. \tag{2.3}$$

After some elementary calculations, we have

$$\begin{aligned}
&\zeta^{nd}q^{hnd}e^{ndt}\int_X \chi(x)\zeta^x q^{(h-1)x}e^{xt}d\mu_{-q}(x) + \int_X \chi(x)\zeta^x q^{(h-1)x}e^{xt}d\mu_{-q}(x) \\
&= \left(1+\zeta^{nd}q^{hnd}e^{ndt}\right)\frac{[2]_q\sum_{a=0}^{d-1}\chi(a)(-1)^a\zeta^a q^{ha}e^{at}}{\zeta^d q^{hd}e^{dt}+1}.
\end{aligned}$$

From the above, we get

$$\begin{aligned}
&q^{hnd}\int_X \chi(x)\zeta^{x+nd}q^{(h-1)x}e^{(x+nd)t}d\mu_{-q}(x) + \int_X \chi(x)\zeta^x q^{(h-1)x}e^{xt}d\mu_{-q}(x) \\
&= \frac{[2]_q\int_X \chi(x)\zeta^x q^{(h-1)x}e^{xt}d\mu_{-q}(x)}{\int_{\mathbb{Z}_p}\zeta^{ndx}q^{(hnd-1)x}e^{ndtx}d\mu_{-q}(x)}.
\end{aligned} \tag{2.4}$$

By substituting Taylor series of e^{xt} into (2.2), we obtain

$$\begin{aligned}
&\sum_{m=0}^{\infty}\left(\zeta^{nd}q^{hnd}\int_X \chi(x)\zeta^x q^{(h-1)x}(x+nd)^m d\mu_{-q}(x)\right. \\
&\qquad \left. + \int_X \chi(x)\zeta^x q^{(h-1)x}x^m d\mu_{-q}(x)\right)\frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty}\left([2]_q\sum_{j=0}^{nd-1}(-1)^j\chi(j)\zeta^j q^{hj}j^m\right)\frac{t^m}{m!}.
\end{aligned}$$

By comparing coefficients $\frac{t^m}{m!}$ in the above equation, we obtain

$$\begin{aligned}
&\zeta^{nd}q^{hnd}\sum_{k=0}^{m}\binom{m}{k}(nd)^{m-k}\int_X \chi(x)\zeta^x q^{(h-1)x}x^k d\mu_{-q}(x) \\
&\quad + \int_X \chi(x)\zeta^x q^{(h-1)x}x^m d\mu_{-q}(x) = [2]_q\sum_{j=0}^{nd-1}(-1)^j\chi(j)\zeta^j q^{hj}j^m.
\end{aligned}$$

By using (2.3), we have

$$\zeta^{nd}q^{hnd}\sum_{k=0}^{m}\binom{m}{k}(nd)^{m-k}\int_X \chi(x)\zeta^x q^{(h-1)x}x^k d\mu_{-q}(x)$$
$$+\int_X \chi(x)\zeta^x q^{(h-1)x}x^m d\mu_{-q}(x)=[2]_q T^{(h)}_{m,\chi,q,\zeta}(nd-1).$$

Hence, we arrive at the following theorem:

Theorem 2.1. *Let n be odd positive integer. Then we have*

$$\frac{\int_X \chi(x)\zeta^x q^{(h-1)x}e^{xt}d\mu_{-q}(x)}{\int_{\mathbb{Z}_p}\zeta^{ndx}q^{(hnd-1)x}e^{ndtx}d\mu_{-q}(x)}=\sum_{m=0}^{\infty}\left(T^{(h)}_{m,\chi,q,\zeta}(nd-1)\right)\frac{t^m}{m!}.$$

Let w_1 and w_2 be odd positive integers. By Theorem 2.1 and after some elementary calculations, we have the following theorem.

Theorem 2.2. *Let w_1 and w_2 be odd positive integers. Then we have*

$$\frac{\int_X \chi(x)\zeta^{w_2x}q^{(w_2-1)x}e^{w_2xt}d\mu_{-q}(x)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2dx}q^{(w_1w_2d-1)x}e^{w_1w_2tdx}d\mu_{-q}(x)}=\sum_{m=0}^{\infty}\left(T^{(w_2)}_{m,\chi,q,\zeta^{w_2}}(w_1d-1)w_2^m\right)\frac{t^m}{m!}.$$

Then we set

$$S(w_1,w_2)=$$
$$\frac{\int_X\int_X \chi(x_1)\chi(x_2)\zeta^{(w_1x_1+w_2x_2)}q^{(w_1-1)x_1}q^{(w_2-1)x_2}e^{(w_1x_1+w_2x_2+w_1w_2x)t}d\mu_{-q}(x_1)d\mu_{-q}(x_2)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2dx}q^{(w_1w_2d-1)x}e^{w_1w_2dxt}d\mu_{-q}(x)}.$$

By Theorem 2.2 and $S(w_1,w_2)$, after elementary calculations, we have

$$S(w_1,w_2)=\left(\int_X \chi(x_1)\zeta^{w_1x_1}q^{(w_1-1)x_1}e^{(w_1x_1+w_1w_2x)t}d\mu_{-q}(x_1)\right)$$
$$\times\left(\frac{\int_X \chi(x_2)\zeta^{w_2x_2}q^{(w_2-1)x_2}e^{x_2w_2t}d\mu_{-q}(x_2)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2dx}q^{(w_1w_2d-1)x}e^{w_1w_2dxt}d\mu_{-q}(x)}\right)$$
$$=\left(\sum_{m=0}^{\infty}E^{(w_1)}_{m,\chi,q,\zeta^{w_1}}(w_2x)w_1^m\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}T^{(w_2)}_{m,\chi,q,\zeta^{w_2}}(w_1d-1)w_2^m\frac{t^m}{m!}\right).$$

By using Cauchy product in the above, we have

$$S(w_1,w_2)=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}E^{(w_1)}_{j,\chi,q,\zeta^{w_1}}(w_2x)w_1^jT^{(w_2)}_{m-j,\chi,q,\zeta^{w_2}}(w_1d-1)w_2^{m-j}\right)\frac{t^m}{m!}.$$

From the symmetry of $S(w_1,w_2)$ in w_1 and w_2, we also see that

$$\begin{aligned}S(w_1,w_2)&=\left(\int_X\chi(x_2)\zeta^{w_2x_2}q^{(w_2-1)x_2}e^{(w_2x_2+w_1w_2x)t}d\mu_{-q}(x_2)\right)\\&\qquad\times\left(\frac{\int_X\chi(x_1)\zeta^{w_1x_1}q^{(w_1-1)x_1}e^{x_1w_1t}d\mu_{-q}(x_1)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2dx}q^{(w_1w_2d-1)x}e^{w_1w_2dxt}d\mu_{-q}(x)}\right)\\&=\left(\sum_{m=0}^{\infty}E^{(w_2)}_{m,\chi,q,\zeta^{w_2}}(w_1x)w_2^m\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}T^{(w_1)}_{m,\chi,q,\zeta^{w_1}}(w_2d-1)w_1^m\frac{t^m}{m!}\right).\end{aligned}$$

Thus we have

$$S(w_1,w_2)=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}E^{(w_2)}_{j,\chi,q,\zeta^{w_2}}(w_1x)w_2^jT^{(w_1)}_{m-j,\chi,q,\zeta^{w_1}}(w_2d-1)w_1^{m-j}\right)\frac{t^m}{m!}.$$

Thus we arrive at the following theorem:

Theorem 2.3. *Let w_1 and w_2 be odd positive integers. Then we obtain*

$$\begin{aligned}&\sum_{j=0}^{m}\binom{m}{j}w_1^{m-j}w_2^jE^{(w_2)}_{j,\chi,q,\zeta^{w_2}}(w_1x)T^{(w_1)}_{m-j,\chi,q,\zeta^{w_1}}(w_2d-1)\\&=\sum_{j=0}^{m}\binom{m}{j}w_1^jw_2^{m-j}E^{(w_1)}_{j,\chi,q,\zeta^{w_1}}(w_2x)T^{(w_2)}_{m-j,\chi,q,\zeta^{w_2}}(w_1d-1),\end{aligned}$$

where $E^{(w_1)}_{k,\chi,q,\zeta}(x)$ and $T^{(w_1)}_{m,\chi,q,\zeta}(k)$ denote generalized twisted (h,q)-Euler polynomials and p-adic functional, respectively.

By Theorem 2.3, we have the following corollary.

Corollary 2.4. *Let w_1 and w_2 be odd positive integers. Then we obtain*

$$\begin{aligned}&\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{m-k}w_2^jx^{j-k}E^{(w_2)}_{k,\chi,q,\zeta^{w_2}}T^{(w_1)}_{m-j,\chi,q,\zeta^{w_1}}(w_2d-1)\\&=\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^jw_2^{m-k}x^{j-k}E^{(w_1)}_{k,\chi,q,\zeta^{w_1}}T^{(w_2)}_{m-j,\chi,q,\zeta^{w_2}}(w_1d-1).\end{aligned}$$

Now we will derive another interesting identities for generalized twisted (h, q)-Euler polynomials using the symmetric property of $S(w_1, w_2)$.

$$\begin{aligned}
&S(w_1, w_2) \\
&= \sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) q^{w_2 j} \int_X \chi(x_1) \zeta^{w_1 x_1} q^{(w_1-1)x_1} e^{\left(x_1+w_2x+j\frac{w_2}{w_1}\right)(w_1 t)} d\mu_{-q}(x_1) \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) \zeta^{w_2 j} q^{w_2 j} E_{n,\chi,q,\zeta^{w_1}}^{(w_1)} \left(w_2 x + j\frac{w_2}{w_1} \right) w_1^n \right) \frac{t^n}{n!}.
\end{aligned}$$

By using the symmetry property in the above equation, we also have

$$\begin{aligned}
&S(w_1, w_2) \\
&= \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) \zeta^{w_1 j} q^{w_1 j} \int_X \chi(x_2) \zeta^{w_2 x_2} q^{(w_2-1)x_2} e^{\left(x_2+w_1x+j\frac{w_1}{w_2}\right)(w_2 t)} d\mu_{-q}(x_2) \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) \zeta^{w_1 j} q^{w_1 j} E_{n,\chi,q,\zeta^{w_2}}^{(w_2)} \left(w_1 x + j\frac{w_1}{w_2} \right) w_2^n \right) \frac{t^n}{n!}.
\end{aligned}$$

Thus we obtain the following theorem.

Theorem 2.5. *Let w_1 and w_2 be odd positive integers and $\zeta \in T_p$. Then we obtain*

$$\begin{aligned}
&\sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) \zeta^{w_2 j} q^{w_2 j} E_{n,\chi,q,\zeta^{w_1}}^{(w_1)} \left(w_2 x + j\frac{w_2}{w_1} \right) w_1^n \\
&= \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) \zeta^{w_1 j} q^{w_1 j} E_{n,\chi,q,\zeta^{w_2}}^{(w_2)} \left(w_1 x + j\frac{w_1}{w_2} \right) w_2^n.
\end{aligned}$$

If we take $x = 0$ in Theorem 2.5, we also derive the interesting identity for generalized twisted (h, q)-Euler numbers as follows:

Corollary 2.6. *Let w_1 and w_2 be odd positive integers. Then we obtain*

$$\sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) \zeta^{w_2 j} q^{w_2 j} E^{(w_1)}_{n,\chi,q,\zeta^{w_1}} \left(\frac{j w_2}{w_1}\right) w_1^n$$
$$= \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) \zeta^{w_1 j} q^{w_1 j} E^{(w_2)}_{n,\chi,q,\zeta^{w_2}} \left(\frac{j w_1}{w_2}\right) w_2^n.$$

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