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ON THE SEGRE UPPER BOUND OF THE REGULARITY FOR NON-FAT POINT SCHEMES IN PROJECTIVE SPACES

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Abstract: We study the generalized Segre bound in projective space (mainly in the plane) with respect to zero-dimensional schemes which are more general that the fat point schemes.

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Fix a finite set $S \subset \mathbb{P}^n$. For each $p \in S$ fix an integer $m_p > 0$ and set $Z := \sum_{p\in S} m_p p$ (a fat point scheme). For any set $E \subseteq \mathbb{P}^n$ set $w(E) := \sum_{p\in S\cap E} m_p$. Fix an integer $d > 0$. The generalized Segre condition says that $h^1(\mathcal{I}_Z(d)) = 0$ (i.e. the zero-dimensional scheme Z has index of regularity $\leq d$) if for all $c = 1, \dots, n$ and all c-dimensional linear spaces $A \subseteq \mathbb{P}^n$ we have $w(A) \leq cd + 1$. The conditions for lines implies $m_p \leq d + 1$ for all p and that if $m_p = d + 1$ for

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some p, then $\sharp(S) = 1$ and $h^i(\mathcal{I}_Z(d)) = 0$, $i = 0, 1$. It is also easy to prove the generalized Segre conjecture if $m_p = d$ for some p. This conjecture is known only in some cases: if S is in linearly general position ([5]), if $n \le 3$ ([4], [6], [7], [8]), if $\sharp(S) = n + 2$ and S spans $\mathbb{P}^n$ ([3]) and if $\sharp(S) = n + 3$, S spans $\mathbb{P}^n$ and Z is almost-equimultiple, i.e. $|m_p - m_q| \le 1$ for all $p, q \in S$ ([9]). In this note we look at a similar conjecture for zero-dimensional schemes whose connected components are more general than fat points $m_p p$. Fix $P \in \mathbb{P}^n$, and integers $m > 0$ and e with $0 \le e < \binom{m+n-1}{n-1}$. The coherent sheaf $(\mathcal{I}_P)^m/(\mathcal{I}_P)^{m+1}$ is an $\binom{m+n-1}{n-1}$-dimensional vector space over the algebraically closed base field. Let $\pi : (\mathcal{I}_P)^m \to (\mathcal{I}_P)^m/(\mathcal{I}_P)^{m+1}$ be the quotient map. Take a linear subspace $V \subseteq (\mathcal{I}_P)^m/(\mathcal{I}_P)^{m+1}$ with codimension e and set $W := \pi^{-1}(V)$. W is a zero-dimensional scheme, $mP \subseteq W \subsetneq (m+1)P$ and $\deg(W) = \deg(mP) + e$. We have $W = mP$ if and only if $e = 0$. We say that W has type (m, e) and write $t_{W,p} = (m, e)$. The set $W(P, m, e)$ of all schemes of type (m, e) with P as their support is isomorphic to a Grassmannian and hence it is irreducible. Fix a finite set $S \subset \mathbb{P}^n$ For each $p \in S$ fix two integers (k_p, e_p) with $k_p > 0$ and $0 \le e_p < \binom{k_p+n-1}{n-1}$. Fix $W_p \in W(p, k_p, e_p)$, $p \in S$, and set $Z := \cup_{p\in S} W_p$. We say that Z is *tame* if $e_p \le 1$ for all $p \in S$. We say that Z is *general* if $(W_p)_{p\in S}$ is general in $\prod_{p\in S} W(p, k_p, e_p)$. We say that Z satisfies $\diamond$ if for all $p \in S$ we have $k_p \le d + 1$ and $e_p = 0$ if $k_p = d + 1$. Since $\deg(W_p) = \binom{n+k_p-1}{n} + e$, condition $\diamond$ is a necessary condition to have $h^1(\mathcal{I}_Z(d)) = 0$. For each $c = 1, \dots, n$, set $w(m, e, c) := m$ if $c + e \le n$ and $w(m, e, c) := m + 1$ if $e + c > n$. For any c-dimensional linear space $L \subseteq \mathbb{P}^n$ set $u_Z(L) := \sum_{p\in S\cap L} w(p, k_p, e_p)$. We say that L satisfies the G-condition for Z in degree d or that Z satisfies the G-condition at L in degree d if $u_Z(L) \le cd + 1$. We say that Z satisfies the G-conditions in degree d or that (Z, d) satisfies the G-conditions if Z satisfies $\diamond$ and it satisfies the G-condition in degree d with respect to all positive dimensional linear subspaces of $\mathbb{P}^n$ (as in the case of the generalized Segre conditions we need to test also $\mathbb{P}^n$, i.e. the improper linear subspace).

Conjecture 1. *Take (Z, d) satisfying the G-conditions. Find additional condition which assure that $h^1(\mathcal{I}_Z(d)) = 0$. We conjecture that $h^1(\mathcal{I}_Z(d)) = 0$ if Z is tame.*

Theorem 1. *Assume $n = 2$, Z tame and that the G-conditions are satisfied in degree d. Then $h^1(\mathcal{I}_Z(d)) = 0$.*

Then we consider the next step, when we weak the bound for $\mathbb{P}^n$, but we preserve the bounds for the lower dimensional linear spaces.

Theorem 2. *Take $n = 2$ and Z, S, d such that $d \ge 4$, $\sharp(S) \ge 5$, $k_p \le d$ for* *all p, the G-conditions are satisfied by all lines, but $u_Z(\mathbb{P}^2) = 2d+2$. Assume $h^1(\mathcal{I}_Z(d)) > 0$. Then one of the following cases occur:*

1. *$e_p = 0$ for all $p \in S$ and there is a conic D with S contained in the smooth locus of D;*
2. *$\sharp(S) = d+3$, $e_q = 0$ for all $q \in S$, there is $p \in S$ with $k_p = d$, $k_q = 1$ for all $q \notin S$ and no line through p contains 3 points of S;*
3. *there is $p \in S$ with $t_{Z,p} = (d,1)$, $k_q = 1$ for all $q \in S \setminus \{p\}$ and no line through p contains 3 points of S.*

The last condition in cases (2) and (3) of Theorem 2 is for free, because it comes from the G-condition for lines, which say that $\deg(L \cap Z) \le d+1$ for every line L.

Remark 1. In the 3 cases listed we have $h^1(\mathcal{I}_Z(d)) > 0$ for the following reasons. If Z is a fat point scheme with $Z \subset D$, D a reduced conic, then $\deg(Z) \ge 2d+2 > 2d+1 = h^0(\mathcal{O}_D(d))$. In the second and third case we have $\deg(Z) = \binom{d+2}{2} + 1 > h^0(\mathcal{O}_{\mathbb{P}^2}(d))$ and hence $h^1(\mathcal{I}_Z(d)) > 0$.

In [1] we introduced some conjectures and questions on the homogeneous ideal and the minimal free resolution of Z, up to a certain level. These questions may be translated into questions of the schemes introduced in this note using the function $u_Z(L)$, obvious generalizations of condition $\diamond$ and avoiding the cases (k_p, e_p) with $k_p = d$, but we have no evidence for them.

Proof of Theorem 1: By [4] orr [7] we may assume that Z is not a fat point scheme, i.e. the existence of $p \in S$ such that $a_p = (m_p, 1)$. For $i = 0, 1$ set $S_i := \{p \in S : t_p = (m_p, i)$. First assume $d = 1$. By assumption $S_1 \ne \emptyset$. We get $\sharp(S_1) = 1$ and $\sharp(S_0) \le 1$. It is sufficient to take $Z = v \cup e$ with v a connected degree 2 scheme, say with support p, and with e either $\emptyset$ or a point. If $e = \emptyset$, then $h^1(\mathcal{I}_v(1)) = 0$ for any v. If e is a point, then $h^1(\mathcal{I}_Z(1)) = 0$ for a general v with its support, i.e. for a v such that the line spanned by v does not contain e.

From now on we assume $d > 1$ and that Theorem 1 is true for all positive integers $< d$. Since $h^1_A(d)) = 0$ if either $A = (d+1)p$ or $A \in W(p,d,1))$, we may assume $\sharp(S) \ge 2$. Fix a line L such that $u_Z(L)$ is maximal and set $B := \mathrm{Res}_L(Z)$. If Z is general, then B is general with the restriction that B_{red} is fixed and each point of B_{red} has a type related to the one of Z. We have $B_{\mathrm{red}} \subseteq S$. Fix $p \in S$ and let Z_p be the connected component of Z with p as support. Set $B_p := B \cap Z_p$. Set $(k_p, e_p) := t_{Z,p}$. If $p \notin L$, then $t_{B,p} = t_{Z,p}$.

If $p \in L$, then $t_{B,p} = (k_p - 1, e_p)$, except in the case $(k_p, e_p) = (1,1)$ in which $t_{B,p} = (1,0)$.

(i) Assume the existence of $p \in S$, such that $t_{B,p} = (d,1)$. First assume $p \notin L$. We get $u_Z(\mathbb{P}^2) \geq d+1+u_Z(L)$. Hence $u_Z(L) \leq d$. Since $u_Z(L) > 0$, there is $q \in L \cap S$. Let D be the line spanned by p and q. We get $u_Z(D) \geq d + k_q > u_Z(L)$, contradicting the maximality property of L. Now assume $p \in L$. Since $d > 1$, we get $(k_p, e_p) = (d+1, 1)$ and so Z does not satisfies $\diamond$, a contradiction. Hence B satisfies condition $\diamond$.

(ii) We have $\deg(L \cap Z) = \sum_{p \in L \cap S} k_p$ and $\deg(B) = \deg(Z) - \deg(Z \cap L)$. The G-condition for Z and L is equivalent to $\deg(Z \cap L) \leq d+1$. Hence $h^1(L, \mathcal{I}_{Z \cap L}(d)) = 0$. By the residual exact sequence

$$0 \to \mathcal{I}_B(d-1) \to \mathcal{I}_Z(d) \to \mathcal{I}_{Z \cap L, L}(d) \to 0 \qquad (1)$$

it is sufficient to prove that B satisfies the G-conditions in degree $d-1$. We have $\sharp(S \cap L) \geq 2$, because $\sharp(S) \geq 2$. Note that $u_B(\mathbb{P}^2) = u_Z(\mathbb{P}^2) - \sharp(S \cap L)$(even in the case in which $(k_p, e_p) = (1,1)$). Hence $u_B(\mathbb{P}^2) \leq 2(d-1)+1$. Fix a line $D \subset \mathbb{P}^2$. Since $u_B(D) \leq u_Z(D)$, to test the G-condition for all lines, it is sufficient to test it for the lines D with $u_Z(D) \geq d+1$. The G-conditions for Z give $u_Z(D) = d+1$. Since $u_Z(L) \geq u_Z(D)$, we get $u_Z(L) = d+1$. Assume for the moment $u_B(L) = d+1$. This is the case if and only if $(k_p, e_p) = (1,1)$ for all $p \in S \cap L$. Since $u_Z(L) = d+1$, we get $u_Z(\mathbb{P}^2) \geq 2d+2$, a contradiction. Now assume $D \neq L$ and set $\{p\} := L \cup D$. If $p \notin S$, then we get $u_Z(\mathbb{P}^2) \geq u_Z(D) + w_Z(L) = 2d+2$, a contradiction. Hence $p \in S$. If $t_{Z,p} = (k_p, e_p) \neq (1,1)$, then $t_{B,p} = (k_p - 1, e_p)$. If $t_{Z,p} = (1,1)$, then $t_{Z,p} = (1,0)$. In the latter case we get $w(\mathbb{P}^2) \geq u_Z(L) + u_Z(D) \geq 2d+2$, a contradiction. □

Proof of Theorem 2: The case in which Z is a fat point scheme, i.e. $e_p = 0$ is well-known (see [2] for much more or look at the proof of the general case below). Now assume $e_p > 0$ for some $p \in S$. Take (k_p, e_p), L and B as in the proof of Theorem 1. Since Z satisfies the G-conditions for all lines, we have $h^1(L, \mathcal{I}_Z(d)) = 0$. The residual exact sequence (1) gives $h^1(\mathcal{I}_B(d-1)) > 0$. Note that $\sharp(S \cap L) \geq 2$.

(i) First assume $\sharp(S \cap L) = 2$.

(i1) Assume the existence of a line T such that $\sharp(S \cap T) > 2$. Set $E := \mathrm{Res}_T(Z)$. For any $q \in S$ set $(m_q, g_q) := t_{E,q}$. If $q \notin T$, then $(m_q, g_q) = (k_q, e_q)$. Now assume $q \in T$. Since T is a smooth curve, we have $(m_q, g_q) = (k_q - 1, e_q)$ if $(k_q, e_q) \neq (1,1)$, and $(m_q, g_q) = (1,0)$ if $(k_q, e_q) = (1,1)$. We have $u_E(\mathbb{P}^2) = u_Z(\mathbb{P}^2) - \sharp(S \cap T) \leq 2(d-1)+1$, i.e. E satisfies the G-condition in degree $d-1$ with respect to $\mathbb{P}^2$. Since $k_p \leq d$ for all p, E satisfies $\diamond$ in degree $d-1$, unless there is $p \in S$ with $(k_p, e_p) = (d, 1)$. Assume the existence of $p \in S$ with $(k_p, e_p) = (d, 1)$. Let $\ell : \mathbb{P}^2 \setminus \mathbb{P}^1$ denote the linear projection from p. Let Z_p be the connected component of Z containing p. We have $h^1(\mathcal{I}_{Z_p}(d)) = 0$ and $h^0(\mathcal{I}_{Z_p}(d)) = d$. Set $Z' := Z \setminus Z_p$ and $S' := S \setminus \{p\}$. Since Z satisfies the G-conditions for lines, we have $\deg(D \cap Z) \leq d+1$ for all lines. Hence $k_q = 1$ for all $q \neq p$ (i.e. Z' is a union of points and tangent vectors) and no line through p contains 3 points of S. We are as in case (3).

(i2) Assume $\sharp(S \cap T) = 2$ for all lines T. Since $\sharp(S) \geq 5$ and no 3 of the points of S are collinear, there is a smooth conic $D \subset \mathbb{P}^2$ such that $\sharp(D \cap S) \geq 5$. We may find D with the additional condition that it contains 4 points of Z with maximal multiplicity., i.e. calling $m_1 \geq \cdots \geq m_s$, $s = \sharp(S)$, the multiplicities of the points of S in Z, we assume the existence of $p_1, p_2, p_3, p_4, p_5 \in S \cap D$ such that $p_i \neq p_j$ for all $i \neq j$ and $k_{p_i} = m_i$ for all i. We have $h^1(D, \mathcal{I}_{D \cap Z}(d)) = 0$ if and only if $\deg(D \cap Z) \geq 2d + 2$. Since D is a smooth curve and Z is general, this is the case if and only if $\sum_{q \in S \cap D} k_q \geq 2d + 2$. Since $u_Z(\mathbb{P}^2) = 2d + 2$ we get $S \subset D$ and $e_q = 0$ for all $q \in S$, i.e. we are in case (2) of Theorem 2. Set $A := \mathrm{Res}_D(Z)$. For any $q \in S$ set $(n_q, f_q) := t_{A,q}$. If $q \notin D$, then $(n_q, f_q) = (k_q, e_q)$. Now assume $q \in D$. Since D is a smooth curve, we have $(n_q, f_q) = (k_q - 1, e_q)$ if $(k_q, e_q) \neq (1, 1)$, and $(n_q, f_q) = (1, 0)$ if $(k_q, e_q) = (1, 1)$. We get $u_A(\mathbb{P}^2) \leq u_Z(\mathbb{P}^2) - \sharp(S \cap D) \leq 2(d-2)+1$. Fix a line R and assume that A does not satisfies the G-condition for A with respect to R, i.e. that $\deg(A \cap R) \geq d$. Since $A \subseteq Z$, we get $\deg(Z \cap R) \geq d$. First assume $\deg(Z \cap R) > d$. Since $d+1 \geq \deg(Z \cap L) \geq \deg(Z \cap R)$, we get $\deg(Z \cap R) = \deg(Z \cap L) = d+1$ and hence (since $\sharp(S \cap R) \leq 2$ and $\sharp(S \cap R) \geq 2$) $m_1 + m_2 = d + 1$ and

Now assume $\deg(Z \cap R) = d$, i.e. $S \cap D \cap R = \emptyset$. Assume for the moment $\sharp(S \cap R) = 1$, say $S \cap R = \{u\}$. We have $k_u = d \geq m_1$. Hence $m_1 = d$. Since $u_Z(\mathbb{P}^2) = 2d+2$ and $d \geq 3$, we have $m_3 < d$. Hence $u \in D$ and $(k_u, e_u) \neq (1, 1)$. Therefore $u_A(R) < u_Z(R) = d$, a contradiction. Therefore $\sharp(S \cap R) = 2$. Let $\{o_1, o_2\}$ denote the points of $S \cap R$ with $m_{o_1} \geq m_{o_2}$. Since $m_{o_1} + m_{o_2} = d$, we have $m_{o_1} \geq \lceil d/2 \rceil$. Since $o_1 \notin S \cap D$, we have $m_{o_1} \leq m_6$. Hence $2d+2 \geq 6\lceil d/2 \rceil$, a contradiction.

Now we check when A satisfies condition $\diamond$ in degree $d-2$. Fix $q \in S$ whose multiplicity in A is maximal. First assume $q \notin D$ and hence $t_{Z,q} = t_{A,q}$. If $k_q \geq d-2$, the choice of $m_1, \dots, m_5$ and D gives $2d+2 \geq 6(d-2)$, contradicting the assumption $d \geq 3$. Now assume $q \in S$. We have $t_{A,q} = (k_q - 1, e_q)$ unless $t_{Z,q} = (1, 1)$ and hence $t_{A,q} = (1, 0)$. Since $d - 2 > 0$, the latter case gives no obstruction to $\diamond$. Now assume that $k_q - 1 \geq d - 2$. We may assume $k_q \leq d$. First assume $k_q = d$. In this case we use the linear projection from q to see that we are either in case (2) or in case (3) with respect to the point $q \in S$.

Now assume $(k_q, e_q) = (d-1, 0)$. In this case q satisfies $\diamond$ for A in degree $d-2$. Now assume $(k_q, e_q) = (d-1, 1)$. Set $Z' := Z \setminus Z_q$, $Z'' := Z' \cup (d-1)q$ and $Z^+ := Z' \cup dq$. We have $u_{Z''}(\mathbb{P}^2) = 2d+1$. Since $Z'' \subset Z'$ and Z satisfies $\diamond$ and the G-conditions in degree d for lines, Z'' satisfies $\diamond$ and the G-conditions in degree d for lines, Theorem 1 gives $h^1(\mathcal{I}_{Z''}(d)) = 0$. Since $h^1(\mathcal{I}_Z(d)) > 0$ and $\deg(Z) = \deg(Z') + 1$, we get $H^0(\mathcal{I}_Z(d)) = h^0(\mathcal{I}_{Z'}(d)$. Since Z_q is general in $W(q, d-1, 1)$, we get $H^0(\mathcal{I}_{Z^+}(d)) = H^0(\mathcal{I}_{Z'}(d)$. We have $u_{Z'}(\mathbb{P}^2) = u_Z(\mathbb{P}^2) - d = d+2$, i.e. $\sum_{u \in S\setminus\{q\}} k_u + e_u = d+2$. Let $\mu : \mathbb{P}^2 \setminus \{q\} \to \mathbb{P}^1$ be the linear projection from q. Since $\sharp(S \cap T) = 2$ for all lines T, $\mu|S \setminus \{q\}$ is injective. For each $u \in S \setminus \{q\}$ let Z_u the connected of Z containing u. Z_q is the connected component of Z' containing u. Since each Z_u is general in $W(u, k_u, e_u)$, the scheme $\mu(Z_q)$ has degree $k_u + e_u$. Since $\mu|S\setminus\{q\}$ is injective and $\sum_{u \in S\setminus\{q\}} k_u + e_u = d+2$, we have $h^0(\mathbb{P}^1, \mathcal{I}_{\mu(Z')}(d)) = 0$. Hence $h^0(\mathcal{I}_{Z^+}(d)) = 0$ and so $h^0(\mathcal{I}_{Z''}(d)) = 0$. Since $h^1(\mathcal{I}_{Z''}(d)) = 0$, we get $\deg(Z'') = \binom{d+2}{2}$. We have $\deg(Z'') = \binom{d}{2} + \sum_{u \in S\setminus\{q\}}(\binom{k_u+1}{2} + e_u)$. Since $k_u \le 2$ for all u, we have $\binom{k_u+1}{2} + e_u) \le 3(\sum_{u \in S\setminus\{q\}} k_u + e_u) \le 3(d+2)/2$, with equality if and only if $e_u = 0$ and $k_u = 2$ for all u. Hence $\binom{d+2}{2} \le \binom{d}{2} + 3(d+2)/2$, a contradiction if $d \ge 5$. Now assume $d = 4$. It remains to handle only the case $\sharp(S) = 3$ and $(k_u, e_u) = (2, 0)$ for all $u \in S \setminus \{q\}$. In this case we have $h^0(\mathcal{I}_{Z''}(4)) = 3$ and $h^0(\mathcal{I}_{Z'}(4)) \ge 5$.

(ii) Now assume $\sharp(S \cap L) > 2$. Since $u_B(\mathbb{P}^2) = u_Z(\mathbb{P}^2) - \sharp(S \cap L)$, B satisfies the G-condition in degree $d-1$ with respect to $d-1$. First assume that B does not satisfies $\diamond$ in degree $d-1$, i.e. there is $p \in S$ with $t_{B,p} = (d, 1)$. Since Z satisfies $\diamond$ in degree d, we have $p \notin L$. Fix any $q \in S \setminus \{q\}$ and let D be the line spanned by p and q. We have $u_Z(D) > d$. The maximality property of L gives $\deg(Z \cap L) = d+1$. Since $w(\mathbb{P}^2) \ge d + 1 + \deg(Z) + \sum_{q \in S \cap L} e_q$ with strict inequality if $S \setminus \{p\} \nsubseteq L$, we get $S \setminus \{p\} \subset L$ and $e_q = 0$ for all $q \ne p$, i.e. we are in the second case listed in Theorem 2. Now assume that B satisfies $\diamond$ in degree $d-1$. Since B satisfies $\diamond$ and the G-condition in degree $d-1$ with respect to $\mathbb{P}^2$ and $h^1(\mathcal{I}_B(d-1)) > 0$, it does not satisfies the G-condition for some line D, i.e. $\deg(D \cap B) \ge d+1$. The maximality property of L gives $\deg(L \cap Z) \ge d+1$. Since $u_Z(\mathbb{P}^2) = 2d+2$, we get $S \subset R \cup L$ and $e_a = 0$ for all $a \in S$, i.e. we are in case (1) with a reducible conic; the point $L \cap R$ is not in S, because we also get $\deg(R \cap B) = \deg(R \cap Z)$. □

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