ON A SEQUENCE OF TRIDIAGONAL MATRICES WHOSE DETERMINANTS ARE FIBONACCI NUMBERS $F_{n+1}$

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Abstract: In this paper, we generalize two previous individual results on connection special tridiagonal matrices to Fibonacci numbers, as we found a sequence of tridiagonal matrices which are equal to Fibonacci numbers.

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1. Introduction

The Fibonacci sequence (or sequence of the Fibonacci numbers) $(F_n)_{n \geq 0}$ is the sequence of non-negative integers satisfying the recurrence $F_{n+2} = F_{n+1} + F_n$ with the initial conditions $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence has many surprising properties (see e. g. [5]), but this paper deals with its connections to determinants of matrices only. Strang [10] include, probably the first example of determinant of $n \times n$ matrix, which is equal to the Fibonacci number, as he showed that the following holds

for any \( n \geq 1 \). Cahill et al. [1] showed that the following holds

\[
\begin{vmatrix}
1 & -1 & 0 & \cdots & 0 & 0 \\
1 & 1 & -1 & 0 & \cdots & 0 \\
0 & 1 & 1 & -1 & 0 & \cdots & 0 \\
\vdots & 0 & 1 & 1 & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & -1 & 0 \\
0 & 0 & \cdots & 0 & 1 & 1 & -1 \\
0 & 0 & \cdots & 0 & 1 & 1
\end{vmatrix} = F_{n+1} \tag{1}
\]

for any \( n \geq 1 \). Matrices in (1) and (2) are the special cases of a tridiagonal matrix, what is a square matrix \( A = (a_{jk}) \) of the order \( n \), with entries \( a_{jk} = 0 \) for \( |k - j| > 1 \) and \( 1 \leq j, k \leq n \), i. e.

\[
\begin{vmatrix}
1 & i & 0 & \cdots & \cdots & 0 \\
i & 1 & i & \ddots & \ddots & \vdots \\
0 & i & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & i \\
\vdots & \ddots & \ddots & i & 1 & i \\
0 & \cdots & \cdots & 0 & i & 1
\end{vmatrix} = F_{n+1} \tag{2}
\]

for any \( n \geq 1 \) (where \( i = \sqrt{-1} \)). Many authors derived the similar types of matrices which determinants are related to Fibonacci numbers or different kinds of their generalizations, e. g. \( k \)-generalized Fibonacci numbers, see [2], [4], [7], [6], [3], [8], [9] and [11].

Now we turn our attention to the relation of determinants of special tridiagonal matrices with Fibonacci numbers. We show that matrix in (1) can be easily changed into a matrix, whose determinant is equal to Fibonacci numbers too.
2. Preliminary Results

Cahill et al. [1] proved the following lemma, which can be easily used for finding the recurrence relation for determinants of a sequence of tridiagonal matrices.

Lemma 1. (Lemma 1 of [1]) Let \( \{H(n), n = 1, 2, \ldots \} \) be a sequence of tridiagonal matrices of the form

\[
H(n) = \begin{pmatrix}
  h_{1,1} & h_{1,2} & 0 & \cdots & 0 \\
  h_{2,1} & h_{2,2} & h_{2,3} & \ddots & \vdots \\
  0 & h_{3,2} & h_{3,3} & \ddots & 0 \\
  \vdots & \ddots & \ddots & \ddots & h_{n-1,n} \\
  0 & \cdots & 0 & h_{n,n-1} & h_{n,n}
\end{pmatrix}.
\]

Then the successive determinants of \( H(n) \) are given by recursive formula

\[
\begin{align*}
det H(1) &= h_{1,1}; \\
det H(2) &= h_{1,1}h_{2,2} - h_{1,2}h_{2,1}; \\
det H(n) &= h_{n,n} det H(n-1) - h_{n-1,n}h_{n,n-1} det H(n-2).
\end{align*}
\]

3. Main Results

We formulate theorem which generalize identities (1) and (2).

Theorem 2. Let \( (\varepsilon_n)_{n \geq 0}, (\delta_n)_{n \geq 0} \) be any sequences of complex numbers, with property \( \varepsilon_k\delta_k = -1 \) for any \( k, 1 \leq k \leq n \). Let \( \{B(n), n = 1, 2, 3, \ldots \} \) be a sequence of tridiagonal matrices in the form

\[
b_{jk} = \begin{cases}
  1, & j = k \\
  \varepsilon_j, & k = j + 1 \\
  \delta_j, & k = j - 1 \\
  0, & \text{otherwise}
\end{cases}
\]
i. e.

\[ B(n) = \begin{pmatrix}
1 & \varepsilon_1 & 0 & \cdots & \cdots & 0 \\
\delta_1 & 1 & \varepsilon_2 & \ddots & \ddots & \vdots \\
0 & \delta_2 & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \varepsilon_1 & 0 \\
\vdots & \ddots & \ddots & \varepsilon_n & 1 & \varepsilon_{n-1} \\
0 & \cdots & \cdots & 0 & \delta_{n-1} & 1 \\
\end{pmatrix}. \]

Then

\[ \det B(n) = F_{n+1}. \]

**Proof.** We use the mathematical induction on \( n \). The assertion holds for \( n = 1 \) and \( n = 2 \) as

\[ \det B(1) = 1 = F_2, \]
\[ \det B(2) = \det \begin{pmatrix} 1 & \varepsilon_1 \\ \delta_1 & 1 \end{pmatrix} = 1 - \varepsilon_1 \delta_1 = 2 = F_3. \]

Suppose that the assertion holds for every \( k, 3 \leq k \leq n \). Then we have to show that the assertion is true for \( n + 1 \). We use recurrence (3)

\[ \det B(n+1) = b_{n+1,n+1} \det B(n) - b_{n,n+1}b_{n+1,n} \det B(n-1) = 1 \cdot \det B(n) - \varepsilon_{n-1} \delta_{n-1} \det B(n-1) = \det B(n) + \det B(n-1) = F_{n+1} + F_n = F_{n+2}. \]

**Corollary 3.** Setting \( \varepsilon_k = -1, \delta_k = 1 \) and \( \varepsilon_k = \delta_k = i = \sqrt{-1} \) in Theorem 2, for \( 1 \leq k \leq n \), we directly obtain (1) and (2) respectively.

Similarly we can obtain infinitely many interesting \( n \)-square matrices, whose determinants are equal to the Fibonacci number \( F_{n+1} \), using Theorem 2, but there are integer matrices of this type only for entries \( \varepsilon_k = \pm 1, \delta_k = - \varepsilon_k \), where \( 1 \leq k \leq n \). For example, we obtain the following sequence of integer matrices.
Corollary 4. Let \( \{ C(n) = (c_{jk})_{1 \leq j,k \leq n}, n = 1,2,3,\ldots \} \) be a sequence of tridiagonal matrices in the form

\[
c_{jk} = \begin{cases} 
1, & j = k; \\
(-1)^j, & j = k \pm 1; \\
0, & \text{otherwise},
\end{cases}
\]

i. e.

\[
C(n) = \begin{pmatrix}
1 & -1 & 0 & \cdots & 0 & \cdots & 0 \\
1 & 1 & 1 & \cdots & \cdot & \cdot & \cdot \\
0 & -1 & 1 & -1 & \cdot & \cdot & \cdot \\
\vdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdots & \cdots & 1 & (-1)^{n-2} & 0 \\
\vdots & \cdots & (-1)^{n-1} & 1 & (-1)^{n-1} \\
0 & \cdots & 0 & \cdots & 0 & (-1)^n & 1
\end{pmatrix}
\]

Then

\[
\det C(n) = F_{n+1}.
\]

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References


