THE GRAPH OF DIVISOR FUNCTION $D(n)$

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Abstract: Starting from 1980, the concepts of Number Theory were used in Graph theory and several types of graphs have been introduced. In this sequel we have introduced divisor function graph and studied its properties. Further, divisor function subgraph, complete divisor function graph and Eulerian properties have been discussed.

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Key Words: divisor function graph, divisor function subgraph, divisor graph, Cayley graph, Cayley arithmetic graph

1. Introduction

The notion of divisor graphs was introduced by Singh and Santhosh [10] in 2000. A divisor graph G is an ordered pair (V, E) where V is a subset of $\mathbb{Z}$ and $uv \in E$ if and only if either $u|v$ or $v|u$ for all $u \neq v$. A graph which is isomorphic to a divisor graph is also called a divisor graph. Moreover, they proved that every odd cycle of length greater than or equal to five is not a divisor graph whereas all even cycles, complete graphs, and caterpillars are divisor graphs. Further properties were studied by Chartrand, Muntean, Saenpholphant and

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Zhang [3]. Moreover, Le Anh Vinh [5] proved that there exists a divisor graph of order n and size m where m is an integer between 0 and any positive integer n . Christopher Frayer investigated on the condition for Cartesian product of graphs to be a divisor graph.

Yu-ping Tsao [11] investigates several properties such as the vertex-chromatic number, the clique number, the clique cover number, and the independence number of $D([n])$ and its complement, where $[n] = \{i : 1 \leq i \leq n, n \in N\}$. Nathanson [9] introduced the concepts of Number Theory, in particular the notion of congruences in Graph Theory and initiated the new way for the emergence of a new class of graphs, namely, arithmetic graphs. An arithmetic graph is the graph whose vertex set V is the set of first n positive integers 1, 2, 3, ..., n and any two vertices a and b are adjacent if and only if $a + b \equiv c \pmod{n}$ where $c \in S$, a pre-assigned subset of V .

There is another class of graphs, called, Cayley graphs [6]. A Cayley graph is the graph whose vertex set V is the set of elements of a finite group $(X, .)$ and two vertices a and b of X are adjacent if and only if $a^{-1}b$ or $b^{-1}a$ is in some symmetric subset S of X (a subset S of a group $(X, .)$ is called symmetric subset of X if s^{-1} is in S for all s in S). This Cayley graph is denoted by $G(X, S)$. If the group $(X, .)$ is the additive group $(Z_n, \oplus)$ of integers 0, 1, 2, ..., $n - 1$ modulo n and the symmetric set S is associated with some arithmetic function, then the Cayley graph may be treated as an arithmetic graph and such graphs are called arithmetic Cayley graphs.

Dejter and Giudici [4], Berrizabeitia and Giudici [2] and others have studied the cycle structure of Cayley graphs associated with certain arithmetic functions. Madhavi and Maheswari [7, 8] studied the Hamilton cycles and triangles (the cycles of longest and shortest length) of the arithmetic Cayley graphs associated with Euler totient function $\varphi(n)$, $n \geq 1$, an integer and quadratic residues modulo a prime p . The cycle structure of these graphs has many applications in Engineering and Communication Networks. Madhavi (2002) also studied the arithmetic Cayley graph associated with divisor function $d(n)$, $n \geq 1$, an integer. Chalapati, Madhavi and Venkataramana [1] studied the enumeration of triangles in these graphs.

In the present paper, we have introduced divisor function graph and studied its properties. Further, divisor function subgraph, complete divisor function graph and Eulerian properties have been discussed.

2. The Divisor Function Graph $G_{D(n)}$

Now, Let us define the divisor function $D(n)$, defined by $D(n) = \{d : d \mid n, n \text{ is any positive integer} \}$

Definition 2.1. For any positive integer $n \geq 1$ with r divisors $d_1, d_2, d_3, \dots, d_r$, the graph of divisor function $D(n)$ of n is a graph $G_{D(n)}$ with the vertex set $\{d_1, d_2, d_3, \dots, d_r\}$ such that two vertices d_i and d_j are adjacent if and only if either $d_i \mid d_j$ or $d_j \mid d_i, i \neq j$.

Example 2.2. The divisor function graph for $n = 1$ to 10 are given below.

Theorem 2.3. For any $G_{D(n)}$, minimum degree $\delta =$ maximum degree $\Delta = 1$ when n is prime. Otherwise, $\delta \geq 2$ and $\Delta = (n_1 + 1)(n_2 + 1) \dots (n_r + 1) - 1$ where $n_1, n_2, \dots, n_r$ are the powers of prime factors of n .

Proof. When n is prime, $G_{D(n)} = K_2$. Hence $\delta = \Delta = 1$. When n is composite, let $n = p_1^{n_1} p_2^{n_2} \dots p_r^{n_r}$. Then the total number of divisors of n are $(n_1 + 1)(n_2 + 1) \dots (n_r + 1)$. Also since 1 divides all the divisors of n , degree of 1 is one less than the total number of divisors. Hence, $\Delta = (n_1 + 1)(n_2 + 1) \dots (n_r + 1) - 1$. Similarly, $\delta \geq 2$ is obvious since n has at least one proper divisor. $\square$

Theorem 2.4. For any positive integer n , the divisor function graph

$G_{D(n)}$ is connected.

Proof. It is obvious since 1 is a divisor for any n , the vertex 1 is adjacent with remaining all the vertices d_i . $\square$

Corollary 2.5. For any positive integer n , not a prime, the divisor function graph $G_{D(n)}$ has at least one cycle of length 3.

Proof. Since n is not prime, there is at least one proper divisor d of n such that d is adjacent with 1 and n in $G_{D(n)}$. Hence we have the cycle $1dn1$ of length 3. $\square$

Theorem 2.6. For any positive integer n , the divisor function graph $G_{D(n)}$ is a complete graph if and only if no two proper divisors in $D(n)$ are relatively prime.

Proof. Necessity: Suppose that $G_{D(n)}$ is complete. Then any two vertices d_i and d_j are adjacent in $G_{D(n)}$, $i \neq j$. Consequently, either $d_i \mid d_j$ or $d_j \mid d_i$. This implies that d_i and d_j are not relatively prime.

Sufficient: Let no two distinct proper divisors in $D(n)$ be relatively prime. Suppose $G_{D(n)}$ is not complete. Then, there exists d_i and d_j such that $d_i \nmid d_j$ and $d_j \nmid d_i$. Consequently, either $\gcd(d_i, d_j) = 1$ or $\gcd(d_i, d_j) = d$, $d > 1$. By our hypothesis, $\gcd(d_i, d_j) \neq 1$. Therefore, $\gcd(d_i, d_j) = d$, $d > 1$. Then, $\gcd\left(\frac{d_i}{d}, \frac{d_j}{d}\right) = 1$ and $\frac{d_i}{d} \mid n$ and $\frac{d_j}{d} \mid n$ which is a contradiction to the fact that no two distinct proper divisors in $D(n)$ be relatively prime. Hence $G_{D(n)}$ is complete. $\square$

Corollary 2.7. For any positive integer n , the divisor function graph $G_{D(p^n)}$ is complete.

Proof. Since $D(p^n) = \{1, p, p^2, \dots, p^{n-1}, p^n\}$ and no two proper divisors in $D(p^n)$ are relatively prime, $G_{D(p^n)}$ is complete. $\square$

Remark 2.8. $G_{D(p^n)}$ is a n -regular graph.

Remark 2.9. Every subgraph of a divisor function graph is neither a divisor function graph nor a divisor graph. The following example supports our

claim.

Example 2.10. Clearly H is a subgraph of $G_{D(12)}$, but it is neither a divisor function graph nor a divisor graph.

Definition 2.11. A subgraph H of a divisor function graph $G_{D(n)}$ is called a divisor function subgraph if H is itself a divisor function graph.

Example 2.12. Here H is a divisor function subgraph $G_{D(4)}$ of a divisor function graph $G_{D(12)}$.

Theorem 2.13. The divisor function graph $G_D(n)$ is planar if every vertex of $G_D(n)$ has degree at most degree 4.

Proof. Since 1 divides all divisors on n , the proof is obvious. □

Theorem 2.14. If n is composite, then $G_{D(n)}$ is not a bipartite graph.

Proof. Assume that n is composite. Suppose, $G_{D(n)}$ is a bipartite graph.

Then, there exist a bipartition (X, Y) . Without loss of generality, we assume $1 \in X$ and $n \in Y$. Since n is composite, there exists at least one proper divisor d . Clearly, $1 \mid d$ and $d \mid n$. Therefore, d can be placed neither in X nor in Y . This violates the condition of bipartite graph. Hence $G_{D(n)}$ is not a bipartite graph. $\square$

Theorem 2.15. *For any composite number n , the following statements are equivalent.*

- (a) $G_{D(n)}$ is a block.
- (b) Any two vertices of $G_{D(n)}$ lie in common cycle.
- (c) Any vertex and an edge lie in a common cycle.
- (d) Any two edges of $G_{D(n)}$ lie on a common cycle.

Proof. **(a) $\Rightarrow$ (b)** Assume that (a) is true. By definition, $G_{D(n)}$ has no cut vertices. since n is composite, there exists at least one proper divisor n_i such that $n_i \mid n$. let n_1, n_2 be any divisors of n .

Case(i): Suppose, either $n_1 \mid n_2$ or $n_2 \mid n_1$. Then, $1n_1n_21$ is a cycle C containing both n_1 and n_2 .

Case(ii): Suppose both $n_1 \nmid n_2$ or $n_2 \nmid n_1$. Then, $n_1n_2 \notin E(G_{D(n)})$. But, $1 \mid n_1$ and $1 \mid n_2$. Then, the $1n_1nn_21$ forms a cycle containing vertices n_1 and n_2 . Therefore, (b) is true.

(b) $\Rightarrow$ (a) Suppose any two vertices lie on a common cycle. Let n_1 be the cutpoint of $G_{D(n)}$. Then, $G_{D(n)} - n_1$ is a disconnected graph. Since $1 \in V[G_{D(n)}]$ and 1 divides all the vertices of $G_{D(n)}$, 1 is adjacent to all the vertices of $G_{D(n)} - n_1$ which contradicts $G_{D(n)} - n_1$ is a disconnected graph. Hence, $G_{D(n)}$ has no cutpoints. Therefore, $G_{D(n)}$ is a block.

(b) $\Rightarrow$ (c) Assume that (b) is true. Let n_1 be any vertex and n_2n_3 be any edge of $G_{D(n)}$. Since, $1 \mid n_1, n_1 \mid n$ and $n_3 \mid n$. Then $1n_1nn_2n_31$ forms a cycle C containing n_1 and n_2n_3 . Consequently, any vertex and an edge lie in a common cycle.

(c) $\Rightarrow$ (b) is trivial.

(c) $\Rightarrow$ (d) Assume that (c) is true. Let $n_1n_2, n_3n_4 \in E(G_{D(n)})$. Then, $1 \mid n_1, n_2 \mid n$ and $1n_1$ and n_2n are the edges of $G_{D(n)}$. Now, $1n_1n_2nn_3n_41$ forms a cycle containing both n_1n_2 and n_3n_4 . Since n_1n_2 and n_3n_4 is arbitrary, any two edges of $G_{D(n)}$ lie on a common cycle.

(d) $\Rightarrow$ (c) is trivial. $\square$

Theorem 2.16. For any n , $L(G_{D(n)})$ is connected.

Proof. **case (i):** n is prime. The result is trivial.

case (ii): Suppose, n is composite. Then, there exist at least one proper divisor, say n_i . For our convenience, we represent the vertices in $L(G_{D(n)})$ as the name given to that of edges in $G_{D(n)}$. (i.e) If $V(G_{D(n)}) = \{1, n_1, n_2, \dots, n\}$, then $V(L(G_{D(n)})) \subset \{1n_1, 1n_2, n_1n_2, n_1n_3, \dots, n_1n, \dots, 1n\}$. Now, the edges in $L(G_{D(n)})$ has any one of the following possibilities:

(i) The labelled vertices having one of its index as 1 in $L(G_{D(n)})$ (say) $1n_i$ is adjacent with $1n_{i+1}$, for all i (or) $1n_i$ is adjacent with n_in_{i+1} if and only if either $n_i \mid n_{i+1}$ or $n_{i+1} \mid n_i$.

(ii) For the edges which is not covered in (i) the labelled vertices having one of its index as n in $L(G_{D(n)})$ (say) nn_i is adjacent with nn_{i+1} , for all i (or) nn_i is adjacent with n_in_{i+1} if and only if either $n_i \mid n_{i+1}$ or $n_{i+1} \mid n_i$.

(iii) For the labelled vertices which are having its index as neither 1 nor n in $G_{D(n)}$ say n_in_{i+1} is adjacent with $n_{i+2}n_{i+3}$ in $L(G_{D(n)})$ if and only if $(n_i, n_{i+2}) > 1$ and $(n_{i+1}, n_{i+4}) > 1$.

From (i), (ii) and (iii), we obtain $L(G_{D(n)})$, line graph of $G_{D(n)}$ with no vertex being isolated in $L(G_{D(n)})$. Hence, there exists a path between every pair of distinct vertices in $L(G_{D(n)})$. Therefore, $L(G_{D(n)})$ is connected, for any composite number. $\square$

Theorem 2.17. For any positive integer n , the divisor function graph $G_{D(n)}$ is an Euler graph if and only if n has odd number of divisors and for each divisor d_i of n , the number of d_j such that $d_i \mid d_j$ and $d_j \mid d_i$ is even.

Proof. **Necessity:** Suppose that the divisor function graph $G_{D(n)}$ with r vertices is an Euler graph. Then each divisor d_i in $G_{D(n)}$ has even degree and the divisor 1 has the degree $r - 1$ since 1 divides remaining all divisors of n and hence $r - 1$ is even. This shows that $G_{D(n)}$ has odd number of divisors. Moreover, by the definition of $G_{D(n)}$, two vertices d_i and d_j are adjacent if and only if either $d_i \mid d_j$ or $d_j \mid d_i$, $i \neq j$. Hence for each d_i , degree of d_i = the number of d_j such that $d_i \mid d_j$ and $d_j \mid d_i$ = even since each divisor d_i in $G_{D(n)}$ has even degree.

Sufficiency: Similar to necessity. $\square$

Theorem 2.18. For any positive integer n , the divisor function graph

$G_{D(n^2)}$ is an Euler graph.

Proof. Suppose that $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$. Then $n^2 = p_1^{2a_1} p_2^{2a_2} \dots p_k^{2a_k}$. Now, the number of divisors of n^2 is $d(n^2) = (2a_1 + 1)(2a_2 + 1) \dots (2a_k + 1)$ = a product of odd integers which is odd. Hence the total number of vertices of the divisor function graph $G_{D(n^2)}$ is odd. Now, it is enough to prove that the degree of all the vertices of $G_{D(n^2)}$ is even.

Clearly 1 and n^2 are the divisors of n^2 of which 1 divides all the remaining even number of divisors of n^2 and n^2 is divisible by all the remaining even number of divisors of n^2 . Hence the degree of 1 and n^2 in $G_{D(n^2)}$ are even.

Now, the proper divisors of n^2 is of the form $d = p_1^{d_1} p_2^{d_2} \dots p_k^{d_k}$ where any one of the following holds.

- (1) all $d_i s$ satisfy $0 < d_i < 2a_i$, $i = 1$ to k ;
- (2) some $d_i s = 0$ and remaining $d_i s$ satisfy $0 < d_i < 2a_i$;
- (3) some $d_i s = 0$ and remaining $d_i s = 2a_i$;
- (4) some $d_i s = 2a_i$ and remaining $d_i s$ satisfy $0 < d_i < 2a_i$;
- (5) some $d_i s = 0$, some $d_i s = 2a_i$ and some $d_i s$ satisfy $0 < d_i < 2a_i$.

Case (1): All $d_i s$ satisfy $0 < d_i < 2a_i$, $i = 1$ to k .

Then, the total number of divisors of d other than d is $(d_1 + 1)(d_2 + 1) \dots (d_k + 1) - 1$ and the number of divisors of n^2 other than d which are divisible by d is $(2a_1 - d_1 + 1)(2a_2 - d_2 + 1) \dots (2a_k - d_k + 1) - 1$. Hence the degree of d is given by $\deg(d) = \{(d_1 + 1)(d_2 + 1) \dots (d_k + 1) - 1\} + \{(2a_1 - d_1 + 1)(2a_2 - d_2 + 1) \dots (2a_k - d_k + 1) - 1\}$.

Subcase i: Suppose that all $d_i s$ are even. Then $(d_1 + 1)(d_2 + 1) \dots (d_k + 1)$ and $(2a_1 - d_1 + 1)(2a_2 - d_2 + 1) \dots (2a_k - d_k + 1)$ are the product of odd integers. Consequently, $\deg(d)$ is the sum of two even integers and hence it is even.

Subcase ii: Suppose that all $d_i s$ are not even. Then there exist at least one d_i , say d_m is odd, where $1 < m < k$. Therefore, $(d_m + 1)$, $(2a_m - d_m + 1)$ are even and hence $(d_1 + 1)(d_2 + 1) \dots (d_k + 1)$ and $(2a_1 - d_1 + 1)(2a_2 - d_2 + 1) \dots (2a_k - d_k + 1)$ are the product of even integers. Hence, $\deg(d)$ is the sum of two odd integers and it is even.

Case (2): Some $d_i s = 0$ and remaining of them satisfy $0 < d_i < 2a_i$.

Suppose that there is l $d_i s$ are zero and remaining $k - l$ $d_i s$ satisfy $0 < d_i < 2a_i$. Then, we can rewrite d in the form $d = p_{i_1}^0 p_{i_2}^0 \dots p_{i_n}^0 p_{j_1}^{r_1} p_{j_2}^{r_2} \dots p_{j_{k-l}}^{r_{k-l}}$. Then, the

total number of divisors of d other than d is $(r_1 + 1)(r_2 + 1) \dots (r_{k-l} + 1) - 1$ and the number of divisors of n^2 other than d which are divisible by d is $(2a_1 - r_1 + 1)(2a_2 - r_2 + 1) \dots (2a_{k-l} - r_{k-l} + 1) - 1$. Hence the degree of d is given by $\deg(d) = \{(r_1 + 1)(r_2 + 1) \dots (r_{k-l} + 1) - 1\} + \{(2a_1 - r_1 + 1)(2a_2 - r_2 + 1) \dots (2a_{k-l} - r_{k-l} + 1) - 1\}$.

Subcase i: Suppose that all r_i s are even. Then $(r_1 + 1)(r_2 + 1) \dots (r_{k-l} + 1)$ and $(2a_1 - r_1 + 1)(2a_2 - r_2 + 1) \dots (2a_{k-l} - r_{k-l} + 1)$ are the product of odd integers. Consequently, $\deg(d)$ is the sum of two even integers and hence it is even.

Subcase ii: Suppose that all r_i s are not even. Then there exist at least one r_i , say r_m is odd, where $1 < m < k - l$. Therefore, $(r_m + 1), (2a_m - r_m + 1)$ are even and hence $(r_1 + 1)(r_2 + 1) \dots (r_{k-l} + 1)$ and $(2a_1 - r_1 + 1)(2a_2 - r_2 + 1) \dots (2a_{k-l} - r_{k-l} + 1)$ are the product of even integers. Consequently, $\deg(d)$ is the sum of two odd integers and it is even.

Case (3): Some d_i s = 0 and remaining d_i s = $2a_i$.

Suppose that there is l d_i s are zero and remaining $k - l$ d_i s satisfy $d_i = 2a_i$. Then, we can rewrite d in the form $d = p_{i_1}^0 p_{i_2}^0 \dots p_{i_l}^0 p_{j_1}^{2a_{j_1}} p_{j_2}^{2a_{j_2}} \dots p_{j_{k-l}}^{2a_{j_{k-l}}}$. Then the total number of divisors of d other than d is $(2a_{j_1} + 1)(2a_{j_2} + 1) \dots (2a_{j_{k-l}} + 1) - 1$ and the number of divisors of n^2 which are divisible by d is $(2a_{i_1} + 1)(2a_{i_2} + 1) \dots (2a_{i_l} + 1) - 1$. Hence the degree of d is given by $\deg(d) = \{(2a_{j_1} + 1)(2a_{j_2} + 1) \dots (2a_{j_{k-l}} + 1) - 1\} + \{(2a_{i_1} + 1)(2a_{i_2} + 1) \dots (2a_{i_l} + 1) - 1\}$. Consequently, $\deg(d)$ is the sum of two even integers and it is even.

Case (4): Some d_i s = $2a_i$ and remaining d_i s satisfy $0 < d_i < 2a_i$

Suppose that there is l d_i s are $2a_i$ and remaining $k - l$ d_i s satisfy $0 < d_i < 2a_i$. Then, we can rewrite d in the form $d = p_{i_1}^{r_1} p_{i_2}^{r_2} \dots p_{i_{k-l}}^{r_{k-l}} p_{j_1}^{2a_{j_1}} p_{j_2}^{2a_{j_2}} \dots p_{j_l}^{2a_{j_l}}$. Then the total number of divisors of d other than d is $(r_1 + 1)(r_2 + 1) \dots (r_{k-l} + 1)(2a_{j_1} + 1)(2a_{j_2} + 1) \dots (2a_{j_l} + 1) - 1$ and the number of divisors of n^2 other than d which are divisible by d is $(2a_{i_1} - r_1 + 1)(2a_{i_2} - r_2 + 1) \dots (2a_{i_{k-l}} - r_{k-l} + 1) - 1$. Hence the degree of d is given by $\deg(d) = \{(r_1 + 1)(r_2 + 1) \dots (r_{k-l} + 1)(2a_{j_1} + 1)(2a_{j_2} + 1) \dots (2a_{j_l} + 1) - 1\} + \{(2a_{i_1} - r_1 + 1)(2a_{i_2} - r_2 + 1) \dots (2a_{i_{k-l}} - r_{k-l} + 1) - 1\}$. Therefore, $\deg(d)$ is even either all r_i s are even or all r_i s are not even.

Case (5): Some d_i s = 0, some d_i s = $2a_i$ and some d_i s satisfy $0 < d_i < 2a_i$

Suppose that there is l d_i s are $2a_i$, m d_i s satisfy $0 < d_i < 2a_i$ and the remaining $k - l - m$ d_i s are 0. Then, we can rewrite d in the form $d =$

$p_1^{r_1} p_2^{r_2} \dots p_m^{r_m} p_{i_m+1}^{2a_{i_m+1}} p_{i_m+2}^{2a_{i_m+2}} \dots p_{i_m+l}^{2a_{i_m+l}}$. Then the total number of divisors of d other than d is $(r_1+1)(r_2+1)\dots(r_m+1)(2a_{i_m+1}+1)(2a_{i_m+2}+1)\dots(2a_{i_m+l}+1)-1$ and the number of divisors of n^2 other than d which are divisible by d is $(2a_{i_1}-r_1+1)(2a_{i_2}-r_2+1)\dots(2a_{i_m}-r_m+1)(2a_{i_m+l+1}+1)(2a_{i_m+l+2}+1)\dots(2a_{i_k}+1)-1$. Hence the degree of d is given by $\deg(d) = \{(r_1+1)(r_2+1)\dots(r_{k-l}+1)(2a_{j_1}+1)(2a_{j_2}+1)\dots(2a_{j_{k-l}}+1)-1\} + \{(2a_{i_1}-r_1+1)(2a_{i_2}-r_2+1)\dots(2a_{i_{k-l}}-r_{k-l}+1)-1\}$. Therefore, $\deg(d)$ is even either all r_i s are even or not. This completes the proof. $\square$

Theorem 2.19. For any prime p , the divisor function graph $G_{D(p)}$ is a tree with one edge.

Proof. The proof is obvious since 1 and p are the only divisors of p . $\square$

Theorem 2.20. If n is prime, then $G_{D(n)}$ is 2-colourable.

Proof. Since n is prime, its divisors are 1 and n itself. Hence $G_{D(n)}$ has unique edge $e = 1n = K_2$. Hence it is 2-colourable graph. $\square$

Theorem 2.21. The necessary and sufficient condition for $G_{D(n)}$ is 3-colourable is that any two proper divisors of n are relatively prime.

Proof. Necessity: Suppose that $G_{D(n)}$ is 3-colourable. Suppose that there exists two proper divisors p_1, p_2 for n which are not relatively prime. Then, there is a common divisor d for p_1 and p_2 that also divides n . It contradicts our assumption.

Sufficiency: Assume that any two proper divisors of n are relatively prime. Then, n has exactly one or two proper divisors which are prime. Otherwise, if n has more than two proper divisors, say p_1, p_2, p_3 . Then, p_1p_2, p_2p_3, p_3p_1 are also proper divisors of n which are not relatively prime to n . Hence $G_{D(n)}$ is 3-colourable. $\square$

Theorem 2.22. The divisor function graph $G_{D(n)}$ is a 3-colourable if and only if all proper divisors of n are prime.

Proof. Suppose that $G_{D(n)}$ is 3-colourable. Suppose there exists a proper divisor d of n which is not prime. Then, $d = ab$ and a, b divide both n and d . Hence the colouring of $G_{D(n)}$ requires more than 3 colours which contradicts

the fact that $G_{D(n)}$ is 3-colourable.

Conversely, assume that all proper divisors of n are prime. Then they are relatively prime. Therefore, $G_{D(n)}$ is 3-colourable {By Theorem 2.21}. $\square$

Remark 2.23. For a prime number p , the following statements are true:

- (i) $G_{D(p)}$ is a bipartite graph.
- (ii) $G_{D(p)}$ has no cut points.
- (iii) $G_{D(p)}$ has a bridge.
- (iv) $G_{D(p)}$ has a cut edge.

Remark 2.24. The following are some observations in $G_{D(n)}$.

- 1 $G_{D(p)}$ is the only triangle free graph, for a given prime p .
- 2 Adjacency and incidence matrix of divisor function graph $G_{D(n)}$ can be defined and its properties are analogous to a simple Graph G .
- 3 Line graph of $G_{D(p)}$ is a trivial graph or null graph.
- 4 The above theorem is also one of the standard algorithms to find out line graph of $G_{D(n)}$.

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