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Γ*-DERIVATION ACTING AS AN ENDOMORPHISM AND AS AN ANTI-ENDOMORPHISM IN SEMIPRIME Γ-RING M WITH INVOLUTION

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Abstract: Let M be a semiprime Γ-ring with involution satisfying the condition that $a\alpha b\beta c = a\beta b\alpha c$ ($a,\ b,\ c \in M$ and $\alpha, \beta \in \Gamma$). An additive mapping $d : M \to M$ is called Γ^{*}-derivation if $d(x\alpha y) = d(x)\alpha y^{*} + x\alpha d(y)$. In this paper we will prove that if d is Γ^{*}-derivation of a semiprime Γ-ring with involution which is either an endomorphism or anti-endomorphism, then d=0.

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1. Introduction

Let M and Γ be additive abelian groups. If there exists a mapping $M \times \Gamma \times$

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$M \longrightarrow M$ defined by $(x, \alpha, y) \longrightarrow (x\alpha y)$ which satisfies the conditions

(i) $x\alpha y \in M$.

(ii) $(x+y)\alpha z = x\alpha z + y\alpha z,\ x(\alpha+\beta)y = x\alpha y + x\beta y,\ x\alpha(y+z) = x\alpha y + x\alpha z$.

(iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$. (see [5], [7])

Then M is called a Γ-ring. Every ring M is a Γ-ring with $M = \Gamma$. However a Γ-ring need not be a ring. Γ-rings, more general than rings, were introduced by Nobusawa [10]. Bernes [12] slightly weakened the conditions in the definition of Γ-ring in the sense of Nobusawa. Let M be a Γ-ring. Then an additive subgroup U of M is called a left (right) ideal of M if $M\Gamma U \subset U (U\Gamma M \subset U)$. If U is both a left and a right ideal, then we say U is an ideal of M. Suppose again that M is a Γ-ring. Then M is said to be 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in M$. An ideal P_1 of a Γ-ring M is said to be prime if for some ideals A and B of M , $A\Gamma B \subseteq P_1$ implies $A \subseteq P_1$ or $B \subseteq P_1$. An ideal P_2 of a Γ-ring M is said to be semiprime if for any ideal U of M, $U\Gamma U \subseteq P_2$ implies $U \subseteq P_2$. A Γ-ring M is said to be prime if $a\Gamma M\Gamma b = (0)$ with $a, b \in M$, implies $a = 0$ or $b = 0$ and semiprime if $a\Gamma M\Gamma a = (0)$ with $a \in M$ implies $a = 0$. Furthermore, M is said to be a commutative Γ-ring if $x\alpha y = y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover, the set $Z(M) = \{x \in M : x\alpha y = y\alpha x \text{ for all } \alpha \in \Gamma,\ y \in M\}$ is called the center of the Γ-ring M. If M is a Γ-ring, then $[x, y]_\alpha = x\alpha y - y\alpha x$ is known as the commutator of x and y with respect to α, where $x,\ y \in M$ and $\alpha \in \Gamma$. We make the basic commutator identities:

$$[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x[\alpha, \beta]_z y + x\alpha[y, z]_\beta \tag{1}$$

$$[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y[\alpha, \beta]_x z + y\alpha[x, z]_\beta \tag{2}$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Now, we consider the following assumption:

(A) $x\alpha y\beta z = x\beta y\alpha z$ for all $x,\ y,\ z \in M$ and $\alpha,\ \beta \in \Gamma$.

According to assumption (A), the above commutator identities reduce to $[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x\alpha[y, z]_\beta$ and $[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y\alpha[x, z]_\beta$. which we will extensively used.

Definition 1. [7] An additive mapping $d : M \longrightarrow M$ is called a derivation if $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$ which holds for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 2. [2] An additive mapping $\phi : M \longrightarrow M$ is said to be homomorphism if $\phi(x\alpha y) = \phi(x)\alpha\phi(y)$ which holds for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 3. [2] An additive mapping $\psi : M \longrightarrow M$ is called an anti-homomorphism if $\psi(x\alpha y) = \psi(y)\alpha\psi(x)$ which holds for all $x, y \in M$ and $\alpha \in \Gamma$.

Note: A derivation d of M is said to act as a homomorphism [resp. as an anti-homomorphism] on a subset S of M if $d(x\alpha y) = d(x)\alpha d(y)$[resp.$d(x\alpha y) = d(y)\alpha d(x)$] for all $x, y \in S$ and $\alpha \in \Gamma$.

Definition 4. [11] Let M be a Γ-ring, then an additive mapping $f : M \longrightarrow M$ is said to be endomorphism if $f(x\alpha y) = f(x)\alpha f(y)$ which holds for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 5. [11] Let M be a Γ-ring, then an additive mapping $f : M \longrightarrow M$ is called an anti-endomorphism if $f(x\alpha y) = f(y)\alpha f(x)$ which holds for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 6. [8] An additive mapping $(x\alpha x) \longrightarrow (x\alpha x)^*$ on a Γ-ring M is called an involution if $(x\alpha y)^* = y^*\alpha x^*$ and $(x\alpha x)^{**} = x\alpha x$ which holds for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 7. An additive mapping $d : M \longrightarrow M$ is called a Γ^*-derivation if $d(x\alpha y) = d(x)\alpha y^* + x\alpha d(y)$ which holds for all $x, y \in M$ and $\alpha \in \Gamma$.

In [3], Bell and Kappe proved that if d is a derivation of a semiprime ring R which is either an endomorphism or an anti-endomorphism on R, then $d = 0$; whereas, the behavior of d is somewhat restricted in case of prime rings in the way that if d is a derivation of a prime ring R acting as a homomorphism or an anti-homomorphism on a non-zero right ideal U of R, then $d = 0$. Asma et. al. [1] extended this result of prime rings on square closed Lie ideals. Afterwards, the said result was extended to σ-prime rings by Oukhtite et. al. in [6].
In Γ-rings, Dey and Paul[4] proved that if D is a generalized derivation of a prime Γ-ring M with an associated derivation d of M which acts as a homomorphism and an anti-homomorphism on a non-zero ideal I of M, then $d = 0$ or M is commutative. Afterwards, Chakraborty and Paul[11] worked on k-derivation of a semiprime Γ-ring in the sense of Nobusawa [10] and proved that $d = 0$ where d is a k-derivation acting as a k-endomorphism and as an anti-k-endomorphism, the above mentioned results following [6] in classical rings are extended to those in gamma rings with derivation acting as a homomorphism and as an anti-homomorphism on σ-prime Γ-rings. In this paper we will prove that if d is Γ^*-derivation of a semiprime Γ-ring with involution which is either an endomorphism or anti-endomorphism, then d=0.

2. Γ*-Derivation Acting as an Endomorphism and as an Anti-Endomorphism of Γ-Ring M with Involution

To prove our main result we need the following lemmas.

Lemma 2.1. *Let M be a semiprime Γ-ring satisfying assumption (A) and let a be an element in M. If $a\alpha[x,y]_\beta = 0$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$, then there exists an ideal U of M such that $a \in U \subset Z(M)$ holds.*

Proof. The lemma has been proven by Hoque and Paul[9].

Lemma 2.2. *Let M be a semiprime Γ-ring with involution and let $d : M \to M$ be a nonzero $\Gamma^\star$-derivation, then $d(x) \in Z(M)$ for all $x \in M$ and if $d(x) = [a,x]_\alpha$ for all $x \in M$ and $\alpha \in \Gamma$, then $d = 0$.*

Proof. We have

$$d(x\alpha y) \;=\; d(x)\alpha y^\star + x\alpha d(y) \tag{3}$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Replace y by $y\beta z$ in (3)we get

$$d(x\alpha(y\beta z)) \;=\; d(x)\alpha z^\star\beta y^\star + x\alpha d(y)\beta z^\star + x\alpha y\beta d(z) \tag{4}$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. And

$$d((x\alpha y)\beta z) \;=\; d(x)\alpha y^\star\beta z^\star + x\alpha d(y)\beta z^\star + x\alpha y\beta d(z) \tag{5}$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. If we comport(4) and (5), we get

$$d(x)\alpha[y^\star, z^\star]_\beta = 0 \tag{6}$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. By using Lemma 2.1, we get $d(x) \in Z(M)$ for all $x \in M$ and if $d(x) = [a,x]_\alpha$, then from relation (6) one can show that $[a,x]_\alpha\gamma M\gamma[a,x]_\alpha = 0$ for all $x \in M$ and $\alpha, \gamma \in \Gamma$. Since M is a semiprime Γ-ring with involution, we get $[a,x]_\alpha = 0$, hence $d = 0$.

Lemma 2.3. *Let $d : M \to M$ be a $\Gamma^\star$-derivation which act-endomorphism of Γ-ring M with involution and satisfying assumption (A) , then*

$$d(y)\alpha x\beta d(y) \;=\; y\alpha x\beta d(y) \tag{7}$$

$$d(x)\alpha y^\star\beta d(x) \;=\; d(x)\alpha x^\star\beta y^\star \tag{8}$$

Proof. We have

$$d(x)\alpha d(y) \quad = \quad d(x)\alpha y^{*} + x\alpha d(y) \tag{9}$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Replace x by $y\alpha x$ in (9)we get

$$d(y\alpha x)\beta d(y) \quad = \quad d(y\alpha x)\beta y^{*} + y\alpha x\beta d(y) \tag{10}$$

for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. Since d is act-endomorphism of Γ-ring M, then

$$d(y\alpha x)\beta d(y) \quad = \quad d(y)\beta d(x\alpha y)$$

Then from above relation we get

$$d(y\alpha x)\beta d(y) \quad = \quad d(y)\beta d(x)\alpha y^{*} + d(y)\beta x\alpha d(y) \tag{11}$$

for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. According to (10) and (11)we arrive at (7). Now replace y by $y\beta x$ in (9)we get

$$d(x)\alpha d(y\beta x) = d(x)\alpha x^{*}\beta y^{*} + x\alpha d(y\beta x) \tag{12}$$

for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. Since d is act-endomorphism on M, then

$$d(x)\alpha d(y\beta x) \quad = \quad d(x\beta y)\alpha d(x)$$

Therefore ,

$$d(x)\alpha d(y\beta x) \quad = \quad d(x)\beta y^{*}\alpha d(x) + x\beta d(y)\alpha d(x) \tag{13}$$

for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. By comparing (12) and (13) we arrive at (8).

Theorem 2.4. *Let M be asemiprime Γ-ring with involution satisfying assumption (A) and let $d : M \to M$ be a Γ^{*}-derivation, then*
a-If d is act-endomorphism on M, then d=0 on M.
b-If d is act anti-endomorphism on M, then d=0 on M.

Proof. a- putting $d(y)\gamma x$ for x in((7) lemma 2.3) we get

$$d(y)\alpha d(y)\gamma x\beta d(y) \quad = \quad y\alpha d(y)\gamma x\beta d(y) \tag{14}$$

for all $x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Since d is act-endomorphism on M and d is a Γ^{*}-derivation , then we get

$$d(y)\alpha y^{*}\gamma x\beta d(y) + y\alpha d(y)\gamma x\beta d(y) \quad = \quad y\alpha d(y)\gamma x\beta d(y) \tag{15}$$

for all $x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Hence

$$d(y)\alpha y^{\star}\gamma x\beta d(y) \;=\; 0 \tag{16}$$

for all $x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Right multiplication of (16) by $y^{\star}$ gives

$$d(y)\alpha y^{\star}\gamma x\beta d(y)\alpha y^{\star} \;=\; 0 \tag{17}$$

for all $x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Since M is semiprime Γ-ring with involution we get

$$d(y)\alpha y^{\star} \;=\; 0$$

for all $y \in M$ and $\alpha \in \Gamma$. By using Lemma(2.3) we get

$$d(x)\alpha y^{\star}\beta d(x) \;=\; 0$$

for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. Then by semiprimness of Γ-ring M we obtain $d = 0$.

b- If d is act anti-endomorphism on M, then

$$d(x\alpha y) \;=\; d(y)\alpha d(x) \tag{18}$$

for all $x, y \in M$ and $\alpha \in \Gamma$. By using Lemma(2.2) we get

$$d(x\alpha y) \;=\; d(x)\alpha d(y) \tag{19}$$

Then from relation (19) we get d is act-endomorphism on M, then by the same way of (a) we get d=0 on M, this completes the proof of (b).

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