Γ*-DERIVATION ACTING AS AN ENDOMORPHISM AND AS AN ANTI-ENDOMORPHISM IN SEMIPRIME Γ-RING M WITH INVOLUTION

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Abstract: Let $M$ be a semiprime Γ-ring with involution satisfying the condition that $aαbβc = aβbαc$ ($a, b, c \in M$ and $α, β \in Γ$). An additive mapping $d : M \to M$ is called Γ*-derivation if $d(xαy) = d(x)αy* + xαd(y)$. In this paper we will prove that if $d$ is Γ*-derivation of a semiprime Γ-ring with involution which is either an endomorphism or anti-endomorphism, then $d=0$.

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1. Introduction

Let $M$ and Γ be additive abelian groups. If there exists a mapping $M \times Γ \times
\( M \rightarrow M \) defined by \((x, \alpha, y) \rightarrow (x\alpha y)\) which satisfies the conditions

(i) \( x\alpha y \in M \).

(ii) \((x + y)\alpha z = x\alpha z + y\alpha z, \ x(\alpha + \beta)y = x\alpha y + x\beta y, \ x\alpha(y + z) = x\alpha y + x\alpha z.\)

(iii) \((x\alpha y)\beta z = x\alpha(y\beta z). \) (see [5], [7])

Then \( M \) is called a \( \Gamma \)-ring. Every ring \( M \) is a \( \Gamma \)-ring with \( M = \Gamma \). However a \( \Gamma \)-ring need not be a ring. \( \Gamma \)-rings, more general than rings, were introduced by Nobusawa [10]. Bernes [12] slightly weakened the conditions in the definition of \( \Gamma \)-ring in the sense of Nobusawa. Let \( M \) be a \( \Gamma \)-ring. Then an additive subgroup \( U \) of \( M \) is called a left (right) ideal of \( M \) if \( M\Gamma U \subset U \) (\( U\Gamma M \subset U \)). If \( U \) is both a left and a right ideal, then we say \( U \) is an ideal of \( M \). Suppose again that \( M \) is a \( \Gamma \)-ring. Then an additive subgroup \( U \) of \( M \) is called a left (right) ideal of \( M \) if \( M\Gamma U \subset U \) (\( U\Gamma M \subset U \)). If \( U \) is both a left and a right ideal, then we say \( U \) is an ideal of \( M \). Suppose again that \( M \) is a \( \Gamma \)-ring. Then \( M \) is said to be 2-torsion free if \( 2x = 0 \) implies \( x = 0 \) for all \( x \in M \). An ideal \( P \) of a \( \Gamma \)-ring \( M \) is said to be prime if \( \alpha \Gamma M \Gamma b = (0) \) with \( a, b \in M \), implies \( a = 0 \) or \( b = 0 \) and semiprime if \( a\Gamma M\Gamma a = (0) \) with \( a \in M \) implies \( a = 0 \). Furthermore, \( M \) is said to be a commutative \( \Gamma \)-ring if \( x\alpha y = y\alpha x \) for all \( x, y \in M \) and \( \alpha \in \Gamma \). Moreover, the set \( Z(M) = \{x \in M : x\alpha y = y\alpha x \text{ for all } \alpha \in \Gamma, \ y \in M \} \) is called the center of the \( \Gamma \)-ring \( M \). If \( M \) is a \( \Gamma \)-ring, then \([x, y]_\alpha = x\alpha y - y\alpha x \) is known as the commutator of \( x \) and \( y \) with respect to \( \alpha \), where \( x, y \in M \) and \( \alpha \in \Gamma \). We make the basic commutator identities:

\[
[x\alpha y, z]_\beta = [x, z]_\beta x\alpha y + x[\alpha, \beta]_\beta y + x\alpha[y, z]_\beta \tag{1}
\]

\[
[x, y\alpha z]_\beta = [x, y]_\beta y\alpha z + y[\alpha, \beta]_\beta xz + y\alpha[x, z]_\beta \tag{2}
\]

for all \( x, y, z \in M \) and \( \alpha, \beta \in \Gamma \). Now, we consider the following assumption:

(A) \( x\alpha y\beta z = x\beta y\alpha z \) for all \( x, y, z \in M \) and \( \alpha, \beta \in \Gamma \).

According to assumption (A), the above commutator identities reduce to

\[
[x\alpha y, z]_\beta = [x, z]_\beta x\alpha y + x\alpha[y, z]_\beta \quad \text{and} \quad [x, y\alpha z]_\beta = [x, y]_\beta y\alpha z + y\alpha[x, z]_\beta.
\]

which we will extensively used.

**Definition 1.** [7] An additive mapping \( d : M \rightarrow M \) is called a derivation if \( d(x\alpha y) = d(x)\alpha y + x\alpha d(y) \) which holds for all \( x, y \in M \) and \( \alpha \in \Gamma \).

**Definition 2.** [2] An additive mapping \( \phi : M \rightarrow M \) is said to be homomorphism if \( \phi(x\alpha y) = \phi(x)\alpha\phi(y) \) which holds for all \( x, y \in M \) and \( \alpha \in \Gamma \).
Definition 3. [2] An additive mapping $\psi : M \rightarrow M$ is called an anti-homomorphism if $\psi(x\alpha y) = \psi(y)\alpha\psi(x)$ which holds for all $x, y \in M$ and $\alpha \in \Gamma$.

Note: A derivation $d$ of $M$ is said to act as a homomorphism [resp. as an anti-homomorphism] on a subset $S$ of $M$ if $d(x\alpha y) = d(x)\alpha d(y)$ [resp. $d(x\alpha y) = d(y)\alpha d(x)$] for all $x, y \in S$ and $\alpha \in \Gamma$.

Definition 4. [11] Let $M$ be a $\Gamma$-ring, then an additive mapping $f : M \rightarrow M$ is said to be endomorphism if $f(x\alpha y) = f(x)\alpha f(y)$ which holds for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 5. [11] Let $M$ be a $\Gamma$-ring, then an additive mapping $f : M \rightarrow M$ is called an anti-endomorphism if $f(x\alpha y) = f(y)\alpha f(x)$ which holds for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 6. [8] An additive mapping $(x\alpha x) \rightarrow (x\alpha x)^*$ on a $\Gamma$-ring $M$ is called an involution if $(x\alpha y)^* = y^*\alpha^* x$ and $(x\alpha x)^{**} = x\alpha x$ which holds for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 7. An additive mapping $d : M \rightarrow M$ is called a $\Gamma^*$-derivation if $d(x\alpha y) = d(x)\alpha y^* + x\alpha d(y)$ which holds for all $x, y \in M$ and $\alpha \in \Gamma$.

In [3], Bell and Kappe proved that if $d$ is a derivation of a semiprime ring $R$ which is either an endomorphism or an anti-endomorphism on $R$, then $d = 0$; whereas, the behavior of $d$ is somewhat restricted in case of prime rings in the way that if $d$ is a derivation of a prime ring $R$ acting as a homomorphism or an anti-homomorphism on a non-zero right ideal $U$ of $R$, then $d = 0$. Asma et. al. [1] extended this result of prime rings on square closed Lie ideals. Afterwards, the said result was extended to $\sigma$-prime rings by Oukhtite et. al. in [6].

In $\Gamma$-rings, Dey and Paul [4] proved that if $D$ is a generalized derivation of a prime $\Gamma$-ring $M$ with an associated derivation $d$ of $M$ which acts as a homomorphism and an anti-homomorphism on a non-zero ideal $I$ of $M$, then $d = 0$ or $M$ is commutative. Afterwards, Chakraborty and Paul [11] worked on $k$-derivation of a semiprime $\Gamma$-ring in the sense of Nobusawa [10] and proved that $d = 0$ where $d$ is a $k$-derivation acting as a $k$-endomorphism and as an anti-$k$-endomorphism, the above mentioned results following [6] in classical rings are extended to those in gamma rings with derivation acting as a homomorphism and as an anti-homomorphism on $\sigma$-prime $\Gamma$-rings. In this paper we will prove that if $d$ is $\Gamma^*$-derivation of a semiprime $\Gamma$-ring with involution which is either an endomorphism or anti-endomorphism, then $d = 0$. 
2. $\Gamma^*$-Derivation Acting as an Endomorphism and as an Anti-Endomorphism of $\Gamma$-Ring $M$ with Involution

To prove our main result we need the following lemmas.

**Lemma 2.1.** Let $M$ be a semiprime $\Gamma$-ring satisfying assumption (A) and let $a$ be an element in $M$. If $aa[x, y]_\beta = 0$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$, then there exists an ideal $U$ of $M$ such that $a \in U \subset Z(M)$ holds.

*Proof.* The lemma has been proven by Hoque and Paul[9].

**Lemma 2.2.** Let $M$ be a semiprime $\Gamma$-ring with involution and let $d : M \to M$ be a nonzero $\Gamma^*$-derivation, then $d(x) \in Z(M)$ for all $x \in M$ and if $d(x) = [a, x]_\alpha$ for all $x \in M$ and $\alpha \in \Gamma$, then $d = 0$.

*Proof.* We have

$$d(xy) = d(xy^* + xad(y)) \quad (3)$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Replace $y$ by $y\beta z$ in (3) we get

$$d(x(y\beta z)) = d(x)\alpha y^* + xad(y)\beta z^* + x\alpha y\beta d(z) \quad (4)$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. And

$$d((x\alpha y)\beta z) = d(x)\alpha y^* \beta z^* + xad(y)\beta z^* + x\alpha y\beta d(z) \quad (5)$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. If we comport (4) and (5), we get

$$d(x)\alpha [y^*, z^*]_\beta = 0 \quad (6)$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. By using Lemma 2.1, we get $d(x) \in Z(M)$ for all $x \in M$ and if $d(x) = [a, x]_\alpha$, then from relation (6) one can show that $[a, x]_\alpha \gamma M \gamma [a, x]_\alpha = 0$ for all $x \in M$ and $\alpha, \gamma \in \Gamma$. Since $M$ is a semiprime $\Gamma$-ring with involution, we get $[a, x]_\alpha = 0$, hence $d = 0$.

**Lemma 2.3.** Let $d : M \to M$ be a $\Gamma^*$-derivation which act-endomorphism of $\Gamma$-ring $M$ with involution and satisfying assumption (A), then

$$d(y)\alpha x \beta d(y) = y \alpha x \beta d(y) \quad (7)$$

$$d(x)\alpha y^* \beta d(x) = d(x)\alpha x^* \beta y^* \quad (8)$$
\[ d(x)\alpha d(y) = d(x)\alpha y^* + x\alpha d(y) \quad (9) \]

for all \( x, y \in M \) and \( \alpha \in \Gamma \). Replace \( x \) by \( y\alpha x \) in (9) we get

\[ d(y\alpha x)\beta d(y) = d(y\alpha x)\beta y^* + y\alpha x\beta d(y) \quad (10) \]

for all \( x, y \in M \) and \( \alpha, \beta \in \Gamma \). Since \( d \) is act-endomorphism of \( \Gamma \)-ring \( M \), then

\[ d(y\alpha x)\beta d(y) = d(y)\beta d(x\alpha y) \]

Then from above relation we get

\[ d(y\alpha x)\beta d(y) = d(y)\beta d(x\alpha y) \quad (11) \]

for all \( x, y \in M \) and \( \alpha, \beta \in \Gamma \). According to (10) and (11) we arrive at (7).

Now replace \( y \) by \( y\beta x \) in (9) we get

\[ d(x)\alpha d(y\beta x) = d(x)\alpha x^*\beta y^* + x\alpha d(y\beta x) \quad (12) \]

for all \( x, y \in M \) and \( \alpha, \beta \in \Gamma \). Since \( d \) is act-endomorphism on \( M \), then

\[ d(x)\alpha d(y\beta x) = d(x\beta y)\alpha d(x) \]

Therefore,

\[ d(x)\alpha d(y\beta x) = d(x)\beta y^*\alpha d(x) + x\beta d(y)\alpha d(x) \quad (13) \]

for all \( x, y \in M \) and \( \alpha, \beta \in \Gamma \). By comparing (12) and (13) we arrive at (8).

**Theorem 2.4.** Let \( M \) be a semiprime \( \Gamma \)-ring with involution satisfying assumption (A) and let \( d : M \to M \) be a \( \Gamma^* \)-derivation, then

a- If \( d \) is act-endomorphism on \( M \), then \( d = 0 \) on \( M \).

b- If \( d \) is act anti-endomorphism on \( M \), then \( d = 0 \) on \( M \).

**Proof.** a- Putting \( d(y)\gamma x \) for \( x \) in (7) lemma 2.3 we get

\[ d(y)\alpha d(y)\gamma x\beta d(y) = y\alpha d(y)\gamma x\beta d(y) \quad (14) \]

for all \( x, y \in M \) and \( \alpha, \beta, \gamma \in \Gamma \). Since \( d \) is act-endomorphism on \( M \) and \( d \) is a \( \Gamma^* \)-derivation, then we get

\[ d(y)\alpha y^* \gamma x\beta d(y) + y\alpha d(y)\gamma x\beta d(y) = y\alpha d(y)\gamma x\beta d(y) \quad (15) \]
for all $x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Hence
\[
d(y)\alpha y^* \gamma x \beta d(y) = 0
\] (16)
for all $x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Right multiplication of (16) by $y^*$ gives
\[
d(y)\alpha y^* \gamma x \beta d(y)\alpha y^* = 0
\] (17)
for all $x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Since $M$ is semiprime $\Gamma$-ring with involution we get
\[
d(y)\alpha y^* = 0
\]
for all $y \in M$ and $\alpha \in \Gamma$. By using Lemma(2.3) we get
\[
d(x)\alpha y^* \beta d(x) = 0
\]
for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. Then by semiprimness of $\Gamma$-ring $M$ we obtain $d = 0$.

b- If $d$ is act anti-endomorphism on $M$, then
\[
d(x\alpha y) = d(y)\alpha d(x)
\] (18)
for all $x, y \in M$ and $\alpha \in \Gamma$. By using Lemma(2.2) we get
\[
d(x\alpha y) = d(x)\alpha d(y)
\] (19)
Then from relation (19) we get $d$ is act-endomorphism on $M$, then by the same way of (a) we get $d=0$ on $M$, this completes the proof of (b).

References


