INVARIANT CORE FOR DOUBLE SEQUENCES IN NON-ARCHIMEDEAN FIELDS

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Abstract: In this paper, K denotes a complete, non-trivially valued, non-archimedean field. The entries of sequences, series and infinite matrices are in K . In the present paper, we prove the Knopp's core theorem for double sequences in K and also the necessary and sufficient conditions for the core to be invariant under the four dimensional matrix transformation in such fields.

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1. Introduction

Throughout this paper K denotes a complete, non-trivially valued, non-archimedean field, where the non-archimedean valuation satisfies the following axioms:

- (i) $|x| \geq 0$ and $|x| = 0$ iff $x = 0$.
- (ii) $|xy| = |x||y|$.

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(iii) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$.

For analysis in the classical case a general reference is [3], while for analysis in non-archimedean fields, a general reference is [2].

Patterson [8] has proved the Invariant core theorem for double sequences in classical case. We prove here its analogue for double sequences in a complete, non-trivially valued, non-archimedean field.

For a given infinite matrix $A = (a_{mnkl})$, we define

$$Ax = (Ax)_{mn} = \sum_{k=1, l=1}^{\infty, \infty} a_{mnkl} x_{kl}, \quad m, n = 1, 2, 3, \dots,$$

assuming that the series on the right converge and $\{(Ax)_{mn}\}$ is called the A -transform of the double sequence $x = \{x_{kl}\}$.

If, whenever $\{x_{mn}\}$ converges to a limit, $\{(Ax)_{mn}\}$ converges to the same limit, then the matrix $A = (a_{mnkl})$ is said to be regular.

Definition 1. Let $C_n(x)$, $n = 1, 2, 3, \dots$ be the smallest closed convex set containing all points x_{kl} for $k, l > n$. We define the core of the double sequence $x = \{x_{kl}\}$ as

$$\mathcal{K}(x) = \bigcap_{n=1}^{\infty} C_n(x).$$

Also define $C_r(u) = \{u' \in K \mid |u' - u| \leq r\}$, where $C(0) = \{u \in K \mid |u| \leq 1\}$.

In this context, we refer to Silvermann-Toeplitz theorem for double sequences and series in non-archimedean fields [6].

Definition 2. Let $\{x_{kl}\}$ be a double sequence in K and $x \in K$. We say that $\lim_{k+l \rightarrow \infty} x_{kl} = x$ if for each $\epsilon > 0$, the set $\{(k, l) \in \mathbb{N}^2 : |x - x_{kl}| \geq \epsilon\}$ is finite. In such a case we say that x is the limit of $\{x_{kl}\}$.

Definition 3. Let $\{x_{kl}\}$ be a double sequence in K and $s \in K$. We say that

$$s = \sum_{k=1, l=1}^{\infty, \infty} x_{kl},$$

if $s = \lim_{k+l \rightarrow \infty} s_{kl}$ where $s_{kl} = \sum_{i=1, j=1}^{k, l} x_{ij}$, $k, l = 1, 2, \dots$

Remark 4. If $\lim_{k+l \rightarrow \infty} x_{kl} = x$, then the sequence $\{x_{kl}\}$ is automatically bounded.

Lemma 5. $\lim_{k+l \rightarrow \infty} x_{kl} = x$ if and only if

- (i) $\lim_{l \rightarrow \infty} x_{kl} = x, k = 1, 2, \dots$
- (ii) $\lim_{k \rightarrow \infty} x_{kl} = x, l = 1, 2, \dots$ and
- (iii) for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x - x_{kl}| < \epsilon$, for all $k, l \geq N$, which we write as $\lim_{k, l \rightarrow \infty} x_{kl} = x$.

Lemma 6. $\lim_{k+l \rightarrow \infty} s_{kl}$ exists if and only if $\lim_{k+l \rightarrow \infty} x_{kl} = 0$.

Theorem 7. In order that whenever a sequence $\{x_{mn}\}$ has a limit x , $\sum_{k=1, l=1}^{\infty, \infty} a_{mnkl} x_{kl}$, shall converge and $\lim_{m+n \rightarrow \infty} \sum_{k=1, l=1}^{\infty, \infty} a_{mnkl} x_{kl} = x$ i.e., for $A = (a_{mnkl})$ to be regular it is necessary and sufficient that

- (a) $\lim_{m+n \rightarrow \infty} a_{mnkl} = 0, k, l = 1, 2, \dots,$
- (b) $\lim_{m+n \rightarrow \infty} \sum_{k=1, l=1}^{\infty, \infty} a_{mnkl} = 1,$
- (c) $\lim_{m+n \rightarrow \infty} \sup_{k \geq 1} |a_{mnkl}| = 0, l = 1, 2, \dots,$
- (d) $\lim_{m+n \rightarrow \infty} \sup_{l \geq 1} |a_{mnkl}| = 0, k = 1, 2, \dots,$
- (e) $\sup_{m, n, k, l} |a_{mnkl}| < \infty.$

2. Main Result

Theorem 8. An infinite matrix $A = (a_{mnkl}), a_{mnkl} \in K$ is such that $\mathcal{K}(Ax) \subset \mathcal{K}(x)$ for a double sequence $x = \{x_{kl}\}$ iff A is regular and satisfies

$$\lim_{m+n \rightarrow \infty} \sup_{k, l \geq 1} |a_{mnkl}| = 1. \quad (1)$$

Proof. Necessary part:

Let $x = \{x_{kl}\}$ be a double sequence converging to L in K such that $A = (a_{mnkl})$ is regular and satisfies (1). If v is any point of $\mathcal{K}(Ax)$, then

$$\begin{aligned}
 |v - L| &\leq \lim_{m+n \rightarrow \infty} \sup_{k,l \geq 1} |(Ax)_{mn} - L|, \quad L \in K. \\
 &= \lim_{m+n \rightarrow \infty} \sup_{k,l \geq 1} \left| \sum_{k,l=1}^{\infty, \infty} a_{mnkl} x_{kl} - L \right|, \quad L \in K. \\
 &= \lim_{m+n \rightarrow \infty} \sup_{k,l \geq 1} \left| \sum_{k,l=1}^{\infty, \infty} a_{mnkl} x_{kl} - \sum_{k,l=1}^{\infty, \infty} a_{mnkl} L \right| \quad (\text{using (b) of Theorem 7}) \\
 &= \lim_{m+n \rightarrow \infty} \sup_{k,l \geq 1} \left| \sum_{k,l=1}^{\infty, \infty} a_{mnkl} (x_{kl} - L) \right| \\
 &\leq \lim_{m+n \rightarrow \infty} \sup_{k,l \geq 1} |x_{kl} - L| \quad \text{using (i)} \\
 \text{i.e., } |v - L| &\leq \lim_{m+n \rightarrow \infty} \sup_{k,l \geq 1} |x_{kl} - L|
 \end{aligned}$$

$\Rightarrow v \in \mathcal{K}(x)$.

Thus, $v \in \mathcal{K}(Ax)$ implies $v \in \mathcal{K}(x)$ gives, $\mathcal{K}(Ax) \subset \mathcal{K}(x)$.

Sufficient part:

Conversely, if $\mathcal{K}(Ax) \subset \mathcal{K}(x)$, then it is clear that $\{(Ax)_{mn}\}$ is a convergent sequence and has a limit in $\mathcal{K}(x)$ Therefore, A is regular. (see [4]).

Now to Prove (i), we shall suppose that (i) does not hold

$$\lim_{m+n \rightarrow \infty} \sup_{k,l \geq 1} |a_{mnkl}| \neq 1.$$

and arrive at a contradiction.

Let if possible $\lim_{m+n \rightarrow \infty} \sup_{k,l \geq 1} |a_{mnkl}| > 1$.

Since K is non-trivially valued, there exists $\pi \in K$ such that $0 < \rho = |\pi| < 1$.

Let us choose $m_1 = n_1 = 1$. Using (a), (b) of Theorem 7, choose $m_2 + n_2 > m_1 + n_1$, there exists $\epsilon > 0$, such that

$$\begin{aligned}
 \sup_{1 \leq k+l \leq m_1+n_1} |a_{m_2 n_2 kl}| &< \frac{\epsilon}{8} \\
 \sup_{k+l \geq 1} |a_{m_2 n_2 kl}| &> \frac{\epsilon}{\rho} \quad \text{and}
 \end{aligned}$$

$$\sup_{k+l \geq m_2+n_2} |a_{m_2 n_2 kl}| < \frac{\epsilon}{8}$$

Inductively choose $m_p + n_p > m_{p-1} + n_{p-1}$ such that,

$$\sup_{1 \leq k+l \leq m_{p-1}+n_{p-1}} |a_{m_p n_p kl}| < \frac{\epsilon}{8} \quad (2)$$

$$\sup_{k+l \geq 1} |a_{m_p n_p kl}| > \frac{\epsilon}{\rho} \quad \text{and} \quad (3)$$

$$\sup_{k+l > m_p+n_p} |a_{m_p n_p kl}| < \frac{\epsilon}{8} \quad (4)$$

In view of (2), (3), (4), we have

$$\begin{aligned} \sup_{m_{p-1}+n_{p-1} < k+l \leq m_p+n_p} |a_{m_p n_p kl}| &> \frac{\epsilon}{\rho} - \frac{\epsilon}{8} - \frac{\epsilon}{8} \\ &> \frac{\epsilon}{\rho} - \frac{\epsilon}{8\rho} - \frac{\epsilon}{8\rho} = \frac{3\epsilon}{4\rho} \end{aligned}$$

Thus, there exist k_p and l_p , $m_{p-1} + n_{p-1} < k_p + l_p \leq m_p + n_p$ such that $|a_{m_p n_p k_p l_p}| > \frac{3\epsilon}{4\rho}$.

Now, define the sequence $\{x_{kl}\}$ as follows:

$$x_{kl} = \begin{cases} \rho, & \text{if } k = k_p, l = l_p, p = 1, 2, \dots \\ 0, & \text{otherwise.} \end{cases}$$

For this sequence x , $\lim_{k+l \rightarrow \infty} x_{kl} = 0$ implies $\mathcal{K}(x) \subset C(0)$. Now,

$$\begin{aligned} |(Ax)_{m_p n_p}| &= \left| \sum_{k=1, l=1}^{\infty, \infty} a_{m_p n_p kl} x_{kl} \right| \\ &\geq \left| \sum_{k+l=(m_{p-1}+n_{p-1})+1}^{m_p+n_p} a_{m_p n_p kl} x_{kl} \right| - \left| \sum_{k+l=1}^{m_{p-1}+n_{p-1}} a_{m_p n_p kl} x_{kl} \right| \\ &\quad - \left| \sum_{k+l=(m_p+n_p)+1}^{\infty} a_{m_p n_p kl} x_{kl} \right| \\ &> |a_{m_p n_p k_p l_p}| |x_{k_p l_p}| - \sup_{1 \leq k+l \leq m_{p-1}+n_{p-1}} |a_{m_p n_p kl}| - \sup_{k+l > m_p+n_p} |a_{m_p n_p kl}| \\ &> \frac{3\epsilon}{4\rho} \cdot \rho - \frac{\epsilon}{8} - \frac{\epsilon}{8} = \frac{\epsilon}{2} \end{aligned}$$

$$\Rightarrow |(Ax)_{m_p n_p}| > \frac{\epsilon}{2}$$

Therefore, $\lim_{m+n \rightarrow \infty} (Ax)_{mn} = 0$ does not hold, which implies that $(Ax)_{mn} \notin C(0)$.

i.e., $\mathcal{K}(Ax) \not\subset C(0)$, which is a contradiction.

This completes the proof of the theorem. $\square$

Theorem 9. *The core of a double sequence $\{x_{kl}\}$ is invariant under the regular A -transformation iff*

$$(i) \quad \lim_{m+n \rightarrow \infty} \sup_{k,l \geq 1} |a_{mnkl}| = 1.$$

(ii) *for every given pair of index sequences (k_j, l_j) , $i, j = 1, 2, \dots$ there exist a pair of index sequences (m_p, n_q) , $p, q = 1, 2, \dots$ such that*

$$\lim_{m_p + n_q \rightarrow \infty} \sum_{i,j=1}^{\infty} a_{m_p n_q k_i l_j} = 1.$$

Proof. Necessary part:

Assume $\mathcal{K}(Ax) = \mathcal{K}(x)$. Then by Theorem 8 condition (i) holds.

The proof of condition (ii) is as follows.

Let (k_i, l_j) for $i, j = 1, 2, \dots$ be given pair of index sequences and define $\{x_{kl}\}$ as follows:

$$x_{kl} = \begin{cases} 1, & \text{if } k = k_i \text{ and } l = l_j \\ 0, & \text{otherwise.} \end{cases}$$

Then we find that if there exists an N such that $x_{kl} \neq 0$ whenever $k, l > N$, then $\mathcal{K}(Ax) = \mathcal{K}(x) = 1$. This implies that there exist a double subsequence of $\{(Ax)_{mn}\}$ with limit 1. i.e., there exist a pair of index sequences (m_p, n_q) , $p, q = 1, 2, \dots$ such that

$$\lim_{m_p + n_q \rightarrow \infty} \sum_{i,j=1}^{\infty} a_{m_p n_q k_i l_j} = 1.$$

This completes the necessary part of this theorem.

Sufficient part:

Let $\{x_{k_i l_j}\}$ be a subsequence of $\{x_{kl}\}$ such that

$$\lim_{i+j \rightarrow \infty} x_{k_i l_j} = \alpha$$

Also, let (k_i, l_j) for $i, j = 1, 2, \dots$ denote the pair of indices from $\{x_{kl}\}$ used in the construction of $\{(Ax)_{mn}\}$ and let $(\bar{k}_i, \bar{l}_j)$ for $i, j = 1, 2, \dots$ be the remaining pair of indices of $\{x_{kl}\}$.

Let us define $\{w_{kl}\}$ and $\{\bar{w}_{kl}\}$ as follows:

$$w_{kl} = \begin{cases} x_{k_i l_j}, & \text{if } k = k_i \text{ and } l = l_j \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\bar{w}_{kl} = \begin{cases} x_{\bar{k}_i \bar{l}_j}, & \text{if } k = \bar{k}_i \text{ and } l = \bar{l}_j \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\{x_{kl}\} = \{w_{kl}\} + \{\bar{w}_{kl}\}$ which gives the following:

$$\begin{aligned} (Ax)_{mn} &= \sum_{k=1, l=1}^{\infty, \infty} a_{mnkl} x_{kl} \\ &= \sum_{k=1, l=1}^{\infty, \infty} a_{mnkl} w_{kl} + \sum_{k=1, l=1}^{\infty, \infty} a_{mnkl} \bar{w}_{kl} \\ &= \sum_{i=1, j=1}^{\infty, \infty} a_{mnk_i l_j} x_{k_i l_j} + \sum_{i=1, j=1}^{\infty, \infty} a_{mn\bar{k}_i \bar{l}_j} x_{\bar{k}_i \bar{l}_j} \end{aligned}$$

Let (m_p, n_q) be the pair of index sequence defined earlier such that

$$(Ax)_{m_p n_q} = \sum_{i=1, j=1}^{\infty, \infty} a_{m_p n_q k_i l_j} x_{k_i l_j} + \sum_{i=1, j=1}^{\infty, \infty} a_{m_p n_q \bar{k}_i \bar{l}_j} x_{\bar{k}_i \bar{l}_j}$$

Since $\lim_{i+j \rightarrow \infty} x_{k_i l_j} = \alpha$, if $\epsilon > 0$, we can choose sufficiently large s, t such that $x_{k_i l_j} = \alpha + \epsilon_{k_i l_j}$ with $|\epsilon_{k_i l_j}| < \frac{\epsilon}{16}$, whenever, $i + j > s + t$. Then,

$$\begin{aligned} (Ax)_{m_p n_q} - \alpha &= \left[\sum_{i=1, j=1}^{\infty, \infty} a_{m_p n_q k_i l_j} x_{k_i l_j} + \sum_{i=1, j=1}^{\infty, \infty} a_{m_p n_q \bar{k}_i \bar{l}_j} x_{\bar{k}_i \bar{l}_j} \right] - \alpha \\ &= \left[\sum_{i=1, j=1}^{s, t} a_{m_p n_q k_i l_j} x_{k_i l_j} + \sum_{i=1, j=t+1}^{s, \infty} a_{m_p n_q k_i l_j} x_{k_i l_j} + \sum_{i=s+1, j=1}^{\infty, t} a_{m_p n_q k_i l_j} x_{k_i l_j} \right. \\ &\quad \left. + \sum_{i=s+1, j=t+1}^{\infty, \infty} a_{m_p n_q k_i l_j} x_{k_i l_j} + \sum_{i=1, j=1}^{\infty, \infty} a_{m_p n_q \bar{k}_i \bar{l}_j} x_{\bar{k}_i \bar{l}_j} \right] - \alpha \end{aligned}$$

$$\begin{aligned}
&= \left[\sum_{i=1, j=1}^{s, t} a_{m_p n_q k_i l_j} x_{k_i l_j} + \sum_{i=1, j=t+1}^{s, \infty} a_{m_p n_q k_i l_j} x_{k_i l_j} + \sum_{i=s+1, j=1}^{\infty, t} a_{m_p n_q k_i l_j} x_{k_i l_j} \right. \\
&\quad \left. + \sum_{i=s+1, j=t+1}^{\infty, \infty} a_{m_p n_q k_i l_j} (\alpha + \epsilon_{k_i l_j}) + \sum_{i=1, j=1}^{\infty, \infty} a_{m_p n_q \bar{k}_i \bar{l}_j} x_{\bar{k}_i \bar{l}_j} \right] - \alpha \\
&= \sum_{i=1, j=1}^{s, t} a_{m_p n_q k_i l_j} x_{k_i l_j} + \sum_{i=1, j=t+1}^{s, \infty} a_{m_p n_q k_i l_j} x_{k_i l_j} + \sum_{i=s+1, j=1}^{\infty, t} a_{m_p n_q k_i l_j} x_{k_i l_j} \\
&\quad + \sum_{i=s+1, j=t+1}^{\infty, \infty} a_{m_p n_q k_i l_j} \epsilon_{k_i l_j} + \alpha \left(\sum_{i=s+1, j=t+1}^{\infty, \infty} a_{m_p n_q k_i l_j} - 1 \right) \\
&\quad + \sum_{i=1, j=1}^{\infty, \infty} a_{m_p n_q \bar{k}_i \bar{l}_j} x_{\bar{k}_i \bar{l}_j}
\end{aligned}$$

Since $\{x_{kl}\}$ is bounded, there exists an M such that $|x_{kl}| < M$. Therefore, the following holds:

$$\begin{aligned}
&|(Ax)_{m_p n_q} - \alpha| \\
&\leq \left| \sum_{i=1, j=1}^{s, t} a_{m_p n_q k_i l_j} x_{k_i l_j} \right| + \left| \sum_{i=1, j=t+1}^{s, \infty} a_{m_p n_q k_i l_j} x_{k_i l_j} \right| + \left| \sum_{i=s+1, j=1}^{\infty, t} a_{m_p n_q k_i l_j} x_{k_i l_j} \right| \\
&\quad + \left| \sum_{i=s+1, j=t+1}^{\infty, \infty} a_{m_p n_q k_i l_j} \epsilon_{k_i l_j} \right| + |\alpha| \left| \sum_{i=s+1, j=t+1}^{\infty, \infty} a_{m_p n_q k_i l_j} - 1 \right| \\
&\quad + \left| \sum_{i=1, j=1}^{\infty, \infty} a_{m_p n_q \bar{k}_i \bar{l}_j} x_{\bar{k}_i \bar{l}_j} \right| \tag{5}
\end{aligned}$$

Therefore, (e) of Theorem 7 implies that we can choose N_1 such that, whenever $m_p + n_q > N_1$,

$$\begin{aligned}
&\sup_{1 \leq i+j \leq s+t} |a_{m_p n_q k_i l_j}| < \frac{\epsilon}{16M}, \\
&\sup_{i > s} |a_{m_p n_q k_i l_j}| < \frac{\epsilon}{16M}, \quad j = 1, 2, \dots, t \text{ and} \\
&\sup_{j > t} |a_{m_p n_q k_i l_j}| < \frac{\epsilon}{16M}, \quad i = 1, 2, \dots, s.
\end{aligned}$$

We also have from (ii)

$$(Ax)_{m_p n_q} = \sum_{i=1, j=1}^{s, t} a_{m_p n_q k_i l_j} + \sum_{i=s+1, j=t+1}^{\infty, \infty} a_{m_p n_q k_i l_j} \quad (6)$$

And if $\alpha \neq 0$, we determine $N_2 > N_1$, so that $|(Ax)_{m_p n_q} - 1| \leq \frac{\epsilon}{16|\alpha|}$ for $m_p + n_q > N_2$. Then from (6),

$$\begin{aligned} & \sum_{i=s+1, j=t+1}^{\infty, \infty} a_{m_p n_q k_i l_j} \\ &= (Ax)_{m_p n_q} - \sum_{i=1, j=1}^{s, t} a_{m_p n_q k_i l_j} - \sum_{i=1, j=t+1}^{s, \infty} a_{m_p n_q k_i l_j} - \sum_{i=s+1, j=1}^{\infty, t} a_{m_p n_q k_i l_j} \\ & \left| \sum_{i=s+1, j=t+1}^{\infty, \infty} a_{m_p n_q k_i l_j} - 1 \right| \\ &= \left| (Ax)_{m_p n_q} - 1 - \sum_{i=1, j=1}^{s, t} a_{m_p n_q k_i l_j} - \sum_{i=1, j=t+1}^{s, \infty} a_{m_p n_q k_i l_j} - \sum_{i=s+1, j=1}^{\infty, t} a_{m_p n_q k_i l_j} \right| \\ &\leq |(Ax)_{m_p n_q} - 1| + \left| \sum_{i=1, j=1}^{s, t} a_{m_p n_q k_i l_j} \right| + \left| \sum_{i=1, j=t+1}^{s, \infty} a_{m_p n_q k_i l_j} \right| \\ &\quad + \left| \sum_{i=s+1, j=1}^{\infty, t} a_{m_p n_q k_i l_j} \right| \\ &\leq |(Ax)_{m_p n_q} - 1| + \sup_{1 \leq i+j \leq s+t} |a_{m_p n_q k_i l_j}| + \sup_{j > t} |a_{m_p n_q k_i l_j}| + \sup_{i > s} |a_{m_p n_q k_i l_j}| \\ &\leq \frac{\epsilon}{4|\alpha|}. \end{aligned}$$

Also, by (i) and (ii), there exists an $N_3 > N_2$ such that

$$\sup_{i+j > s+t} |a_{m_p n_q k_i l_j}| < 1 \quad \text{for } m_p + n_q > N_3.$$

Now from (5),

$$|(Ax)_{m_p n_q} - \alpha| \leq \frac{\epsilon}{2} + M \sup_{i+j \geq 1} |a_{m_p n_q \bar{k}_i \bar{l}_j}|, \text{ if } \alpha \neq 0.$$

$$|(Ax)_{m_p n_q}| \leq \frac{\epsilon}{4} + M \sup_{i+j \geq 1} |a_{m_p n_q \bar{k}_i \bar{l}_j}|, \text{ if } \alpha = 0.$$

There exists N_4 such that,

$$\sup_{i+j \geq 1} |a_{m_p n_q \bar{k}_i \bar{l}_j}| \leq \frac{\epsilon}{2M}$$

Therefore, for $m_p + n_q > N_4$, we have,

$$|(Ax)_{m_p n_q} - \alpha| \leq \epsilon \text{ if } \alpha \neq 0,$$

and

$$|(Ax)_{m_p n_q}| \leq \frac{3\epsilon}{4} \text{ if } \alpha = 0.$$

$$\Rightarrow \lim_{m+n \rightarrow \infty} (Ax)_{mn} = \alpha.$$

$\therefore \alpha \in \mathcal{K}(x)$ implies $\alpha \in \mathcal{K}(Ax)$. Hence $\mathcal{K}(x) \subset \mathcal{K}(Ax)$.

By Theorem 8, we conclude that $\mathcal{K}(Ax) = \mathcal{K}(x)$.

This completes the proof of the theorem. $\square$

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