HYERS-ULAM-RASSIAS STABILITY
OF FRACTIONAL DIFFERENTIAL EQUATION

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Abstract: In this paper, we proved Hyers-Ulam and Hyers-Ulam-Rassias stability for the following fractional differential equation with boundary condition

\[ D^\alpha y(t) = F(t, y(t)), \]
\[ ay(0) + by(T) = c. \]

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1. Introduction

Let \( Y \) be a normed space and \( I = [0, T] \) be a given interval. Assume that for a continuously differentiable function \( f : I \to Y \) satisfying fractional differential inequality \( \|{}^c D^\alpha y(t) - F(t, y(t))\| \leq \epsilon \) for all \( t \in I \) and for some \( \epsilon > 0 \), where \( {}^c D^\alpha \)
is the Caputo fractional derivative of order $\alpha \in (0, 1)$, there exists a solution $f_0 : I \to Y$ of the fractional boundary value problem (0.1) and (0.2) such that $\|f(t) - f_0(t)\| \leq K(\epsilon)$ for all $t \in I$, where $K(\epsilon)$ is only dependent on $\epsilon$. Then, we say that the above fractional boundary value problem (0.1) and (0.2) has the Hyers-Ulam stability.

If the above statement is also true when we replace $\epsilon$ and $K(\epsilon)$ by $\varphi(t)$ and $\Phi(t)$, where $\varphi, \Phi : I \to [0, \infty)$ are functions not depending on $f$ and $f_0$ explicitly, then we say that the corresponding fractional boundary value problem (0.1) and (0.2) has the Hyers-Ulam-Rassias stability.

Fractional differential equations is the area of concentration of recent research and there has been significant progress in this area. However, the concept of fractional derivative is not new and is very much as old as differential equations. In 1695, L Hospital raised the question about fractional derivative in a letter written to Leibniz and related his generalization of differentiation. Recently, the differential equations of fractional order has proved to be a valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find many applications in electromagnetic, control, electrochemistry etc. (see [22, 23, 24, 25]). For more details on this area, one can see the monographs of Kilbas et al [4], Miller and Ross [6], I.Podulbny [5], Diethelm et al [8, 9], Benchora [2] and the references therein.

In 1940, S. M. Ulam [10] asked the question concerning the stability of group homomorphisms. Next year Hyers [11] gave the first positive answer to the question of Ulam for Banach spaces. Thereafter, the stability of this type is called Hyers-Ulam Stability. In 1978 Th.M. Rassias [12] provided a generalization of the Hyers theorem which allows the Cauchy difference to be unbounded. After this result many mathematicians were attracted and motivated to investigate the Hyers-Ulam stability and Hyers-Ulam-rassias stability in the mathematical analysis area. The stability occurrence that was introduced and proved by Th.M. Rassias in his paper is called the Hyers-Ulam-Rassias stability.

The developement of the analysis of stability of fractional differential equations has been a bit slow. Recently, Li and Zhang [16] made a brief overview on the stability results of the fractional differential equations. Li et al. [17, 18] first proposed the Mittag Leffler stability and the fractional Lyapunov second method. Although, there are some work on the local stability and Mittag Leffler stability for fractional differential equations, to the best of my knowledge, there are very rare works on the Ulam stability of fractional differential equations. A pioneering work on the Ulam stability and data dependence for fractional differential equations with Caputo derivative has been reported by
Wang et al. [19]. Most recently J. Wang [1] proved Hyers Ulam stability as well as the Hyers-Ulam-Rassias stability of fractional differential equations via a generalized fixed point approach, by adopting some part idea of Wang et al. [19], Cadariu and Radu [20] and Jung [21].

In this paper we proved Hyers-Ulam stability and Hyers-Ulam-Rassias stability of fractional differential equation (0.1) with the boundary condition (0.2). This paper is organized as follows: In Section 2, basic definitions and notations are given. In Section 3, the Hyers-Ulam-Rassias stability of fractional differential equation (0.1) with boundary condition (0.2) is proven. In Section 4, the Hyers-Ulam stability of fractional differential equation (0.1) with boundary condition (0.2) is proven.

2. Preliminaries

In this section, we give some basic definition and theorems which we used to prove the results.

Definition 2.1. [2] The fractional order integral of the function $h \in L^1([a, b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I^\alpha_a h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} h(s) ds,$$

where $\Gamma$ is the gamma function.

Definition 2.2. [2] For a function $h$ given on the the interval $[a, b]$, the $\alpha$-th Riemann-Liouville fractional order derivative of $h$, is defined by

$$(D^\alpha_a h)(t) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_a^t (t - s)^{n-\alpha-1} h(s) ds.$$

Here $n = \lfloor \alpha \rfloor + 1$ and $\lfloor \alpha \rfloor$ denotes the integer part of $\alpha$.

Definition 2.3. [2] For a function $h$ given on the interval $[a, b]$, the Caputo fractional order derivative of $h$, is defined by

$$(^{c}D^\alpha_{a+} h)(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - s)^{n-\alpha-1} h^{(n)}(s) ds,$$

where $n = \lfloor \alpha \rfloor + 1$ and $\lfloor \alpha \rfloor$ denotes the integer part of $\alpha$. 
**Definition 2.4.** [2] A function \( y \in C(J, \mathbb{R}) \) is said to be a solution of (0.1)-(0.2) if \( y \) satisfies the equation \( ^cD^\alpha y(t) = f(t, y(t)) \) on \( J \), and the condition \( ay(0) + by(T) = c \)

**Lemma 2.1.** [2] Let \( 0 < \alpha < 1 \) and let \( f : [0, T] \to \mathbb{R} \) be continuous. A function \( y \in C(J, \mathbb{R}) \) is a solution of the fractional integral equation

\[
y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) \, ds - \frac{1}{a+b} \left[ \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s)) \, ds - c \right]
\]

(2.1)

if and only if \( y \) is a solution of the fractional boundary value problem

\[
^cD^\alpha y(t) = f(t, y(t)), \quad t \in [0, T]
\]

\[
ay(0) + by(T) = c.
\]

**Definition 2.5.** For a nonempty set \( X \), a function \( d : X \times X \to [0, \infty] \) is called generalized metric on \( X \) if and only if \( d \) satisfies

1. \( d(x, y) = 0 \) if and only if \( x = y \);
2. \( d(x, y) = d(y, x) \) for all \( x, y \in X \);
3. \( d(x, z) \leq d(x, y) + d(y, z) \) for all \( x, y, z \in X \).

**Theorem 2.1.** Let \( (X, d) \) be a generalised complete metric space. Assume that \( \Lambda : X \to X \) is a strictly contractive operator with the Lipschitz constant \( L < 1 \), If there exists a nonnegative integer \( k \) such that \( d(\Lambda^k x, \Lambda^k y) < \infty \) for some \( x \in X \), then the following are true:

- (a) The sequence \( \{\Lambda^nx\} \) converges to a fixed end point \( x^* \) of \( \Lambda \)
- (b) \( x^* \) is the unique fixed point of \( \Lambda \) in \( X^* = \{ y \in X / d(\Lambda^k x, y) < \infty \} \);
- (c) If \( y \in X^* \), then \( d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y) \)

3. **Hyers-Ulam-Rassias stability**

In this section, we first investigate the Hyers-Ulam-Rassias stability of the fractional differential equation (0.1) with boundary condition (0.2) in the interval \([0, T]\) via theorem (2.1), by using the idea of cadariu and Radu [20], Jung [21], and J.wang [1].
Theorem 3.2. Let $I = [0, T]$ be a closed interval. Let $K, P,$ and $L$ be positive constants with $0 < KPL < 1$. Assume that $F : I \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function which satisfies the standard Lipschitz condition

$$|F(t, y) - F(t, z)| \leq L|y - z|$$  \hspace{1cm} (3.1)

for any $t \in I$ and $y, z \in \mathbb{R}$. If a continuously differential function $y : I \rightarrow \mathbb{R}$ satisfies

$$|^cD^\alpha y(t) - F(t, y(t))| \leq \varphi(t)$$  \hspace{1cm} (3.2)

for all $t \in I$, where $\varphi : I \rightarrow (0, \infty)$ is a continuous function with

$$\left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(\tau)d\tau \right| \leq K \varphi(t)$$  \hspace{1cm} (3.3)

for all $t \in I$, then there exists unique continuous function $y_0 : I \rightarrow \mathbb{R}$ such that

$$y_0(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, y_0(s))ds$$

$$- \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, y_0(s))ds + \frac{c}{a+b}$$  \hspace{1cm} (3.4)

and

$$|y(t) - y_0(t)| \leq \frac{K}{1 - KPL} \varphi(t)$$  \hspace{1cm} (3.5)

Proof. Let us define a set $X$ of all continuous functions $f : I \rightarrow \mathbb{R}$ by

$$X = \{ f : I \rightarrow \mathbb{R} | f \text{ is continuous} \}.$$  \hspace{1cm} (3.6)

Similar to theorem 3.1 of Jung S-M [21], we introduce a generalised complete metric on $X$ as follows

$$d(f, g) = \inf \{ C \in [0, \infty] / |f(t) - g(t)| \leq C \varphi(t) \text{ for all } t \in I \}$$  \hspace{1cm} (3.7)

Define an operator $\Lambda : X \rightarrow X$ by

$$(\Lambda f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, f(s))ds$$

$$- \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, f(s))ds + \frac{c}{a+b}$$  \hspace{1cm} (3.8)

for all $f \in X$. 
It is easy to see that $\Lambda$ is well defined, since $F$ and $f$ are continuous functions.

To achieve our aim, we need to prove that $\Lambda$ is strictly contractive on $X$.
For any $f, g \in X$, let $C_{fg} \in [0, \infty]$ be an arbitrary constant with $d(f, g) \leq C_{fg}$, that is by (3.7) we have

$$|f(t) - g(t)| \leq C_{fg}\varphi(t)$$  \hspace{1cm} (3.9)

for any $t \in I$. It then follows from (3.1), (3.3), (3.7), (3.8) and (3.9) that

$$|(\Lambda f) t - (\Lambda g) t| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} [F(s, f(s)) - F(s, g(s))] \, ds ight|$$

$$- \frac{b}{(a + b)\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1} [F(s, f(s)) - F(s, g(s))] \, ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} |F(s, f(s)) - F(s, g(s))| \, ds$$

$$- \frac{b}{(a + b)\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1} |F(s, f(s)) - F(s, g(s))| \, ds$$

$$\leq \frac{L}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} |f(s) - g(s)| \, ds$$

$$- \frac{bL}{(a + b)\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1} |f(s) - g(s)| \, ds$$

$$\leq \frac{L}{\Gamma(\alpha)} C_{fg} \int_0^t (t - s)^{\alpha-1} \varphi(s) \, ds$$

$$- \frac{bL}{(a + b)\Gamma(\alpha)} C_{fg} \int_0^T (T - s)^{\alpha-1} \varphi(s) \, ds$$

$$\leq KPLC_{fg} \varphi(t),$$

for all $t \in I$. That is

$$d(\Lambda f, \Lambda g) \leq KPLC_{fg}$$

Hence we can conclude that

$$d(\Lambda f, \Lambda g) \leq KPLd(f, g)$$

for any $f, g \in X$, where we note that $0 < KPL < 1$.

It follows from (3.6) and (3.8) that for an arbitrary $g_0 \in X$, there exists a constant $0 < C < \infty$ with

$$|(\Lambda g_0)(t) - g_0(t)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} F(s, f(s)) \, ds \right|$$
\[ -\frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, f(s)) \, ds + \frac{c}{a+b} - g_0(t) \]
\[ \leq C \varphi(t), \]
for all \( t \in I \), since \( F(t, g_0(t)) \) and \( g_0(t) \) are bounded on \( I \) and \( \min_{t\in I} \varphi(t) > 0 \)
Thus (3.7) implies that
\[ d(\Lambda g_0, g_0) < \infty \quad (3.10) \]
Therefore according to theorem 2.1(a), there exists a continuous function \( y_0 : I \to \mathbb{R} \) such that \( \Lambda g_0 \to y_0 \) in \((X, d)\) and \( \Lambda y_0 = y_0 \), that is, \( y_0 \) satisfies equation (3.4) for every \( t \in I \).
We will now verify that \( g \in X/d(g_0, g) < \infty = X \).
For any \( g \in X \), since \( g \) and \( g_0 \) are bounded on \( I \) and \( \min_{t\in I} \varphi(t) > 0 \), there exists a constant \( 0 < C_{fg} < \infty \) such that
\[ |g_0(t) - g(t)| \leq C_{fg} \varphi(t) \]
Hence, we have \( d(g_0, g) < \infty \) for all \( g \in X \), that is \( \{g \in X/d(g_0, g) < \infty\} = X \).

Hence in view of theorem 2.1(b) we conclude that \( y_0 \) is the unique continuous function with the property (3.4). On the other hand, it follows from (3.2) that
\[ -\varphi(t) \leq D^\alpha_{a+y(t)} - F(t, y(t)) \leq \varphi(t), \quad (3.11) \]
for all \( t \in I \).
If we integrate each term in the above inequality and substitute the boundary conditions we obtain
\[ \left| y(t) - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, y(s)) \, ds \right|
\[ -\frac{b}{\Gamma(\alpha)(a+b)} \int_0^t (t-s)^{\alpha-1} F(s, y(s)) \, ds - \frac{c}{a+b} \right| \leq \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \varphi(t) \, ds, \]
for any \( t \in I \).
Thus by (3.3) and (3.8) we get
\[ |y(t) - (\Lambda y)(t)| \leq K \varphi(t) \]
for each \( t \in I \), which implies that
\[ d(y, \Lambda y) \leq K. \quad (3.12) \]
Finally theorem 2.1(c) together with (3.13) implies that
\[ d(y, y_0) \leq \frac{1}{1 - KLd} d(y, \Lambda y) \leq \frac{K}{1 - KLd} \quad (3.13) \]
4. Hyers-Ulam Stability

In this section, we prove the Hyers-Ulam Stability of fractional differential equation (0.1) with the boundary condition (0.2).

**Theorem 4.3.** Let \( I = [0, T] \) be a closed interval and let \( r > 0 \) be a positive constant with \( 0 \leq t \leq r \). Let \( F : I \times \mathbb{R} \to \mathbb{R} \) be a continuous function which satisfies a Lipschitz condition 3.1 for all \( t \in I \) and \( y, z \in \mathbb{R} \), where \( L \) is a constant with \( 0 < \frac{LP_r \alpha}{\Gamma(\alpha+1)} < 1 \). If \( a \) is a continuously differentiable function \( y : I \to \mathbb{R} \) satisfies the differential inequality

\[
\left| ^cD_{a^+}^\alpha y(t) - F(t, y(t)) \right| \leq \epsilon
\]  

for all \( t \in I \) and for some \( \epsilon \geq 0 \), then there exists a unique continuous function \( y_0 : I \to \mathbb{R} \) satisfying equation (3.4) and

\[
|y(t) - y_0(t)| \leq \frac{r^\alpha}{\Gamma(\alpha+1) - LP_r \alpha} \epsilon
\]  

for all \( t \in I \).

**Proof.** First, we define a set \( X \) of all continuous functions \( f : I \to \mathbb{R} \) by

\[
X = \{ f : I \to \mathbb{R} / f \text{ is continuous} \}
\]

and introduce a generalised complete metric on \( X \) as follows

\[
d(f, g) = \inf \{ C \in [0, \infty] / |f(t) - g(t)| \leq C \text{ for all } t \in I \}
\]

Define an operator \( \Lambda : X \to X \) by

\[
(\Lambda f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, f(s)) ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, f(s)) ds + \frac{c}{a+b} \quad (4.3)
\]

for all \( f \in X \).

We now assert that \( \Lambda \) is strictly contractive on \( X \).

For any \( f, g \in X \), let \( C_{fg} \in [0, \infty] \) be an arbitrary constant with \( d(f, g) \leq C_{fg} \), that is, let us assume that

\[
|f(t) - g(t)| \leq C_{fg}
\]  

(4.4)
for any \( t \in I \). Moreover, it follows from (3.1), (4.3) and (4.4) that

\[
| (Af) t - (Ag) t | = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [F(s, f(s)) - F(s, g(s))] \, ds \\
- \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} [F(s, f(s)) - F(s, g(s))] \, ds \right| \\
\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F(s, f(s)) - F(s, g(s))| \, ds \\
- \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |F(s, f(s)) - F(s, g(s))| \, ds \\
\leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s) - g(s)| \, ds \\
- \frac{bL}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |f(s) - g(s)| \, ds \\
\leq \frac{L}{\Gamma(\alpha)} Cfg \int_0^t (t-s)^{\alpha-1} ds - \frac{bL}{(a+b)\Gamma(\alpha)} Cfg \int_0^T (T-s)^{\alpha-1} ds \\
\leq L Cfg \left[ \frac{t^\alpha}{\alpha \Gamma(\alpha)} - \frac{bT^\alpha}{(a+b)\alpha \Gamma(\alpha)} \right] \\
\leq L Cfg \left[ \frac{r^\alpha}{\alpha \Gamma(\alpha)} - \frac{br^\alpha}{(a+b)\alpha \Gamma(\alpha)} \right] \\
\leq \frac{LCfg r^\alpha}{\Gamma(\alpha+1)} \left[ \frac{a}{a+b} \right] \\
\leq \frac{LPCfg r^\alpha}{\Gamma(\alpha+1)},
\]

for all \( t \in I \). That is

\[
(\Lambda f, \Lambda g) \leq \frac{LP_{r^\alpha}}{\Gamma(\alpha+1)} Cfg
\]

Thus, it follows that

\[
d(\Lambda f, \Lambda g) \leq \frac{LP_{r^\alpha}}{\Gamma(\alpha+1)} d(f, g)
\]

for any \( f, r \in X \), and we note that \( 0 < \frac{LP_{r^\alpha}}{\Gamma(\alpha+1)} < 1 \).

Analogously to the proof of theorem 3.1, we can show that each \( g_0 \in X \) satisfies the property \( d(\Lambda g_0, g_0) < \infty \).

Therefore, theorem 2.1(a) implies that there exists a continuous function \( y_0 : I \rightarrow \mathbb{R} \) such that \( \Lambda^n g_0 \rightarrow y_0 \) in \( (X, d) \) as \( n \rightarrow \infty \), and such that \( y_0 = \Lambda y_0 \), that is, \( y_0 \) satisfies equation 3.4 for any \( t \in I \).
If \( g \in X \), then \( g_0 \) and \( g \) are continuous functions defined on a compact interval \( I \). Hence, there exists a constant \( C > 0 \) with \( |g_0(t) - g(t)| \leq C \) for all \( t \in I \). This implies that \( d(g_0, g) < \infty \) for every \( g \in X \), or equivalently, \( \{g \in X / d(g_0, g) < \infty \} = X \). Therefore, according to theorem 2.1(b), \( y_0 \) is a unique continuous function with property (3.4) furthermore, it follows from eqn(4.1) that

\[-\epsilon \leq c D^\alpha_{a+} y(t) - F(t, y(t)) \leq \epsilon\]

for all \( t \in I \). If we integrate each term of the above inequality and applying the boundary conditions, then we have

\[ |(\Lambda y)(t) - y(t)| \leq \frac{r^\alpha}{\Gamma(\alpha + 1)} \epsilon \]

for any \( t \in I \), that is, it holds that \( d(\Lambda y, y) \leq \frac{r^\alpha}{\Gamma(\alpha + 1)} \epsilon \).

It now follows from theorem 2.1(c) that

\[ d(y, y_0) \leq \frac{1}{1 - LP^{r\alpha}} d(\Lambda y, y) \leq \frac{r^\alpha}{\Gamma(\alpha + 1) - LP^{r\alpha}} \epsilon \]

which implies the validity of (4.2) for each \( t \in I \)

\[ \Box \]

References


