International Journal of Pure and Applied Mathematics
Volume 102 No. 4 2015, 631-642
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v102i4.4

ijpam.eu

HYERS-ULAM-RASSIAS STABILITY OF FRACTIONAL DIFFERENTIAL EQUATION

P. Muniyappan[1 §], S. Rajan[2]

[1]Department of Mathematics
Adhiyamaan College of Engineering
Hosur, 635109, Tamil Nadu, INDIA

[2]Department of Mathematics
Erode Arts and Science College
Erode, Tamilnadu, INDIA

Abstract: In this paper, we proved Hyers-Ulam and Hyers-Ulam-Rassias stability for the following fractional differential eqaution wih boundary condition

$$D^{\alpha}y(t) = F(t, y(t)),$$
$$ay(0) + by(T) = c.$$

AMS Subject Classification: 34A08, 34K10, 34K20
Key Words: Hyers-Ulam stability, Hyers-Ulam-Rassias stability, fractional differential equation

1. Introduction

Let Y be a normed space and $I = [0, T]$ be a given interval. Assume that for a continuously differentiable function $f : I \to Y$ satisfying fractional differential inequality $\|{}^{c}D^{\alpha}y(t) - F(t, y(t))\| \le \epsilon$ for all $t \in I$ and for some $\epsilon > 0$, where ${}^{c}D^{\alpha}$

Received: December 16, 2014

[§]Correspondence author

is the Caputo fractional derivative of order $\alpha \in (0,1)$, there exists a solution $f_0 : I \to Y$ of the fractional boundary value problem (0.1)and (0.2) such that $\|f(t) - f_0(t)\| \leq K(\epsilon)$ for all $t \in I$, where $K(\epsilon)$ is only dependent on ϵ. Then, we say that the above fractional boundary value problem (0.1) and (0.2) has the Hyers-Ulam stability.

If the above statment is also true when we replace ϵ and $K(\epsilon)$ by $\varphi(t)$ and $\Phi(t)$, where $\varphi, \Phi : I \to [0, \infty)$ are functions not depending on f and f_0 explicitly, then we say that the corresponding fractional boundary value problem (0.1) and (0.2) has the Hyers-Ulam-Rassias stability.

Fractional differential equations is the area of concentration of recent research and there has been significant progress in this area. However, the concept of fractional derivative is not new and is very much as old as differential eqautions. In 1695, L Hospital raised the question about fractional derivative in a letter written to Leibniz and related his generilization of differentiation. Recently, the differential equations of fractional order has proved to be a valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find many applications in electromagnetic, control, electrochemistry etc. (see [22, 23, 24, 25]). For more details on this area, one can see the monographs of Kilbas et al [4], Miller and Ross [6], I.Podulbny [5], Diethelm at al [8, 9], Benchora [2] and the references therein.

In 1940, S. M. Ulam [10] asked the question concerning the stability of group homomorphisms. Next year Hyers [11] gave the first positive answer to the question of Ulam for Banach spaces. Thereafter, the stability of this type is called Hyers-Ulam Stability. In 1978 Th.M. Rassias [12] provided a generalization of the Hyers theorem which allows the Cauchy difference to be unbounded. After this result many mathematicians were attracted and motivated to investigate the Hyers-Ulam stability and Hyers-Ulam-rassias stability in the mathematical analysis area.The stability occurrence that was introduced and proved by Th.M. Rassias in his paper is called the Hyers-Ulam-Rassias stability.

The developement of the analysis of stability of fractional differential equations has been a bit slow. Recently, Li and Zhang [16] made a brief overview on the stability results of the fractional differential equations. Li et al. [17, 18] first proposed the Mittag Leffler stability and the fractional Lyapunovs second method. Although, there are some work on the local stability and Mittag Leffler stability for fractional differential equations, to the best of my knowledge, there are very rare works on the Ulam stability of fractional differential equations. A pioneering work on the Ulam stability and data dependence for fractional differential equations with Caputo derivative has been reported by

Wang et al. [19]. Most recently J.Wang [1] proved Hyers Ulam stability as well as the Hyers-Ulam-Rassias stability of fractional differential equations via a generalized fixed point approach, by adopting some part idea of Wang et al. [19], Cadariu and Radu [20] and Jung [21].

In this paper we proved Hyers-Ulam stability and Hyers-Ulam-Rassias stability of fractional differential equation (0.1) with the boundary condition (0.2). This paper is organized as follows: In Section 2, basic definitions and notations are given. In Section 3, the Hyers-Ulam-Rassias stability of fractional differential equation (0.1) with boundary condition (0.2) is proven. In Section 4, the Hyers-Ulam stability of fractional differential equation (0.1) with boundary condition (0.2) is proven.

2. Preliminaries

In this section, we give some basic definition and theorems which we used to prove the results.

Definition 2.1. [2] The fractional order integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds,$$

where Γ is the gamma function.

Definition 2.2. [2] For a function h given on the the interval $[a,b]$, the α -th Riemann-Liouville fractinal order derivative of h, is defined by

$$(D_a^{\alpha} h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s) ds.$$

Here $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α .

Definition 2.3. [2] For a function h given on the interval $[a,b]$, the Caputo fractional order derivative of h, is defined by

$$({}^cD_{a+}^{\alpha} h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds,$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α .

Definition 2.4. [2] A function $y \in C(J, \mathbb{R})$ is said to be a solution of (0.1)-(0.2) if y satisfies the equation ${}^{c}D^{\alpha}y(t) = f(t, y(t))$ on J, and the condition $ay(0) + by(T) = c$

Lemma 2.1. *[2] Let $0 < \alpha < 1$ and let $f : [0, T] \to \mathbb{R}$ be continuous. A function $y \in C(J, \mathbb{R})$ is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))ds \\ - \frac{1}{a+b} \left[\frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s))ds - c \right] \quad (2.1)$$

if and only if y is a solution of the fractional boundary value problem

$$\begin{aligned} {}^{c}D^{\alpha}y(t) &= f(t, y(t)), t \in [0, T] \\ ay(0) + by(T) &= c. \end{aligned}$$

Definition 2.5. For a nonempty set X, a function $d : X \times X \to [0, \infty]$ is called generalized metric on X if and only if d satisfies

1. d(x,y)=0 if and only if x=y;
2. d(x,y)=d(y,x) for all $x, y \in X$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 2.1. *Let (X,d) be a generalised complete metric space. Assume that $\Lambda : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$, If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^{k}) < \infty$ for some $x \in X$, then the following are true:*
(a) The sequence $\{\Lambda^n x\}$ converges to a fixed end point x^* of Λ
(b) x^* is the unique fixed point of Λ in $X^* = \{y \in X / d(\Lambda^k x, y) < \infty\}$;
(c) If $y \in X^*$, then $d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y)$

3. Hyers-Ulam-Rassias stability

In this section, we first investigate the Hyers-Ulam-Rassias stability of the fractional differential equation (0.1) with boundary condition (0.2) in the interval [0, T] via theorem (2.1), by using the idea of cadariu and Radu [20], Jung [21], and J.wang [1].

Theorem 3.2. *Let $I = [0, T]$ be a closed interval. Let K, P, and L be positive constants with $0 < KPL < 1$. Assume that $F : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies the standard Lipschitz condition*

$$|F(t, y) - F(t, z)| \leq L\,|y - z| \tag{3.1}$$

for any $t \in I$ and $y, z \in \mathbb{R}$. If a continuously differential function $y : I \to \mathbb{R}$ satisfies

$$|{}^{c}D^{\alpha}y(t) - F(t, y(t))| \leq \varphi(t) \tag{3.2}$$

for all $t \in I$, where $\varphi : I \to (0, \infty)$ is a continuous function with

$$\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}\varphi(\tau)d\tau\right| \leq K\varphi(t) \tag{3.3}$$

for all $t \in I$, then there exists unique continuous function $y_0 : I \to \mathbb{R}$ such that

$$\begin{aligned} y_0(t) = \frac{1}{\Gamma(\alpha)}&\int_0^t (t - s)^{\alpha-1}F(s, y_0(s))ds \\ &- \frac{b}{(a + b)\Gamma(\alpha)}\int_0^T (T - s)^{\alpha-1}F(s, y_0(s))ds + \frac{c}{a + b} \end{aligned} \tag{3.4}$$

and

$$|y(t) - y_0(t)| \leq \frac{K}{1 - KPL}\varphi(t) \tag{3.5}$$

Proof. Let us define a set X of all continuous functions $f : I \to \mathbb{R}$ by

$$X = \{f : I \to \mathbb{R} | \text{f is continuous}\}. \tag{3.6}$$

Similar to theorem 3.1 of Jung S-M [21], we introduce a generalised complete metric on X as follows

$$d(f, g) = inf\,\{C \in [0, \infty]/\,|f(t) - g(t)| \leq C\varphi(t)\text{for all}t \in I\} \tag{3.7}$$

Define an operator $\Lambda : X \to X$ by

$$\begin{aligned} (\Lambda f)(t) = \frac{1}{\Gamma(\alpha)}&\int_0^t (t - s)^{\alpha-1}F(s, f(s))ds \\ &- \frac{b}{(a + b)\Gamma(\alpha)}\int_0^T (T - s)^{\alpha-1}F(s, f(s))ds + \frac{c}{a + b} \end{aligned} \tag{3.8}$$

for all $f \in X$.

It is easy to see that Λ is well defined, since F and f are continuous functions.

To achieve our aim, we need to prove that Λ is strictly contractive on X.

For any $f, g \in X$, let $C_{fg} \in [0, \infty]$ be an arbitrary constant with $d(f,g) \leq C_{fg}$, that is by (3.7) we have

$$|f(t) - g(t)| \leq C_{fg}\varphi(t) \tag{3.9}$$

for any $t \in I$. It then follows from $(3.1), (3.3), (3.7), (3.8)$ and (3.9) that

$$\begin{aligned}
|(\Lambda f)t - (\Lambda g)t| =& \Bigg| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[F(s, f(s)) - F(s, g(s))\right] ds \\
& - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left[F(s, f(s)) - F(s, g(s))\right] ds \Bigg| \\
\leq & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|F(s, f(s)) - F(s, g(s))\right| ds \\
& - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left|F(s, f(s)) - F(s, g(s))\right| ds \\
\leq & \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|f(s) - g(s)\right| ds \\
& - \frac{bL}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left|f(s) - g(s)\right| ds \\
\leq & \frac{L}{\Gamma(\alpha)} C_{fg} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds \\
& - \frac{bL}{(a+b)\Gamma(\alpha)} C_{fg} \int_0^T (T-s)^{\alpha-1} \varphi(s) ds \\
\leq & KPLC_{fg}\varphi(t),
\end{aligned}$$

for all $t \in I$. That is

$$d\left(\Lambda f, \Lambda g\right) \leq KPLC_{fg}$$

Hence we can conclude that

$$d\left(\Lambda f, \Lambda g\right) \leq KPLd\left(f, g\right)$$

for any $f, g \in X$, where we note that $0 < KPL < 1$.

It follows from (3.6) and (3.8)that for an arbitrary $g_0 \in X$, there exists a constant $0 < C < \infty$ with

$$|(\Lambda g_0)(t) - g_0(t)| = \Bigg| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, f(s)) ds$$

$$-\frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}F(s,f(s))ds + \frac{c}{a+b} - g_0(t)\Bigg|$$
$$\leq C\varphi(t),$$

for all $t \in I$, since $F(t, g_0 t)$ and $g_0 t$ are bounded on I and $min_{t\in I}\varphi(t) > 0$ Thus (3.7) implies that

$$d(\Lambda g_0, g_0) < \infty \tag{3.10}$$

Therefore according to theorem 2.1(a), there exists a continuous function $y_0 : I \to \mathbb{R}$ such that $\Lambda g_0 \to y_0$ in (X, d) and $\Lambda y_0 = y_0$, that is, y_0 satisfies equation (3.4) for every $t \in I$.

We will now verify that $g \in X/d(g_0, g) < \infty = X$.

For any $g \in X$, since g and g_0 are bounded on I and $min_{t\in I}\varphi(t) > 0$, there exists a constant $0 < C_{fg} < \infty$ such that $|g_0(t) - g(t)| \leq C_g\varphi(t)$

Hence, we have $d(g_0, g) < \infty$ for all $g \in X$, that is $\{g \in X/d(g_0, g) < \infty\} = X$.

Hence in view of theorem 2.1(b) we conclude that y_0 is the unique continuous function with the property (3.4). On the other hand, it follows from (3.2) that

$$-\varphi(t) \leq^c D^{\alpha}_{a+}y(t) - F(t, y(t)) \leq \varphi(t), \tag{3.11}$$

for all $t \in I$.

If we integrate each term in the above inequality and substitute the boundary conditions we obtain

$$\Bigg|y(t) - \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}F(s,y(s))ds$$
$$-\frac{b}{\Gamma(\alpha)(a+b)}\int_0^t (t-s)^{\alpha-1}F(s,y(s))ds - \frac{c}{a+b}\Bigg| \leq \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\varphi(t)ds,$$

for any $t \in I$.

Thus by (3.3)and (3.8) we get

$$|y(t) - (\Lambda y)(t)| \leq K\varphi(t)$$

for each $t \in I$, which implies that

$$d(y, \Lambda y) \leq K. \tag{3.12}$$

Finally theorem 2.1(c) together with (3.13) implies that

$$d(y, y_0) \leq \frac{1}{1-KPL}d(y, \Lambda y) \leq \frac{K}{1-KPL} \tag{3.13}$$

□

4. Hyers-Ulam Stability

In this section, we prove the Hyers-Ulam Stability of fractional diffrential equation (0.1) with the boundary condition (0.2).

Theorem 4.3. *Let I=[0,T] be a closed interval and let $r > 0$ be a positive constant with $0 \leq t \leq r$. Let $F : I \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies a Lipschitz condition 3.1 for all $t \in I$ and $y, z \in \mathbb{R}$, where L is a constant with $0 < \frac{LPr^\alpha}{\Gamma(\alpha+1)} < 1$. If a is a continuously differentiable function $y : I \to \mathbb{R}$ satisfies the differential inequality*

$$\left|{}^cD^\alpha_{a+}y(t) - F(t, y(t))\right| \leq \epsilon \tag{4.1}$$

for all $t \in I$ and for some $\epsilon \geq 0$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ satisfying equation (3.4) and

$$|y(t) - y_0(t)| \leq \frac{r^\alpha}{\Gamma(\alpha+1) - LPr^\alpha}\epsilon \tag{4.2}$$

for all $t \in I$.

Proof. First, we define a set X of all continuous functions $f : I \to \mathbb{R}$ by

$$X = \{f : I \to \mathbb{R}/\text{f is continuous}\}$$

and introduce a generalised complete metric on X as follows

$$d(f, g) = inf\,\{C \in [0, \infty]/\,|f(t) - g(t)| \leq C\text{for all}t \in I\}$$

Define an operator $\Lambda : X \to X$ by

$$\begin{aligned}(\Lambda f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s, f(s))ds \\ - \frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}F(s, f(s))ds + \frac{c}{a+b}\end{aligned} \tag{4.3}$$

for all $f \in X$.

We now assert that Λ is strictly contractive on X.

For any $f, g \in X$, let $C_{fg} \in [0, \infty]$ be an arbitrary constant with $d(f, g) \leq C_{fg}$, that is, let us assume that

$$|f(t) - g(t)| \leq C_{fg} \tag{4.4}$$

for any $t \in I$. Moreover, it follows from (3.1), (4.3) and (4.4) that

$$\begin{aligned}
|(\Lambda f)t - (\Lambda g)t| = &\left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[F(s,f(s)) - F(s,g(s))\right] ds \right. \\
&\left. - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left[F(s,f(s)) - F(s,g(s))\right] ds \right| \\
\leq &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|F(s,f(s)) - F(s,g(s))\right| ds \\
&- \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left|F(s,f(s)) - F(s,g(s))\right| ds \\
\leq &\frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|f(s) - g(s)\right| ds \\
&- \frac{bL}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left|f(s) - g(s)\right| ds \\
\leq &\frac{L}{\Gamma(\alpha)} C_{fg} \int_0^t (t-s)^{\alpha-1} ds - \frac{bL}{(a+b)\Gamma(\alpha)} C_{fg} \int_0^T (T-s)^{\alpha-1} ds \\
\leq &LC_{fg} \left[\frac{t^\alpha}{\alpha\Gamma(\alpha)} - \frac{bT^\alpha}{(a+b)\alpha\Gamma(\alpha)} \right] \\
\leq &LC_{fg} \left[\frac{r^\alpha}{\alpha\Gamma(\alpha)} - \frac{br^\alpha}{(a+b)\alpha\Gamma(\alpha)} \right] \\
\leq &\frac{LC_{fg} r^\alpha}{\Gamma(\alpha+1)} \left[\frac{a}{a+b} \right] \\
\leq &\frac{LPC_{fg} r^\alpha}{\Gamma(\alpha+1)},
\end{aligned}$$

for all $t \in I$. That is

$$d\left(\Lambda f, \Lambda g\right) \leq \frac{LPr^\alpha}{\Gamma(\alpha+1)} C_{fg}$$

Thus, it follows that

$$d\left(\Lambda f, \Lambda g\right) \leq \frac{LPr^\alpha}{\Gamma(\alpha+1)} d\left(f, g\right)$$

for any $f, r \in X$, and we note that $0 < \frac{LPr^\alpha}{\Gamma(\alpha+1)} < 1$.

Analogously to the proof of theorem 3.1, we can show that each $g_0 \in X$ satisfies the property $d(\Lambda g_0, g_0) < \infty$.

Therefore, theorem 2.1(a) imlies that there exists a continuous function $y_0 : I \to \mathbb{R}$ such that $\Lambda^n g_0 \to y_0$ in (X, d) as $n \to \infty$, and such that $y_0 = \Lambda y_0$, that is, y_0 satisfies equation 3.4 for any $t \in I$.

If $g \in X$, then g_0 and g are continuous functions defined on a compact interval I, Hence, there exists a constant $C > 0$ with $|g_0(t) - g(t)| \leq C$ for all $t \in I$. This implies that $d(g_0, g) < \infty$ for every $g \in X$, or equivalently, $\{g \in X/d(g_0, g) < \infty\} = X$. Therefore, according to theorem 2.1(b), y_0 is a unique continuous function with property (3.4) furtheremore, it follows from eqn(4.1) that

$$-\epsilon \leq^{c} D_{a+}^{\alpha} y(t) - F(t, y(t)) \leq \epsilon$$

for all $t \in I$. If we integrate each term of the above inequality and appling the boundary conditions, then we have

$$\left|\left(\Lambda y\right)(t) - y(t)\right| \leq \frac{r^{\alpha}}{\Gamma(\alpha+1)}\epsilon$$

for any $t \in I$, that is, it holds that $d\left(\Lambda y, y\right) \leq \frac{r^{\alpha}}{\Gamma(\alpha+1)}\epsilon$.

It now follows from theorem 2.1(c) that

$$d(y, y_0) \leq \frac{1}{1 - \frac{LPr^{\alpha}}{\Gamma(\alpha+1)}} d\left(\Lambda y, y\right) \leq \frac{r^{\alpha}}{\Gamma(\alpha+1) - LPr^{\alpha}}\epsilon \tag{4.5}$$

which implies the validity of (4.2) for each $t \in I$ □

References

[1] J.Wang, Linli Lv, Yong Zhou; *New concepts and results in stability of fractional differential equations*, Commun Nonlinear Sci Numer Simulat 17 (2012) 2530-2538

[2] Mouffak Benchohra; Samira Hamani and Sotiris K. Ntouyas, *Boundary Value Problems for Diferential Equations with Fractional Order*, Surveys in Mathematics and its Applications, Volume 3, (2008), 1-12.

[3] Ravi P. Agarwal, Mouffak Benchohra, Samira Hamani, *A Survey on existence Results for Boundary Value Problems of Nonlinear Fractional Differential Equations and Inclusions*, Acta Appl Math (2010) 109: 973-1033.

[4] Kilbas, A. A.; Srivastava, H. M.; Trujillo, J. J.;*Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Volume 204, 2006.

[5] Podlubny, I.; *Fractional Differential Equations*. Academic Press,London, 1999.

[6] Miller, K. S.; Ross, B.; *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, Inc., New, York, 1993.

[7] Samko, S. G.; Kilbas, A. A.; Marichev, O.I.; *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, 1993.

[8] Diethelm, K.; Neville J. Ford; *Analysis of fractional differential equations*, J. Math. Anal. Appl., 265 (2002), 229-248.

[9] Diethelm, K.; *The analysis of fractional differential equations*, Springer, 2010.

[10] S.M. Ulam, *Problems in Modern Mathematics*, Rend. Chap.VI,Wiley, New York, 1940.

[11] D.H.Hyers, *on the stability of the linear functional equations*, Proc.Nat.Acad.Sci., 27(1941), 222-224.

[12] Th.M.Rassias, *on the stability of linear mappig in Banach spaces*, Proc.Amer.Soc., 72(1978), 297-300.

[13] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge (1990).

[14] M. Rosenblum, Generalized Hermite polynomials and the Bose-like oscillator calculus, In: *Operator Theory: Advances and Applications*, Birkhäuser, Basel (1994), 369-396.

[15] D.S. Moak, *The q-analogue of the Laguerre polynomials*, J. Math. Anal. Appl., **81** (1981), 20-47.

[16] Li CP, Zhang FR. *A survey on the stability of fractional differential equations*, Eur Phys J Special Topics 2011;193:27-47.

[17] Li Y, Chen Y, Podlubny I. *Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag Leffler stability*, Comput Math Appl 2010;59:1810-21.

[18] Li Y, Chen Y, Podlubny I. *Mittag Leffler stability of fractional order nonlinear dynamic systems*, Automatica 2009;45:1965-9.

[19] Wang J, Lv L, Zhou Y. *Ulam stability and data dependence for fractional differential equations with Caputo derivative*. Electron J Qualit Th Diff Equat 2011;63:

[20] Cadariu L, Radu V. *Fixed points and the stability of Jensens functional equation* J Inequal Pure Appl Math 2003;4(1):

[21] Jung S-M. *A fixed point approach to the stability of differential equations y'(t) = F(x,y).* Bull Malays Math Sci Soc 2010;33:47-56.

[22] K. Diethelm and A.D. Freed, *On the solution of nonlinear fractional order differential equations used in the modeling of viscoplasticity*, in "Scientifice Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties" (F. Keil, W. Mackens, H. Voss, and J.Werther, Eds), pp 217-224, Springer-Verlag, Heidelberg, 1999.

[23] A. M. A. El-Sayed, *Fractional order diffusion-wave equations*, Intern. J. Theo.Physics 35 (1996), 311-322. MR1372176(96k:34123). Zbl 0846.35001.

[24] W. G. Glockle and T. F. Nonnenmacher, *A fractional calculus approach of self-similar protein dynamics*, Biophys. J. 68 (1995), 46-53.

[25] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000. Zbl 0998.26002.