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ON THE ENESTRÖM KAKEYA THEOREM

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Abstract: A well known Eneström and Kakeya theorem says that , if $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that $0<a_0 \leq a_1 \leq ... \leq a_n$ then all the zeros of P(z) lie in $|z| \leq 1$. Many generalizations of the Enestrom Kakeya Theorem are available in literature. In this paper we prove some results which further generalize some known results by relaxing the hypothesis.

AMS Subject Classification: 30C10, 30C15
Key Words: zeros of polynomial, Eneström-Kakeya theorem

1. Introduction

To estimate the zeros of a polynomial is a long standing classical problem [5-9]. It is an interesting area of research for engineers as well as mathematicians and many results on the same topic are available in literature. The polynomials in various forms have recently come under extensive revision because of their applications in linear control systems, signal processing, electrical networks, coding theory and several areas of physical sciences, where among others location of zeros and stability problems arise in a natural way. Existing results in literature also show that there is a need to find bounds for special polynomials, for example, for those having restrictions on the coefficient, there is always

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need for refinement of results in this subject. The well known results Eneström-Kakeya theorem [1, 2] in theory of the distribution of zeros of polynomials is the following:

Theorem (A_1)**.** *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n *such that*

$$0<a_0 \leq a_1 \leq ... \leq a_{n-1} \leq a_n$$

then all the zeros of P(z) lie in $|z| \leq 1$

A. Joyal, G. Labelle and Q.I. Rahman [3] obtained the following generalization, by considering the coefficients to be real, instead of being only positive.

Theorem (A_2)**.** *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n *such that*

$$a_0 \leq a_1 \leq ... \leq a_{n-1} \leq a_n,$$

then all the zeros of P(z) lie in $|z| \leq \frac{1}{|a_n|}\{a_n - a_0 + |a_0|\}$.

In the literature some attempts have been made to extend and generalize the Eneström-Kakeya Theorem. Aziz and Zargar [4] relaxed the hypothesis of Eneström-Kakeya theorem in a different way and proved the following results.

Theorem (A_3)**.** *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n such that for some* $k \geq 1$, $0<a_0 \leq a_1 \leq ... \leq a_{n-1} \leq ka_n$ *then all the zeros of P(z) lie in* $|z + k - 1| \leq k$.

Theorem (A_4)**.** *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n *such that for some* $k \geq 1$,

$$a_0 \leq a_1 \leq ... \leq a_{n-1} \leq ka_n,$$

then all the zeros of $P(z)$ *lie in* $|z + k - 1| \leq \frac{1}{|a_n|}\{ka_n - a_0 + |a_0|\}$.

By using the above Theorems, We get some more improvized results by imposing some conditions on coefficients of polynomials.

Theorem 1. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n *with real coefficients such that for some* $k \geq 1, \delta \geq 0, a_m \neq 0$,

$$a_0 - \delta \leq a_1 \leq ... \leq a_{m-1} \leq ka_m \geq a_{m+1} \geq ... \geq a_{n-1} \geq a_n$$

then all the zeros of P(z) lie in $|z| \leq \frac{1}{|a_n|}\{2k(a_m + |a_m|) + |a_0| - (|a_n| + 2a_m + a_0) + 2\delta\}$.

Corollary 1. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n *with real coefficients such that for some* $\delta \geq 0$,

$$a_0 - \delta \leq a_1 \leq ... \leq a_{m-1} \leq a_m \geq a_{m+1} \geq ... \geq a_{n-1} \geq a_n$$

then all the zeros of P(z) lie in $|z| \leq \frac{1}{|a_n|}\{2a_m + |a_0| - (a_0 + |a_n|) + 2\delta\}$.

Corollary 2. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n *with real coefficients such that*

$$a_0 \leq a_1 \leq ... \leq a_{m-1} \leq a_m \geq a_{m+1} \geq ... \geq a_{n-1} \geq a_n$$

then all the zeros of P(z) lie in $|z| \leq \frac{1}{|a_n|}\{2a_m + |a_0| - (a_0 + |a_n|)\}$.

Corollary 3. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n *with poisitive real coefficients such that for some* $k \geq 1, \delta \geq 0, a_m \neq 0$,

$$a_0 - \delta \leq a_1 \leq ... \leq a_{m-1} \leq ka_m \geq a_{m+1} \geq ... \geq a_{n-1} \geq a_n$$

then all the zeros of P(z) lie in $|z| \leq \frac{1}{|a_n|}\{2(2k-1)a_m - a_n + 2\delta\}$.

Remark 1. 1. By taking k=1 in Theorem 1, then it is reduces to Corollary 1.

2. By taking k=1 and $\delta = 0$ in Theorem 1, then it reduces to Corollary 2.

3. By taking a_i>0 for $i = 0, 1, ..., n$, in Theorem 1, then it reduces to Corollary 3.

Theorem 2. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n *with real coefficients such that for some* $0<r \leq 1, \delta \geq 0$,

$$ra_n \leq a_{n-1} \leq ... \leq a_{m+1} \leq a_m + \delta \geq a_{m-1} \geq ... \geq a_1 \geq a_0$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\{4\delta + |a_n| + 2a_m + |a_0| - a_0 - r(|a_n| + a_n)\}.$$

Corollary 4. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n *with real coefficients such that for some* $0<r \leq 1$,

$$ra_n \leq a_{n-1} \leq ... \leq a_{m+1} \leq a_m \geq a_{m-1} \geq ... \geq a_1 \geq a_0$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\{|a_n| + 2a_m + |a_0| - a_0 - r(|a_n| + a_n)\}.$$

Corollary 5. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some $\delta \geq 0$,*

$$a_n \leq a_{n-1} \leq ... \leq a_{m+1} \leq a_m + \delta \geq a_{m-1} \geq ... \geq a_1 \geq a_0$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\{4\delta + 2a_m + |a_0| - a_0 - a_n)\}.$$

Corollary 6. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with positive real coefficient such that for some $0<r \leq 1, \delta \geq 0$,*

$$ra_n \leq a_{n-1} \leq ... \leq a_{m+1} \leq a_m + \delta \geq a_{m-1} \geq ... \geq a_1 \geq a_0$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\{4\delta + 2a_m + (1-2r)a_n\}.$$

Remark 2.

1. By taking k=1 and $\delta = 0$ in Theorem 1, then it reduces to Corollary 2.
2. By taking $\delta = 0$ in theorem 2 then it reduces to Corollary 4.
3. By taking k=1 in Theorem 2, then it reduces to Corollary 5.
4. By taking $a_i>0$ for $i = 1, ..., n$, in Theorem 2, then it reduces to Corollary 6.

Theorem 3. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some $0<r \leq 1, \delta \geq 0, a_m \neq 0$,*

$$a_0 + \delta \geq a_1 \geq ... \geq a_{m-1} \geq ra_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq a_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\{2|a_m| + |a_0| + a_0 + a_n - 2r(|a_m| + a_m) + 2\delta\}.$$

Corollary 7. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some $0<r \leq 1, a_m \neq 0$,*

$$a_0 \geq a_1 \geq ... \geq a_{m-1} \geq ra_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq a_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\{2|a_m| + |a_0| + a_0 + a_n - 2r(|a_m| + a_m)\}.$$

Corollary 8. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficient such that for some $\delta \geq 0$,*

$$a_0 + \delta \geq a_1 \geq ... \geq a_{m-1} \geq a_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq a_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\{|a_0| + a_0 + a_n - 2a_m + 2\delta\}.$$

Corollary 9. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that*

$$a_0 \geq a_1 \geq ... \geq a_{m-1} \geq a_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq a_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\{|a_0| + a_0 + a_n - 2a_m\}.$$

Corollary 10. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with positive real coefficients such that for some $0 < r \leq 1, \delta \geq 0, a_m \neq 0$,*

$$a_0 + \delta \geq a_1 \geq ... \geq a_{m-1} \geq r a_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq a_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\{a_n + 2(a_0 + (1 - 2r)a_m + \delta)\}.$$

Remark 3.

1. By taking $\delta = 0$ in Theorem 3, then it reduces to Corollary 7.
2. By taking k=1 in Theorem 3, then it reduces to Corollary 8.
3. By taking k=1, $\delta = 0$ in Theorem 3, then it reduces to Corollary 9.
4. By taking $a_i > 0$ for $i = 0, 1, 2, ..., n$, in Theorem 3, it reduces to Corollary 10.

Theorem 4. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some $k \geq 1, \delta \geq 0$,*

$$k a_n \geq a_{n-1} \geq ... \geq a_{m+1} \geq a_m - \delta \leq a_{m-1} \leq ... \leq a_1 \leq a_0$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\{4\delta + k(|a_n| + a_n) + |a_0| + a_0 - 2a_m - |a_n|\}.$$

Corollary 11. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some $k \geq 1$,*

$$ka_n \geq a_{n-1} \geq ... \geq a_{m+1} \geq a_m \leq a_{m-1} \leq ... \leq a_1 \leq a_0$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\{k(|a_n| + a_n) + |a_0| + a_0 - 2a_m - |a_n|\}.$$

Corollary 12. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that*

$$a_n \geq a_{n-1} \geq ... \geq a_{m+1} \geq a_m \leq a_{m-1} \leq ... \leq a_1 \leq a_0$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\{|a_0| + a_0 + a_n - 2a_m\}.$$

Corollary 13. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with positive real coefficients such that for some $k \geq 1, \delta \geq 0$,*

$$ka_n \geq a_{n-1} \geq ... \geq a_{m+1} \geq a_m - \delta \leq a_{m-1} \leq ... \leq a_1 \leq a_0$$

then all the zeros of P(z) lie in

$$|z + k - 1| \leq \frac{1}{|a_n|}\{ka_n + 2(a_0 - a_m + 2\delta)\}.$$

Remark 4.

1. By taking $\delta = 0$ in theorem 4, then it reduces to Corollary 11.
2. By taking k=1 and $\delta = 0$ in Theorem 4, then it reduces to Corollary 12.

2. Proof of the Theorems

Proof of Theorem 1. Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_{m-1} z^{m-1} + a_m z^m + a_{m+1} z^{m+1} + ... + a_1 z + a_0$ be a polynomial degree n.

Then consider the polynomial $Q(z) = (1 - z)P(z)$ so that

$$Q(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{m+1} - a_m)z^{m+1} + (a_m - a_{m-1})z^m$$

$$+ ... + (a_1 - a_0)z + a_0.$$

Also if $|z|>1$ then $\frac{1}{|z|^{n-i}}<1$ for $i = 1, 2, ..., n-1$. Now

$$\begin{aligned}
|Q(z)| \geq& |a_n||z|^{n+1} - \big\{|a_n - a_{n-1}||z|^n + ... + |a_{m+1} - a_m||z|^{m+1} \\
&+ |a_m - a_{m-1}||z|^m + ... + |a_1 - a_0||z| + a_0\big\} \\
\geq& |a_n||z|^n \Bigg[|z| - \frac{1}{|a_n|}\big\{|a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} \\
&+ ... + \frac{|a_{m+1} - a_m|}{|z|^{n-m-1}} + \frac{|a_m - a_{m-1}|}{|z|^{n-m}} \\
&+ ... + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\big\}\Bigg] \\
\geq& |a_n||z|^n \Bigg[|z| - \frac{1}{|a_n|}\big\{|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| \\
&+ ... + |a_{m+1} - ka_m + ka_m - a_m| + |a_m - ka_m + ka_m - a_{m-1}| \\
&+ ... + |a_1 + \delta - a_0 - \delta| + |a_0|\big\}\Bigg] \\
\geq& |a_n||z|^n \Bigg[|z| - \frac{1}{|a_n|}\{(a_{n-1} - a_n) + (a_{n-2} - a_{n-1}) \\
&+ ... + (ka_m - a_{m+1}) + (k-1)|a_m| + (ka_m - a_{m-1}) + (k-1)|a_m| \\
&+ ... + (a_1 + \delta - a_0) \\
&+ \delta + |a_0|\}\Bigg] \\
=& |a_n||z|^n \Bigg[|z| - \frac{1}{|a_n|}\big\{2k(a_m + |a_m|) + |a_0| \\
&- (|a_n| + 2a_m + a_0) + 2\delta\big\}\Bigg] \\
>& 0.
\end{aligned}$$

Let

$$|z|>\frac{1}{|a_n|}\{2k(a_m + |a_m|) + |a_0| - (|a_n| + 2a_m + a_0) + 2\delta\}.$$

Therefore if $|z|>1$, then $Q(z)>0$.

Let

$$|z|>\frac{1}{|a_n|}\{2k(a_m + |a_m|) + |a_0| - (|a_n| + 2a_m + a_0) + 2\delta\}.$$

Hence all the zeros of $Q(z)$ with $|z|>1$ lie in

$$|z| \leq \frac{1}{|a_n|}\{2k(a_m + |a_m|) + |a_0| - (|a_n| + 2a_m + a_0) + 2\delta\}.$$

But those zeros of $Q(z)$ whose modulus is less than or equal to 1, already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of Theorem 1. □

Proof of Theorem 2. Let

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_{m-1}z^{m-1} + a_m z^m + a_{m+1}z^{m+1} + ... + a_1 z + a_0$$

be a polynomial degree n.

Consider the polynomial $Q(z) = (1-z)P(z)$ so that:

$$Q(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{m+1} - a_m)z^{m+1} + (a_m - a_{m-1})z^m + ... + (a_1 - a_0)z + a_0.$$

If $|z|>1$, then $\frac{1}{|z|^{n-i}}<1$ for $i = 1, 2, ..., n-1$.

Now we have

$$\begin{aligned}
|Q(z)| \geq & |a_n||z|^{n+1} - \Big\{|a_n - a_{n-1}||z|^n + ... + |a_{m+1} - a_m||z|^{m+1} + |a_m - a_{m-1}||z|^m \\
& + ... + |a_1 - a_0||z| + a_0\Big\} \\
\geq & |a_n||z|^n\Bigg[|z| - \frac{1}{|a_n|}\Big\{|a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} \\
& + ... + \frac{|a_{m+1} - a_m|}{|z|^{n-m-1}} + \frac{|a_m - a_{m-1}|}{|z|^{n-m}} \\
& + ... + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\Big\}\Bigg] \\
\geq & |a_n||z|^n\Bigg[|z| - \frac{1}{|a_n|}\Big\{|ra_n - a_{n-1} - ra_n + a_n| + |a_{n-1} - a_{n-2}| \\
& + ... + |a_{m+1} - (\delta + a_m) + \delta| + |a_m + \delta - a_{m-1} - \delta| \\
& + ... + |a_1 - a_0| + |a_0|\Big\}\Bigg] \\
\geq & |a_n||z|^n\Bigg[|z| - \frac{1}{|a_n|}\Big\{(a_{n-1} - ra_n) + (1-r)|a_n| + (a_{n-2} - a_{n-1}) \\
& + ... + (a_m + \delta - a_{m+1}) + \delta + (a_m + \delta - a_{m-1}) + \delta
\end{aligned}$$

$$
\begin{aligned}
&+ ... + (a_1 - a_0) + |a_0|\}\Big] \\
=&|a_n||z|^n\left[|z| - \frac{1}{|a_n|}\{4\delta + |a_n| + 2a_m\right. \\
&\left. + |a_0| - a_0 - r(|a_n| + a_n)\}\right] \\
>&0.
\end{aligned}
$$

Let

$$|z|>\frac{1}{|a_n|}\{4\delta + |a_n| + 2a_m + |a_0| - a_0 - r(|a_n| + a_n)\}.$$

Then if $|z|>1$, then $Q(z)>0$.

Let

$$|z|>\frac{1}{|a_n|}\{4\delta + |a_n| + 2a_m + |a_0| - a_0 - r(|a_n| + a_n)\}.$$

Hence all the zeros of $Q(z)$ with $|z|>1$ lie in

$$|z| \leq \frac{1}{|a_n|}\{4\delta + |a_n| + 2a_m + |a_0| - a_0 - r(|a_n| + a_n)\}.$$

But those zeros of $Q(z)$ whose modulus is less than or equal to 1, already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of Theorem 2. □

Proof of Theorem 3. Let

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_{m-1}z^{m-1} + a_m z^m + a_{m+1}z^{m+1} + ... + a_1 z + a_0$$

be a polynomial degree n.

Consider the polynomial $Q(z) = (1 - z)P(z)$ so that

$$
\begin{aligned}
Q(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{m+1} - a_m)z^{m+1} + (a_m - a_{m-1})z^m \\
+ ... + (a_1 - a_0)z + a_0.
\end{aligned}
$$

Let us mark that if $|z|>1$, then $\frac{1}{|z|^{n-i}}<1$ for $i = 0, 1, 2, ..., n - 1,$.

Now we have

$$
\begin{aligned}
|Q(z)| \geq&|a_n||z|^{n+1} - \Big\{|a_n - a_{n-1}||z|^n + ... + |a_{m+1} - a_m||z|^{m+1} + |a_m - a_{m-1}||z|^m \\
&+ ... + |a_1 - a_0||z| + a_0\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\geq |a_n||z|^n \Bigg[|z| - \frac{1}{|a_n|} \Big\{ |a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} \\
&\quad + ... + \frac{|a_{m+1} - a_m|}{|z|^{n-m-1}} + \frac{|a_m - a_{m-1}|}{|z|^{n-m}} \\
&\quad + ... + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \Big\} \Bigg] \\
&\geq |a_n||z|^n \Bigg[|z| - \frac{1}{|a_n|} \Big\{ |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| \\
&\quad + ... + |a_{m+1} - ra_m + ra_m - a_m| + |a_m - ra_m + ra_m - a_{m-1}| \\
&\quad + ... + |a_1 - \delta - a_0 + \delta| + |a_0| \Big\} \Bigg] \\
&\geq |a_n||z|^n \Bigg[|z| - \frac{1}{|a_n|} \Big\{ (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) \\
&\quad + ... + (a_{m+1} - ra_m) + (1 - r)|a_m| + (a_{m-1} - ra_m) + (1 - r)|a_m| \\
&\quad + ... + (a_0 + \delta - a_1) + \delta + |a_0| \Big\} \Bigg] \\
&= |a_n||z|^n \Bigg[|z| - \frac{1}{|a_n|} \Big\{ 2|a_m| + |a_0| + a_0 \\
&\quad + a_n - 2r(|a_m| + a_m) + 2\delta \Big\} \Bigg] \\
&> 0.
\end{aligned}
$$

Let

$$|z| > \frac{1}{|a_n|} \{2|a_m| + |a_0| + a_0 + a_n - 2r(|a_m| + a_m) + 2\delta\}.$$

Hence if $|z|>1$, then $Q(z)>0$.

Let

$$|z| > \frac{1}{|a_n|} \{2|a_m| + |a_0| + a_0 + a_n - 2r(|a_m| + a_m) + 2\delta\}.$$

Hence all the zeros of $Q(z)$ with $|z|>1$ lie in

$$|z| \leq \frac{1}{|a_n|} \{2|a_m| + |a_0| + a_0 + a_n - 2r(|a_m| + a_m) + 2\delta\}.$$

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of Theorem 1. □

Proof of theorem 4. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_{m-1} z^{m-1} + a_m z^m + a_{m+1} z^{m+1} + ... + a_1 z + a_0$$

be a polynomial degree n.

Consider the polynomial $Q(z) = (1-z)P(z)$ such that

$$Q(z) = -a_n z^{n+1} + (a_n - a_{n-1}) z^n + ... + (a_{m+1} - a_m) z^{m+1} + (a_m - a_{m-1}) z^m + ... + (a_1 - a_0) z + a_0.$$

Also let us note: if $|z|>1$, then $\frac{1}{|z|^{n-i}}<1$ for $i = 0, 1, 2, ..., n-1$.

Therefore

$$\begin{aligned}
|Q(z)| \geq & |a_n||z|^{n+1} - \Big\{ |a_n - a_{n-1}||z|^n + ... + |a_{m+1} - a_m||z|^{m+1} + |a_m - a_{m-1}||z|^m \\
& + ... + |a_1 - a_0||z| + a_0 \Big\} \\
\geq & |a_n||z|^n \Bigg[|z| - \frac{1}{|a_n|} \Big\{ |a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} \\
& + ... + \frac{|a_{m+1} - a_m|}{|z|^{n-m-1}} + \frac{|a_m - a_{m-1}|}{|z|^{n-m}} \\
& + ... + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \Big\} \Bigg] \\
\geq & |a_n||z|^n \Bigg[|z| - \frac{1}{|a_n|} \Big\{ |k a_n - a_{n-1} - k a_n + a_n| + |a_{n-1} - a_{n-2}| \\
& + ... + |a_{m+1} - \delta - a_m + \delta| + |a_m + \delta - a_{m-1} - \delta| \\
& + ... + |a_1 - a_0| + |a_0| \Big\} \Bigg] \\
\geq & |a_n||z|^n \Bigg[|z| - \frac{1}{|a_n|} \Big\{ (k a_n - a_{n-1}) + (k-1)|a_n| + (a_{n-1} - a_{n-2}) \\
& + ... + (a_{m+1} + \delta - a_m) + \delta + (a_{m-1} + \delta - a_m) + \delta \\
& + ... + (a_0 - a_1) + |a_0| \Big\} \Bigg] \\
= & |a_n||z|^n \Bigg[|z| - \frac{1}{|a_n|} \Big\{ 4\delta + k(|a_n| + a_n) \\
& + |a_0| + a_0 - 2a_m - |a_n| \Big\} \Bigg] \\
> & 0.
\end{aligned}$$

Let

$$|z|>\frac{1}{|a_n|}\{4\delta+k(|a_n|+a_n)+|a_0|+a_0-2a_m-|a_n|\}.$$

Hence if $|z|>1$, then $Q(z)>0$.

If

$$|z|>\frac{1}{|a_n|}\{4\delta+k(|a_n|+a_n)+|a_0|+a_0-2a_m-|a_n|\}.$$

Then we have that all zeros of $Q(z)$ with $|z|>1$ lie in

$$|z|\le\frac{1}{|a_n|}\{4\delta+k(|a_n|+a_n)+|a_0|+a_0-2a_m-|a_n|\}.$$

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of Theorem 4. □

Proof of Corollary 13. Let

$$P(z)=a_nz^n+a_{n-1}z^{n-1}+...+a_{m-1}z^{m-1}+a_mz^m+a_{m+1}z^{m+1}+...+a_1z+a_0$$

be a polynomial degree n.

Consider the polynomial $Q(z)=(1-z)P(z)$ such that

$$Q(z)=-a_nz^n(z+k-1)+(ka_n-a_{n-1})z^n+...+(a_{m+1}-a_m)z^{m+1}+(a_m-a_{m-1})z^m \\ +...+(a_1-a_0)z+a_0.$$

If $|z|>1$, then $\frac{1}{|z|^{n-i}}<1$ for $i=1,2,...,n-1$.

Now

$$\begin{aligned}|Q(z)|\ge&|a_n||z|^n|z+k-1|-\big\{|ka_n-a_{n-1}||z|^n\\&+...+|a_{m+1}-a_m||z|^{m+1}+|a_m-a_{m-1}||z|^m\\&+...+|a_1-a_0||z|+a_0\big\}\\\ge&|a_n||z|^n\Bigg[|z+k-1|-\frac{1}{|a_n|}\{|ka_n-a_{n-1}|+\frac{|a_{n-1}-a_{n-2}|}{|z|}\\&+...+\frac{|a_{m+1}-a_m|}{|z|^{n-m-1}}+\frac{|a_m-a_{m-1}|}{|z|^{n-m}}\\&+...+\frac{|a_1-a_0|}{|z|^{n-1}}+\frac{|a_0|}{|z|^n}\}\Bigg]\end{aligned}$$

$$\begin{aligned}
&\geq |a_n||z|^n \Bigg[|z+k-1| - \frac{1}{|a_n|}\big\{ |ka_n - a_{n-1}| + |a_{n-1} - a_{n-2}| \\
&\quad + ... + |a_{m+1} - \delta - a_m + \delta| + |a_m + \delta - a_{m-1} - \delta| \\
&\quad + ... + |a_1 - a_0| + |a_0| \big\} \Bigg] \\
&\geq |a_n||z|^n \Bigg[|z+k-1| - \frac{1}{|a_n|}\big\{ (ka_n - a_{n-1}) + (a_{n-1} - a_{n-2}) \\
&\quad + ... + (a_{m+1} + \delta - a_m) + \delta + (a_{m-1} + \delta - a_m) + \delta \\
&\quad + ... + (a_0 - a_1) + a_0 \big\} \Bigg] \\
&= |a_n||z|^n \Bigg[|z+k-1| - \frac{1}{|a_n|}\big\{ ka_n + 2(a_0 - a_m + 2\delta) \big\} \Bigg] \\
&> 0.
\end{aligned}$$

Let

$$|z+k-1| > \frac{1}{|a_n|}\{ka_n + 2(a_0 - a_m + 2\delta)\}.$$

If $|z|>1$, then $Q(z)>0$.

Let

$$|z+k-1| > \frac{1}{|a_n|}\{ka_n + 2(a_0 - a_m + 2\delta)\}.$$

Hence all zeros of $Q(z)$ with $|z|>1$ lie in

$$|z+k-1| \leq \frac{1}{|a_n|}\{ka_n + 2(a_0 - a_m + 2\delta)\}.$$

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of Corollary 13.

References

[1] G.Eneström,Remarquee sur un théorème relatif aux racines de l'equation $a_n + ... + a_0 = 0$ oü tous les coefficient sont et positifs, Tôhoku Math.J 18 (1920),34-36.

[2] S.KAKEYA, On the limits of the roots of an alegebraic equation with positive coefficient, Tôhoku Math.J 2 (1912-1913),140-142.

[3] A. Joyal, G. Labelle and Q. I. Rahman, On the location of zeros of polynomials, Canad. Math. Bull. 10, (1967), 53-63.

[4] A. Aziz and B. A. Zargar, Some extensions of Eneström-Kakeya Theorem, Glasnik Matematiki, 31, (1996), 239-244..

[5] R. Fricke, Lehrbuch der Algebra, Friedrich Vieweg and Sohn Verlag, braunschweig, Germany, 1924.

[6] W. Heitzinger, W. I. Troch and G. Valentin, Praxis nichtlinearer Gleichungen, Carl Hanser Verlag, munchen, Wien, Germany, Austria, 1985.

[7] W. M. Shah, A. Liman and Shamim Ahmad Bhat, On the Enestrm-Kakeya Theorem, International Journal of Mathematical Science, 7, (1-2) (2008), 111-120.

[8] W. M. Shah and A. Liman, On the zeros of a certain class of polynomials and related analytic functions, Mathematicka Balkanicka, New Series, 19, (2005): Facs., 3-4

[9] Gulshan Singh, On the Eneström-Kakeya Theorem. American Journal of Mathematical Analysis, vol. 2, no. 1 (2014).