

ON THE DIOPHANTINE EQUATION $(p + 1)^{2x} + q^y = z^2$

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Abstract: In this paper, we found that the Diophantine equation $(p + 1)^{2x} + q^y = z^2$ has no non-negative integer solution where p is a Mersenne prime number which $q - p = 2$ and x, y and z are non-negative integers.

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1. Introduction

In 1844, Catalan [2] posed a conjecture that $(a, b, x, y) = (3, 2, 2, 3)$ is a unique solution of the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$. Then Mihailescu [3] proved the Catalan's conjecture in 2004. After that Acu [1] proved that $(3, 0, 3)$ and $(2, 1, 3)$ are only two solutions (x, y, z) for the Diophantine equation $2^x + 5^y = z^2$ where x, y and z are non-negative integers.

In 2011, Suvarnamani, Singta and Chotchaisthit [9] proved that two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution. Then Suvarnamani [5] proved that two Diophantine equations $4^x + 13^y = z^2$ and $4^x + 17^y = z^2$ have no non-negative integer solution. After

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that Suvarnamani [6] proved that the Diophantine equation $2^x + p^y = z^2$ has some non-negative integer solutions where p is a prime number.

In 2012, Suvarnamani [7] found that Diophantine equation $A^x + B^y = C^z$ has some non-negative integer solutions. Then Suvarnamani [8] found that the Diophantine equation $p^x + p^y = z^2$ has some non-negative integer solutions where p is a prime number. After that Sroysang [4] proved that $(0, 1, 3)$ is a unique non-negative integer solution of the Diophantine equation $7^x + 8^y = z^2$.

In 2014, Suvarnamani [10] found the solution of the Diophantine equation $p^x + q^y = z^2$ where p is an odd prime number which $q - p = 2$ and x, y and z are non-negative integers. After that he studied in [11] about the Diophantine equation $p^x + (p + 1)^y = z^2$ where p is an odd prime number and x, y and z are non-negative integers.

In this paper, we will use the Catalan's conjecture to solving $(p + 1)^{2x} + q^y = z^2$ where p is a Mersenne prime number which $q - p = 2$ and x, y and z are non-negative integers.

2. Preliminaries

Lemma 2.1. *$(a, b, x, y) = (3, 2, 2, 3)$ is a unique solution of the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$.*

Proof. See in [4]. □

Lemma 2.2. *If q is an odd prime number and y, z are non-negative integers. Then the Diophantine equation $1 + q^y = z^2$ has no solution.*

Proof. Let q is an odd prime number and y, z be non-negative integers such that $1 + q^y = z^2$. We consider in 3 cases.

Case 1: $y = 0$. Then $z^2 = 2$ which is impossible.

Case 2: $y = 1$. Thus $z^2 = q + 1$. That is $z = 0$ or 2 . It is impossible.

Case 3: $y > 1$. Thus $z^2 = q^y + 1 > q + 1$. Then $z > 2$. By Lemma 2.1, we have $z = 3, q = 2$ and $y = 3$. Contradiction. □

Lemma 2.3. *$(p, x, z) = (3, 1, 2)$ is a unique solution of the Diophantine equation $p^x + 1 = z^2$ where p is an odd prime number and x, z are non-negative integers.*

Proof. Let p be an odd prime number and x, z be non-negative integers such that $p^x + 1 = z^2$. We consider in 3 cases.

Case 1: $x = 0$. Then $z^2 = 2$ which is impossible.

Case 2: $x = 1$. Thus $z^2 = p + 1$. That is $z = 2$. Then we get $p = 3$.

Case 3: $x > 1$. Thus $z^2 = p^x + 1 > p + 1$. Then $z > 2$. By Lemma 2.1, we have $z = 3, p = 2$ and $x = 3$. Contradiction. \square

3. Main Theorem

Main Theorem 3.1. *The Diophantine equation $(p + 1)^{2x} + q^y = z^2$ has no non-negative integer solution where p is a Mersenne prime number which $q - p = 2$ and x, y and z are non-negative integers.*

Proof. Let p is a Mersenne prime number which $q - p = 2$ and x, y and z are non-negative integers such that $(p + 1)^{2x} + q^y = z^2$. By Lemma 2.2, we have $x \geq 1$. Then we consider in 2 cases.

Case 1: $y = 0$.
Then $(p + 1)^{2x} + 1 = z^2$. By Lemma 2.3, we get $p + 1 = 3$ but it is impossible.

Case 2: $y \geq 1$.
Since $z^2 = (p + 1)^{2x} + q^y = (p + 1)^{2x} + (p + 2)^y$. So, z is odd. Then $z^2 \equiv 1 \pmod{4}$. Next, $(p + 2)^y = z^2 - (p + 1)^{2x} = [z - (p + 1)^x][z + (p + 1)^x]$, where $z - (p + 1)^x = (p + 2)^u$ and $z + (p + 1)^x = (p + 2)^{y-u}, y > 2u$. Then $(p + 2)^u((p + 2)^{y-2u} - 1) = 2 \cdot (p + 1)^x$. Since p is a Mersenne prime number, i.e., $p = 2^k - 1$ for some prime k . That is $(p + 2)^u(q^{y-2u} - 1) = 2 \cdot (p + 1)^x = 2^{kx+1}$. By Lemma 2.1, we get $u = 0$. That is $(x, y, z) = (2, 2, 5)$. It is impossible because we get p which is not a Mersenne prime number. \square

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