THE GENERALIZED β –CONFORMAL CHANGE OF FINSLER METRIC

S.K. Tiwari¹, Anamika Rai^{2 §}

^{1,2}Department of Mathematics

K.S. Saket Post Graduate College

Ayodhya, Faizabad, 224 123, INDIA

Abstract: In the present paper, we have considered the generalized β –conformal change of Finsler metric $\bar{L}$ given by

$$\bar{L}(x, y) = f(e^{\sigma(x)} L(x, y), \beta^1, \beta^2, \dots, \beta^m),$$

where f is any positively homogeneous function of degree one in $e^{\sigma} L$ and $\beta^1, \beta^2, \dots, \beta^m, \beta^r = b_i^{(r)}(x) y^i$ and $\sigma(x)$ is a function of position only, known as conformal factor. The relation between the v –curvature tensor of the original Finsler space and Finsler space obtained by generalized β –conformal change has been obtained. Some basic properties of generalized β –conformal change of Finsler metric have been discussed and obtained useful results.

AMS Subject Classification: 53B40, 53C60

Key Words: generalized β –conformal change, generalized β –change, conformal change, β –change, Finsler metric

1. Introduction

In 1971, Matsumoto [6] introduced the transformation of Finsler metric:

Received: April 17, 2015

© 2015 Academic Publications, Ltd.
url: www.acadpubl.eu

[§]Correspondence author

$$\overline{L}(x, y) = L(x, y) + \beta(x, y),$$

where $\beta(x, y) = b_i(x)y^i$ is a one-form. In 1984, C. Shibata [2] introduced the transformation of Finsler metric:

$$\overline{L}(x, y) = f(L, \beta),$$

where $\beta = b_i(x)y^i$, $b_i(x)$ are components of a covariant vector in (M^n, L) and f is positively homogenous function of degree one in L and β . This change of metric is called a β -change. In 2013, Prasad, B. N. and Kumari, Bindu [1] have considered the β -change of Finsler metric. In the year 2014 [10], we studied generalized β -change defining as

$$L(x, y) \rightarrow \overline{L}(x, y) = f(L, \beta^1, \beta^2, \dots, \beta^m),$$

where f is any positively homogeneous function of degree one in $L, \beta^1, \beta^2, \dots, \beta^m$, where $\beta^1, \beta^2, \dots, \beta^m$ are linearly independent one-form.

The conformal theory of Finsler spaces has been initiated by M. S. Knebelman [5] in 1929 and has been investigated in detail by many authors [3, 4, 7, 9] etc. The conformal change is defined as

$$L(x, y) \rightarrow e^{\sigma(x)} L(x, y),$$

where $\sigma(x)$ is a function of position only and known as conformal factor.

In the present paper, we have considered the generalized β -conformal change of Finsler metric by taking

$$\overline{L} = f(e^{\sigma(x)} L(x, y), \beta^1, \beta^2, \dots, \beta^m),$$

where f is any positively homogeneous function of degree one in $e^{\sigma} L, \beta^1, \beta^2, \dots, \beta^m$.

In the present paper, we have found out the relation between the v -curvature tensor of the original Finsler space and the other which is Finsler space given by generalized β -conformal change

$$\overline{L} = f(e^{\sigma(x)} L, \beta^1, \beta^2, \dots, \beta^m).$$

2. Preliminaries

Let M^n be an n -dimensional smooth manifold and $F^n = (M^n, L)$ be an n -dimensional Finsler space equipped with the fundamental function L on M^n . The metric tensor $g_{ij}(x, y)$ and Cartan's C -tensor $C_{ijk}(x, y)$ are given by

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j}, \quad C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k},$$

respectively and we can introduce the Cartan's connection $CT = (F_{jk}^i, N_j^i, C_{jk}^i)$ in F^n . Here, we have considered the following transformation of Finsler metric:

$$\bar{L}(x, y) = f(e^{\sigma(x)} L, \beta^1, \beta^2, \dots, \beta^m), \quad (2.1)$$

and such transformation is called generalized β -conformal change of Finsler metric, where f is positively homogeneous function of degree one in $e^{\sigma(x)} L, \beta^1, \beta^2, \dots, \beta^m$ and $\beta^1, \beta^2, \dots, \beta^m$ all are linearly independent one-form and defined as $\beta^r = b_i^r y^i$. Homogeneity of f gives

$$e^\sigma L f_0 + f_r b^r = f, \quad (2.2)$$

where the subscripts '0' and 'r' denote the partial derivative with respect to L and β^r respectively. The letters r, s, t, r and s vary from 1 to m throughout the paper. Summation convention is applied for the indices r, s, t, r and s . If we write $\bar{F}^n = (M^n, \bar{L})$, then the Finsler space $\bar{F}^n$ is said to be obtained from F^n by generalized β -conformal change. The quantities corresponding to $\bar{F}^n$ are denoted by putting bar on those quantities.

To find the relation between fundamental quantities of (M^n, L) and $(M^n, \bar{L})$, we use the following results:

$$\dot{\partial}_i \beta^r = b_i^r, \quad \dot{\partial}_i L = l_i, \quad \dot{\partial}_j l_i = L^{-1} h_{ij}, \quad (2.3)$$

where $\dot{\partial}_i$ stands for $\frac{\partial}{\partial y^i}$ and h_{ij} are components of angular metric tensor of (M^n, L) given by $h_{ij} = g_{ij} - l_i l_j = L \dot{\partial}_i \dot{\partial}_j L$.

The v -curvature tensor S_{hijk} of (M^n, L) is given by [8],

$$S_{hijk} = C_{ij}^a C_{hka} - C_{ik}^a C_{hja}.$$

3. Generalized β -Conformal Change of Finsler Metric

Differentiating equation (2.2) with respect to L and β^s respectively, we get

$$e^\sigma L f_{00} + f_{0r} \beta^r = 0 \quad (3.1)$$

and

$$e^\sigma L f_{0s} + f_{rs} \beta^r = 0. \quad (3.2)$$

The successive differentiation of (2.1) with respect to y^i and y^j give

$$\bar{l}_i = e^\sigma f_0 l_i + f_r b_i^r, \quad (3.3)$$

$$\bar{h}_{ij} = e^\sigma \frac{f f_0}{L} h_{ij} + e^{2\sigma} f f_{00} l_i l_j + e^\sigma f f_{0r} (b_j^r l_i + b_i^r l_j) + f f_{rs} b_i^r b_j^s. \quad (3.4)$$

Using equations (3.1) and (3.2) in equation (3.4), we have

$$\bar{h}_{ij} = e^\sigma \frac{f f_0}{L} h_{ij} + f f_{rs} \left(b_i^r - \frac{\beta^r}{L} l_i \right) \left(b_j^s - \frac{\beta^s}{L} l_j \right). \quad (3.5)$$

If we put $m_i^r = b_i^r - \frac{\beta^r}{L} l_i$, equation (3.5) may be written as

$$\bar{h}_{ij} = e^\sigma \frac{f f_0}{L} h_{ij} + f f_{rs} m_i^r m_j^s. \quad (3.6)$$

From equations (3.3) and (3.6), we get the following relation between metric tensors of (M^n, L) and $(M^n, \bar{L})$

$$\bar{g}_{ij} = e^\sigma \frac{f f_0}{L} g_{ij} + e^\sigma \left(e^\sigma f_0^2 - \frac{f f_0}{L} \right) l_i l_j + f f_{rs} m_i^r m_j^s + e^\sigma f_0 f_r (b_i^r l_j + b_j^r l_i) + f_r f_s b_i^r b_j^s. \quad (3.7)$$

To find inverse of g_{ij} , let

$$\bar{g}^{hj} \bar{g}_{ij} = \delta_i^h. \quad (3.8)$$

Using equation (3.7) above equation (3.8) can be written as

$$\begin{aligned} \bar{g}^{hj} \left[e^\sigma \frac{f f_0}{L} g_{ij} + e^\sigma \left(e^\sigma f_0^2 - \frac{f f_0}{L} \right) l_i l_j \right. \\ \left. + e^\sigma (f_0 f_r + f f_{0r}) (b_i^r l_j + b_j^r l_i) + (f_r f_s + f f_{rs}) b_i^r b_j^s \right] = \delta_i^h. \end{aligned} \quad (3.9)$$

Contracting equation (3.9) by g^{ik} , we get

$$\begin{aligned} e^\sigma \bar{g}^{hk} \frac{f f_0}{L} = g^{hk} - e^\sigma \left(e^\sigma f_0^2 + f e^\sigma f_{00} - \frac{f f_0}{L} \right) \bar{g}^{hj} l_j l^k - (f_r f_s + \\ f f_{rs}) \bar{g}^{hj} b_j^r b^s{}^k - e^\sigma (f_0 f_r + f f_{0r}) (\bar{g}^{hj} b_j^r l^k + \bar{g}^{hj} b^r{}^k l_j). \end{aligned} \quad (3.10)$$

Again contracting equation (3.9) by l^i , we get

$$e^\sigma f_0 \left(e^\sigma f_0 + \frac{f_r \beta^r}{L} \right) \bar{g}^{hj} l_j + f_r \left(e^\sigma f_0 + \frac{f_s \beta^s}{L} \right) \bar{g}^{hj} b_j^r = l^h. \quad (3.11)$$

Again contracting (3.9) by $b^{r)i}$, we obtain

$$\begin{aligned} & \left\{ e^\sigma \left(e^\sigma f_0^2 + f e^\sigma f_{00} - \frac{f f_0}{L} \right) \frac{\beta^r}{L} + e^\sigma (f_0 f_r + f f_{0r}) b_i^r b^s{}^i \right\} \bar{g}^{hj} l_j \\ & + \left\{ \frac{e^\sigma f f_0}{L} + (f_r f_s + f f_{rs}) b_i^r b^s{}^i + e^\sigma (f_0 f_r + f f_{0r}) \frac{\beta^r}{L} \right\} \bar{g}^{hj} b_j^t = b^t{}^h. \end{aligned} \quad (3.12)$$

With the help of equation (2.2), equation (3.11) may be written as

$$e^\sigma f_0 \bar{g}^{hj} l_j + f_r \bar{g}^{hj} b_j^r = \frac{L}{f} l^h \quad (3.13)$$

Equation (3.12) can also be written as

$$A \bar{g}^{hj} l_j + B \bar{g}^{hj} b_j^s = b^r{}^h, \quad (3.14)$$

where $A = e^\sigma (f_0 f_r + f f_{0r}) \Delta^{rs}$, $B = \frac{f^2}{L^2} + (f_r f_s + f f_{rs}) \Delta^{rs}$, $\Delta^{rs} = b_i^r b^s{}^i - \frac{\beta^r \beta^s}{L^2}$.

Solving equations (3.13) and (3.14), we get

$$\bar{g}^{hj} l_j = \frac{\left[\frac{f}{L} + L(f_r f_s + f f_{rs}) \Delta^{rs} \right] l^h - f_r b^r{}^h}{\frac{f^2}{L^2} (e^\sigma f_0 + L f_{rs} \Delta^{rs})} \quad (3.15)$$

and

$$\bar{g}^{hj} b_j^r = \frac{-e^\sigma [L(f_0 f_r + f f_{0r}) \Delta^{rs} l^h - f_0 b^r{}^h]}{\frac{f^2}{L^2} (e^\sigma f_0 + L f_{rs} \Delta^{rs})}.$$

In view of equations (2.2), (3.1) and (3.15), equation (3.10) can be written as

$$\begin{aligned} \bar{g}^{hk} = & \frac{L}{e^\sigma f f_0} g^{hk} + \frac{L^3}{e^\sigma f^3 f_0 (e^\sigma f_0 + L f_{rs'} \Delta^{rs'})} \left\{ e^\sigma (f_0 f_r + f f_{0r}) \left(\frac{f \beta^r}{L} + f_s \Delta^{rs} \right) l^h l^k \right. \\ & \left. - \frac{f^2}{L} f_{rs} b^r{}^h b^s{}^k - \frac{e^\sigma f}{L} (f_0 f_r + f f_{0r}) (l^h b^r{}^k + l^k b^r{}^h) \right\}, \end{aligned}$$

where we put $b^{r)i} = g^{ij} b_j^r$, $l^i = g^{ij} l_j$ and $\Delta^{rs} = b_i^r b^s{}^i - \frac{\beta^r \beta^s}{L^2}$.

Now,

$$\begin{aligned}
 (a) \quad \dot{\partial}_j m_i^r &= -\frac{1}{L} m_i^r l_j - \frac{\beta^r}{L^2} h_{ij}, \\
 (b) \quad \dot{\partial}_i f &= e^\sigma f_0 l_i + f_r b_i^r, \\
 (c) \quad \dot{\partial}_i f_{rs} &= e^\sigma f_{rs} l_i + f_{rst} b_i^t.
 \end{aligned} \tag{3.17}$$

Differentiating equation (3.6) with respect to y^k and using equations (3.7), (2.3), (3.2), (3.1), (3.3) and (3.17), we get

$$\begin{aligned}
 2\bar{C}_{ijk} &= \frac{2e^\sigma f f_0}{L} C_{ijk} + \frac{e^\sigma (f_0 f_r + f f_{0r})}{L} (h_{ij} m_k^r + h_{jk} m_i^r + h_{ki} m_j^r) \\
 &\quad + (f_{rs} f_t + f_{rt} f_s + f_{st} f_r + f f_{rst}) m_i^r m_j^s m_k^t.
 \end{aligned} \tag{3.18}$$

It is to be noted that

$$m_i^r l^i = 0, \quad m_i^r b^{s)i} = \Delta^{rs} = m_i^r m^s)^i, \quad h_{ij} l^j = 0, \quad h_{ij} m^r)^j = h_{ij} b^{r)j} = m_i^r, \tag{3.19}$$

where $m^r)^i = g^{ij} m_j^r = b^r)^i - \frac{\beta^r}{L} l^i$.

To find $\bar{C}_{ij}^h = \bar{g}^{hk} \bar{C}_{ijk}$, we use equations (3.16), (3.18) and (3.19), we get

$$\begin{aligned}
 \bar{C}_{ij}^h &= C_{ij}^h + \frac{1}{2f f_0} (f_0 f_r + f f_{0r}) (h_{ij} m^r)^h + h_j^h m_i^r + h_i^h m_j^r) \\
 &\quad + \frac{L}{2e^\sigma f f_0} (f_{rs} f_t + f_{st} f_r + f_{tr} f_s + f f_{rst}) m_i^r m_j^s m^t)^h \\
 &\quad - \frac{L}{f(e^\sigma f_0 + L f_{rs} \Delta^{rs})} C_{ij}^r n_r^h - \frac{L \Delta^{rs} (f_0 f_r + f f_{0r})}{2f^2 f_0 (e^\sigma f_0 + L f_{rs} \Delta^{rs})} h_{ij} n_s^h \\
 &\quad - \left\{ \frac{L \left\{ \begin{aligned} &e^\sigma (f_0 f_r + f f_{0r}) + e^\sigma (f_0 f_s + f f_{0s}) \\ &+ L \Delta^{rt} (f_{rs} f_t + f_{st} f_r + f_{tr} f_s + f f_{rst}) \end{aligned} \right\}}{2e^\sigma f^2 f_0 (e^\sigma f_0 + L f_{rs'} \Delta^{rs'})} \right\} m_i^r m_j^s n_r^h,
 \end{aligned} \tag{3.20}$$

where $C_{jk}^r = C_{ijk} b^r)^i$ and $n_r^h = f f_{rs} b^s)^h + e^\sigma (f_0 f_r + f f_{0r}) l^h$.

We have the following corresponding to the vector with components n_r^i and $m^r)^i$

$$C_{ijk} m^r)^i = C_{jk}^r, \quad C_{ijk} n_s^i = f f_{rs} C_{jk}^r, \quad m_i^r n_r^i = f f_{rs} \Delta^{rs}. \tag{3.21}$$

To find v -curvature tensor of $(M^n, \bar{L})$ with respect to Cartan's connection, we use the following:

$$C_{ij}^h h_{hk} = C_{ijk}, \quad h_j^k h_k^i = h_j^i, \quad h_{ij} n_r^i = f f_{rs} m_j^s). \tag{3.22}$$

The v -curvature tensor $\overline{S}_{hijk}$ of $(M^n, \overline{L})$ is defined as

$$\overline{S}_{hijk} = \overline{C}_{ij}^a \overline{C}_{ahk} - \overline{C}_{ik}^a \overline{C}_{ahj}. \quad (3.23)$$

From equations (3.18), (3.19), (3.20), (3.21), (3.22) and (3.23), we get the following relation between v -curvature tensor of the original Finsler space and the other which is Finsler space given by generalized β -conformal change as

$$\begin{aligned} \overline{S}_{hijk} = & \frac{e^\sigma f f_0}{L} S_{hijk} + \frac{e^\sigma f f_0 f_{rs}}{e^\sigma f_0 + L f_{rs'} \Delta^{rs'}} (C_{.ik}^{(r)} C_{.hj}^{(s)} - C_{.ij}^{(r)} C_{.hk}^{(s)}) \\ & + \frac{e^{2\sigma} \Delta^{rs} (f_0 f_r + f f_{0r}) (f_0 f_s + f f_{0s})}{4L f (e^\sigma f_0 + L f_{rs'} \Delta^{rs'})} (h_{ij} h_{hk} - h_{ik} h_{hj}) \\ & + \frac{e^{2\sigma} f_0 (f_0 f_r + f f_{0r})}{2L (e^\sigma f_0 + L f_{rs'} \Delta^{rs'})} (C_{.ij}^{(r)} h_{hk} + C_{.hk}^{(r)} h_{ij} - C_{.ik}^{(r)} h_{hj} - C_{.hj}^{(r)} h_{ik}) \end{aligned} \quad (3.24)$$

$$\begin{aligned} & + \frac{e^\sigma}{(e^\sigma f_0 + L f_{rs'} \Delta^{rs'})} \left\{ \frac{f_0 (f_{rs} f_t + f_{st} f_r + f_{tr} f_s + f f_{rst})}{2} - f_{rs} (f_0 f_t + f f_{0t}) \right\} \\ & \times (C_{.ij}^{(r)} m_h^{(s)} m_k^{(t)} + C_{.hk}^{(r)} m_i^{(s)} m_j^{(t)} - C_{.ik}^{(r)} m_h^{(s)} m_j^{(t)} - C_{.hj}^{(r)} m_i^{(s)} m_k^{(t)}) \\ & + \left\{ \frac{e^{2\sigma} (f_0 f_r + f f_{0r}) (f_0 f_s + f f_{0s})}{2L f (e^\sigma f_0 + L f_{rs'} \Delta^{rs'})} \right. \\ & + \left. \frac{e^\sigma \Delta^{r't} (f_0 f_{r'} + f f_{0r'}) (f_{rs} f_t + f_{st} f_r + f_{tr} f_s + f f_{rst})}{4f (e^\sigma f_0 + L f_{rs'} \Delta^{rs'})} \right\} \\ & \times (h_{ij} m_h^{(r)} m_k^{(s)} + h_{hk} m_i^{(r)} m_j^{(s)} - h_{ik} m_h^{(r)} m_j^{(s)} - h_{hj} m_i^{(r)} m_k^{(s)}). \end{aligned}$$

Theorem 3.1. *The relation between the v -curvature tensor of the original Finsler space (M^n, L) and the other Finsler space $(M^n, \overline{L})$ obtained by generalized β -conformal change is given by equation (3.24).*

References

- [1] B. N. Prasad and Bindu Kumari, The β -change of Finsler metric and imbedding classes of their tangent spaces, *Tensor, N. S.*, **74** (2013), 48-59.
- [2] C. Shibata, On invariant tensors of β -change of Finsler metrics, *J. Math. Kyoto Univ.*, **24** (1984), 163-188.

- [3] H. Izumi, Conformal transformations of Finsler spaces I, *Tensor, N. S.*, **31** (1977), 33-41.
- [4] H. Izumi, Conformal transformations of Finsler spaces II, *Tensor, N. S.*, **33** (1980), 337-359.
- [5] M. S. Knebelman, Conformal geometry of generalized metric spaces, *Proc. Nat. Acad. Sci., USA*, **15** (1929), 33-41, 376-379.
- [6] M. Matsumoto, On transformations of locally Minkowskian space, *Tensor, N. S.*, **22** (1971), 103-111.
- [7] M. Hashiguchi, On conformal transformation of Finsler metric, *J. Math. Kyoto Univ.*, **16** (1976), 25-50.
- [8] M. Matsumoto, Foundations of Finsler geometry and special Finsler spaces, *Kaiseisha Press*, Otsu, Japan, 1986.
- [9] M. Kitayama, Geometry of transformations of Finsler metrics, *Hokkaido University of Education*, Kushiro Campus, Japan (2000).
- [10] S. K. Tiwari and Anamika Rai, The generalized β -change of Finsler metric, *International J. Contemp. Math. Sci.*, **9** (2014), 695-702.