ON $\theta$-$b$-GENERALIZED CLOSED SETS IN TOPOLOGY

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Abstract: The aim of this paper is to introduce and study a new class of generalized closed sets called $\theta$-$b$-generalized closed sets in topological space.

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1. Introduction and Preliminaries

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the variously modified forms of continuity, separation axioms etc. by utilizing generalized open sets. For a subset $A$ of a topological space $(X, \tau)$, $\text{Cl}(A)$ and $\text{Int}(A)$ denote the closure of $A$ and the interior of $A$, respectively. A set $A$ is called $b$–open [1](=$\gamma$-open

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[4]) if \( A \subset \text{Int}(\text{Cl}(A)) \cup \text{Cl}(\text{Int}(A)) \). The complement of a \( b \)-open sets is called a \( b \)-closed set. The intersection of all \( b \)-closed sets of \((X, \tau)\) containing \( A \) is called the \( b \)-closure [1] of \( A \) and is denoted by \( b \text{Cl}(A) \). A set \( A \) is \( b \)-closed if and only if \( A = b \text{Cl}(A) \). The family of all \( b \)-open sets of \((X, \tau)\) is denoted by \( BO(X, \tau) \). For each \( x \in X \), the family of all \( b \)-open sets of \((X, \tau)\) containing a point \( x \) is denoted by \( BO(X, x) \). The \( b \)-interior of \( A \) is the union of all \( b \)-open sets contained in \( A \) and is denoted by \( b \text{Int}(A) \). A subset \( A \) is called \( b \)-regular [5] if it is both \( b \)-open and \( b \)-closed. The \( b \)-\( \theta \)-clouser [5], denoted by \( b \text{Cl}_{\theta}(A) \), is the set of all \( x \in X \) such that \( b\text{Cl}(U) \cap A \neq \emptyset \) for every \( U \in BO(X, x) \). A subset \( A \) is called \( b \)-\( \theta \)-closed [5] if \( A = b \text{Cl}_{\theta}(A) \). The set \( \{x \in X | b\text{Cl}(U) \subset A \} \) is called the \( b \)-\( \theta \)-interior [5] of \( A \) and is denoted by \( b\text{Int}_{\theta}(A) \). By [5], it is proved that, for a subset \( A \), \( b \text{Cl}_{\theta}(A) \) is the intersection of all \( b \)-\( \theta \)-closed sets containing \( A \). The aim of this paper is to introduce and study a new class of generalized closed sets called \( \theta \)- \( b \)-generalized closed sets in topological space.

2. On \( \theta \)-\( b \)-Generalized Closed Sets

**Definition 1.** A subset \( A \) of a topological space \((X, \tau)\) is called \( \theta \)-\( b \)-generalized closed (briefly \( \theta \)-\( bg \)-closed) if \( b \text{Cl}_{\theta}(A) \subset U \) whenever \( A \subset U \) and \( U \in BO(X, \tau) \).

The complement of a \( \theta \)-\( b \)-generalized closed set is called a \( \theta \)-\( b \)-generalized open set (briefly \( \theta \)-\( bg \)-open).

**Lemma 2.1.** Every \( b \)-\( \theta \)-closed set is \( \theta \)-\( bg \)-closed but not conversely.

**Proof.** Let \( A \subset X \) be \( b \)-\( \theta \)-closed. Then \( A = b \text{Cl}_{\theta}(A) \). Let \( A \subset U \) and \( U \in BO(X, \tau) \). It follows that \( b \text{Cl}_{\theta}(A) \subset U \). This means that \( A \) is \( \theta \)-\( bg \)-closed. \( \square \)

**Example 2.2.** Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, \{a\}, X\} \). Then the set \( \{b, c\} \) is \( \theta \)-\( bg \)-closed but not \( b \)-\( \theta \)-closed in \((X, \tau)\).

**Theorem 2.3.** A set \( A \subset (X, \tau) \) is \( \theta \)-\( bg \)-open if and only if \( F \subset b\text{Int}_{\theta}(A) \) whenever \( F \) is \( b \)-closed in \( X \) and \( F \subset A \).

**Proof.** Let \( A \) be \( \theta \)-\( bg \)-open and \( F \subset A \), where \( F \) is \( b \)-closed. It is obvious that \( X \setminus A \) is contained in \( X \setminus F \). This implies that \( b \text{Cl}_{\theta}(X \setminus A) \subset X \setminus F \). Hence
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$b \text{Cl}_{\theta}(X \setminus A) = X \setminus (b \text{Int}_{\theta}(A)) \subset X \setminus F$, that is, $F \subset b \text{Int}_{\theta}(A)$. Conversely, if $F$ is a $b$-closed set with $F \subset b \text{Int}_{\theta}(A)$ whenever $F \subset A$, then it follows that $X \setminus A \subset X \setminus F$ and $X \setminus (b \text{Int}_{\theta}(A)) \subset X \setminus F$, that is, $b \text{Cl}_{\theta}(X \setminus A) \subset X \setminus F$. Therefore, $X \setminus A$ is $\theta$-$bg$-closed and hence $A$ is $\theta$-$bg$-open.

Lemma 2.4. Let $A$ be $\theta$-$bg$-closed subset of $(X, \tau)$. Then: (1) $b \text{Cl}_{\theta}(A) \setminus A$ does not contain a nonempty $b$-closed set.

(2) $b \text{Cl}_{\theta}(A) \setminus A$ is $\theta$-$bg$-open.

Proof. (1). Let $F$ be a $b$-closed set such that $F \subset b \text{Cl}_{\theta}(A) \setminus A$. Since $F^c$ is $b$-open and $A \subset X \setminus F$, $b \text{Cl}_{\theta}(A) \subset X \setminus F$, that is $F \subset X \setminus (b \text{Cl}_{\theta}(A))$. This implies that $F \subset X \setminus (b \text{Cl}_{\theta}(A)) \cap b \text{Cl}_{\theta}(A) = \emptyset$.

(2). If $A$ is $\theta$-$bg$-closed and $F$ is a $b$-closed set such that $F \subset b \text{Cl}_{\theta}(A) \setminus A$, then by (1), $F$ is empty and therefore $F \subset b \text{Int}_{\theta}(b \text{Cl}_{\theta}(A) \setminus A)$. By Theorem 2.3, $b \text{Cl}_{\theta}(A) \setminus A$ is $\theta$-$bg$-open.

Definition 2. A topological space $(X, \tau)$ is called $b$-$T_1$ [2] if for distinct points $x, y \in X$, there exists a $b$-open set containing $x$ but not $y$ and a $b$-open set containing $y$ but not $x$, or equivalently $(X, \tau)$ is $b$-$T_1$ if and only if every singleton is $b$-closed [2].

Theorem 2.5. A topological space $(X, \tau)$ is $b$-$T_1$ if and only if every $\theta$-$bg$-closed set is $b$-$\theta$-closed.

Proof. Let $A \subset X$ be $\theta$-$bg$-closed and $x \in b \text{Cl}_{\theta}(A)$. Since $X$ is $b$-$T_1$, $\{x\}$ is $b$-closed and thus by Lemma 2.4, $x \notin b \text{Cl}_{\theta}(A) \setminus A$. Since $x \in b \text{Cl}_{\theta}(A)$, then $x \in A$. This shows that $b \text{Cl}_{\theta}(A) \subset A$ or equivalently $A$ is $b$-$\theta$-closed. Conversely, let $x \in X$. Assume that $\{x\}$ is not $b$-closed. Then $X \setminus \{x\}$ is not $b$-open but is $\theta$-$bg$-closed since the only $b$-open superset of $X \setminus \{x\}$ is $X$. By hypothesis, $X \setminus \{x\}$ is $b$-$\theta$-closed and thus $\{x\}$ is $b$-$\theta$-open. Since a singleton is $b$-$\theta$-open if and only if it is $b$-regular, $\{x\}$ is $b$-regular.

Definition 3. A subset $A$ of a topological space $(X, \tau)$ is called a $\Lambda_b$-set [2] if $A = A^{b\theta}$, where $A^{b\theta} = \cap \{U | A \subset U, U \in BO(X, \tau)\}$.

Definition 4. A subset $A$ of a topological space $(X, \tau)$ is called a generalized $\Lambda_b$-set (briefly $g.\Lambda_b$-set)[2] if $A^{b\theta} \subset F$ whenever $A \subset F$ and $F$ is a $b$-closed
Lemma 2.6. (1) For a subset A of a topological space \((X, \tau)\), we define \(b\ker_\theta(A)\) as follows \(b\ker_\theta(A) = \{x \in X | b\Cl_\theta(\{x\}) \cap A \neq \emptyset\}\). (2) A subset A of \((X, \tau)\) is called \(\theta\)-generalized \(\Lambda_b\)-set (briefly \(\theta-g.\Lambda_b\)-set) if \(b\ker_\theta(A) \subset F\), whenever \(A \subset F\) and \(F\) is a \(b\)-closed set of \((X, \tau)\).

Lemma 2.6. [5] For any subset A of a topological space \((X, \tau)\), \(b\Cl_\theta(A)\) is \(b-\theta\)-closed.

Lemma 2.7. If A is a \(\theta\)-bg-closed set of a topological space \((X, \tau)\) such that \(A \subset B \subset b\Cl_\theta(A)\), then B is also a \(\theta\)-bg-closed set of \((X, \tau)\).

Proof. Let \(U\) be a \(b\)-open set of \((X, \tau)\) such that \(B \subset U\). Then \(A \subset U\). Since A is \(\theta\)-bg-closed, \(b\Cl_\theta(A) \subset U\). By using Lemma 2.6, \(b\Cl_\theta(B) \subset b\Cl_\theta(b\Cl_\theta(A)) = b\Cl_\theta(A) \subset U\). Therefore, B is also a \(\theta\)-bg-closed set of \((X, \tau)\).

Proposition 2.8. Let \((X, \tau)\) be a topological space and \(A\) be a subset of \(X\). Then, \(b\ker_\theta(A) = \cap\{U | A \subset U, U \text{ is } b-\theta\text{-open}\}\) for any subset \(A \subset (X, \tau)\).

Proof. Let \(H = \cap\{U | A \subset U, U \text{ is } b-\theta\text{-open}\}\) and \(x \in H\). Suppose that \(x \notin b\ker_\theta(A)\) which means \(b\Cl_\theta(\{x\}) \cap A = \emptyset\). Hence \(x \notin X \setminus b\Cl_\theta(\{x\})\), where \(X \setminus b\Cl_\theta(\{x\})\) is \(b-\theta\)-open set containing \(A\) by Lemma 2.6. But this is impossible since \(x \in H\). Consequently \(x \in b\ker_\theta(A)\). If \(x \notin b\ker_\theta(A)\) and \(x \notin H\), then there exists a \(b-\theta\)-open set \(U\) containing \(A\) such that \(x \notin U\). Assume that \(y \in b\Cl_\theta(\{x\}) \cap A\). Thus \(y \in U\) and \(x \notin U\). But this is a contradiction and hence the claim.

Proposition 2.9. (1) For any set \(A \subset X\), \(A \subset A^{\Lambda_b} \subset b\ker_\theta(A) \subset b\Cl_\theta(A)\).

(2) Every \(b-\theta\)-closed set is a \(\Lambda_b\)-set.

(3) Every \(\theta\)-g.\(\Lambda_b\)-set is a \(g.\Lambda_b\)-set.

Proof. (1) Let \(A\) be a subset of \((X, \tau)\). It is shown in [2] that \(A \subset A^{\Lambda_b}\). Now we prove that \(A^{\Lambda_b} \subset b\ker_\theta(A)\). Suppose that \(x\) is not a point of \(b\ker_\theta(A)\). It follows that \(A \subset X \setminus b\Cl_\theta(\{x\}) = U\), say. Since \(b\Cl_\theta(\{x\})\) is \(b-\theta\)-closed by Lemma 2.6, so \(U\) is \(b-\theta\)-open. Hence \(U\) is \(b\)-open since \(U \subset b\Cl(U) \subset b\Cl_\theta(U)\).
Hence there exists a $b$-open set $U$ containing $A$ but not $X$, i.e. $x \notin A^b_{\theta}$. This shows that $A^b_{\theta} \subseteq bKer_{\theta}(A)$. proving $bKer_{\theta}(A) \subseteq bCl_{\theta}(A)$, let $x \in bKer_{\theta}(A)$. Suppose that $x \notin bCl_{\theta}(A)$. Then, there exists a $b$-open set $U$ containing $x$ such that $bCl(U) \cap A = \emptyset$. Since $U$ is a $b$-open subset of $X$, it follows that $bCl_{\theta}(U) = bCl(U)$ [5]. Hence $bCl_{\theta}(U) \cap A = \emptyset$. This implies that $A \subset X \setminus bCl_{\theta}(U)$. Therefore $x \notin X \setminus bCl_{\theta}(U)$, where $X \setminus bCl_{\theta}(U)$ $b$-$\theta$-open set containing $A$ by Lemma 2.6. But this is impossible since $x \in bKer_{\theta}(A)$ by proposition 2.8. Consequently $x \in bCl_{\theta}(A)$.

(2). Let $A$ be $b$-$\theta$-closed set. Since $A^b_{\theta} \subseteq bCl_{\theta}(A)$ and $A = bCl_{\theta}(A)$, then $A^b_{\theta} = A$. Thus $A$ is a $\Lambda_{b}$-set.

(3). Let $A \subset F$, where $F$ is $b$-closed. Then we have $A^b_{\theta} \subseteq bKer_{\theta}(A) \subset F$. This implies that $A$ is a $g.\Lambda_{b}$-set. \(\square\)

**Definition 6.** A topological space $(X, \tau)$ is called $b$-$R_1$ [3] if for $x, y \in X$ with $bCl(\{x\}) \neq bCl(\{y\})$, there exists disjoint $b$-open sets $U$ and $V$ such that $bCl(\{x\}) \subset U$ and $bCl(\{y\}) \subset V$, or equivalently $(X, \tau)$ is $b$-$R_1$ [3] if and only if for each $x \in X$, $bCl(\{x\}) = bCl_{\theta}(\{x\})$.

**Theorem 2.10.** Let $(X, \tau)$ be $b$-$R_1$. A subset $A$ of $(X, \tau)$ is a $g.\Lambda_{b}$-set if and only if $A$ is a $\theta$-$g.\Lambda_{b}$-set.

**Proof.** Sufficiency. It is an immediate consequence of Proposition 2.9 (3).

Necessity. By the fact $X$ is $b$-$R_1$ if and only if $bCl(\{x\}) = bCl_{\theta}(\{x\})$ for each $x \in X$, the proof follows from the observation that $A^b_{\theta} = bKer_{\theta}(A)$. \(\square\)

**Definition 7.** A subset $A$ of a topological space $(X, \tau)$ is called a $\theta$-$\Lambda_{b}$-set if $A = bKer_{\theta}(A)$.

**Definition 8.** A subset $A$ of a topological space $(X, \tau)$ is called $\Lambda_{b}$-$b$-$\theta$-closed if $A = L \cap S$, where $L$ is a $\theta$-$\Lambda_{b}$-set and $S$ is $b$-$\theta$-closed.

**Lemma 2.11.** For a subset $A$ of a topological space $(X, \tau)$ the following conditions are equivalent:

1. $A$ is $\Lambda_{b}$-$b$-$\theta$-closed;
2. $A = L \cap bCl_{\theta}(A)$, where $L$ is a $\theta$-$\Lambda_{b}$-set;
3. $A = bKer_{\theta}(A)$, that is $A$ is $\theta$-$\Lambda_{b}$-set:
Proof. (1) ⇒ (2): Suppose that \( A = L \cap S \), where \( L \) is a \( \theta\)-\( \Lambda_b \)-set and \( S \) is \( b\)-\( \theta \)-closed. Then \( A \subset L \) and \( A \subset b\text{Cl}_\theta(A) \subset S \). Now, we have \( A \subset L \cap b\text{Cl}_\theta(A) \subset L \cap S = A \). This means that \( A = L \cap b\text{Cl}_\theta(A) \).

(2) ⇒ (3): Suppose that \( A = L \cap b\text{Cl}_\theta(A) \) where \( L \) is \( \theta\)-\( \Lambda_b \)-set. We have \( A \subset b\text{Ker}_\theta(A) \subset L \) and \( A \subset b\text{Cl}_\theta(A) \). So \( A = b\text{Ker}_\theta(A) \cap b\text{Cl}_\theta(A) \) and Hence \( A = b\text{Ker}_\theta(A) \) by Proposition 2.9.

(3) ⇒ (1): It is clear that \( b\text{Ker}_\theta(b\text{Ker}_\theta(A)) = b\text{Ker}_\theta(A) \) for any set \( A \). Therefore \( b\text{Ker}_\theta(A) \) is a \( \theta\)-\( \Lambda_b \)-set. Suppose that \( A = b\text{Ker}_\theta(A) \). By Proposition 2.9, \( A = b\text{Ker}_\theta(A) \cap b\text{Cl}_\theta(A) \). Clearly \( A \) is the intersection of a \( \theta\)-\( \Lambda_b \)-set and a \( b\)-\( \theta \)-closed set and hence the result. \( \square \)

**Definition 9.** A subset \( A \) of a topological space \((X, \tau)\) is called quasi \( b\)-\( \theta \)-closed (briefly qbt-closed) if \( b\text{Cl}_\theta(A) \subset U \) whenever \( A \subset U \) and \( U \) is \( b\)-\( \theta \)-open in \((X, \tau)\).

**Lemma 2.12.** A set \( A \) of a topological space \((X, \tau)\) is qbt-closed if and only if \( b\text{Cl}_\theta(A) \subset b\text{Ker}_\theta(A) \).

Proof. Necessity. Let \( x \in X \) such that \( x \notin b\text{Ker}_\theta(A) \). So there exists a \( b\)-\( \theta \)-open subset \( U \) such that \( A \subset U \) with \( x \notin U \). This means that \( x \notin b\text{Cl}_\theta(A) \) since \( A \) is qbt-closed.

Sufficiently. Obvious. \( \square \)

**Theorem 2.13.** For a subset \( A \) of a topological space \((X, \tau)\), the following are equivalent:

(1) \( A \) is \( b\)-\( \theta \)-closed;

(2) \( A \) is qbt-closed and \( \Lambda_b\)-\( b\)-\( \theta \)-closed.

Proof. (1) ⇒ (2): It is obvious that every \( b\)-\( \theta \)-closed set is both qbt-closed and \( \Lambda_b\)-\( b\)-\( \theta \)-closed.

(2) ⇒ (1): Since \( A \) is qbt-closed, then \( b\text{Cl}_\theta(A) \subset b\text{Ker}_\theta(A) \). Now \( A = b\text{Ker}_\theta(A) \cap b\text{Cl}_\theta(A) = b\text{Cl}_\theta(A) \). \( \square \)

**Lemma 2.14.** Let \((X, \tau)\) be a topological space and \( x, y \in X \). Then the following two statement are equivalent:

(1) \( y \in b\text{Ker}_\theta(\{x\}) \);
Hence $bKer Cl \in \{z\} \in y$ such that $bKer Cl Cl \in y$ and $y$ is open set contains the $x$ and $x$ containing similar.

**Lemma 2.15.** The following statements are equivalent for any two points $x$ and $y$ in a topological space $(X, \tau)$:

1. $bKer Cl \{x\} \neq bKer Cl \{y\}$;
2. $bCl Cl \{x\} \neq bCl Cl \{y\}$.

**Proof.** (1) $\Rightarrow$ (2): Let $bKer Cl \{x\} \neq bCl Cl \{y\}$. Then there exists a point $z$ in $X$ such that $z \in bKer Cl \{x\}$ and $z \notin bKer Cl \{y\}$. By $z \in bKer Cl \{x\}$, it follows that $\{x\} \cap bCl Cl \{z\} \neq \emptyset$. This implies $x \in bCl Cl \{z\}$. By $z \notin bKer Cl \{y\}$, we obtain $\{y\} \cap bCl Cl \{z\} = \emptyset$. Since $x \in bCl Cl \{z\}$, $bCl Cl \{x\} \subset bCl Cl \{z\}$ and $\{y\} \cap bCl Cl \{x\} = \emptyset$. Hence it follows that $bCl Cl \{x\} \neq bCl Cl \{y\}$. Now $bKer Cl \{x\} \neq bKer Cl \{y\}$ implies that $bCl Cl \{x\} \neq bCl Cl \{y\}$.

(2) $\Rightarrow$ (1): Let $bCl Cl \{x\} \neq bCl Cl \{y\}$. Then there exists a point $z$ in $X$ such that $z \in bCl Cl \{x\}$ and $z \notin bCl Cl \{y\}$. This means that there exists a $b$-$\theta$-open set containing $z$ and therefore $x$ but not $y$, that is, $y \notin bKer Cl \{x\}$. Hence $bKer Cl \{x\} \neq bKer Cl \{y\}$.

**Definition 10.** A topological space $(X, \tau)$ is a $b$-$\theta$-$R_0$ space if every $b$-$\theta$-open set contains the $b$-$\theta$-closure of each of its singletons.

**Example 2.16.** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then $(X, \tau)$ is a $b$-$\theta$-$R_0$ topological space.

**Theorem 2.17.** A topological space $(X, \tau)$ is $b$-$\theta$-$R_0$ if and only if for $x$ and $y$ in $X$, $bCl Cl \{x\} \neq bCl Cl \{y\}$ implies $bCl Cl \{x\} \cap bCl Cl \{y\} = \emptyset$.

**Proof.** Suppose that $(X, \tau)$ is $b$-$\theta$-$R_0$ and $x, y \in X$ such that $bCl Cl \{x\} \neq bCl Cl \{y\}$. Then, there exists $z \in bCl Cl \{x\}$ such that $z \notin bCl Cl \{y\}$ (or $z \in bCl Cl \{y\}$ such that $z \notin bCl Cl \{x\}$). There exists $V \in BO(X, \tau)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin bCl Cl \{y\}$. Thus $x \in X \setminus bCl Cl \{y\}$, which implies $bCl Cl \{x\} \subset X \setminus bCl Cl \{y\}$ and $bCl Cl \{x\} \cap bCl Cl \{y\} = \emptyset$. The proof for otherwise is similar. Conversely, Let $V$ be $b$-$\theta$-
open and let \( x \in V \). we will show that \( b \text{Cl}_\theta (\{x\}) \subset V \). Let \( y \notin V \) that is, \( y \in X \setminus V \). Then \( x \neq y \) and \( x \notin b \text{Cl}_\theta (\{y\}) \). This shows that \( b \text{Cl}_\theta (\{x\}) \neq b \text{Cl}_\theta (\{y\}) \). By assumption, \( b \text{Cl}_\theta (\{x\}) \cap b \text{Cl}_\theta (\{y\}) = \emptyset \). Hence \( y \notin b \text{Cl}_\theta (\{x\}) \). Therefore, \( b \text{Cl}_\theta (\{x\}) \subset V \). \( \square \)

**Theorem 2.18.** A topological space \((X, \tau)\) is \(b\theta-R_0\) if and only if for any points \( x \) and \( y \) in \( X \), \( b\text{Ker}_\theta (\{x\}) \neq b\text{Ker}_\theta (\{y\}) \) implies \( b\text{Ker}_\theta (\{x\}) \cap b\text{Ker}_\theta (\{y\}) = \emptyset \).

**Proof.** Suppose that \((X, \tau)\) is \(b\theta-R_0\) space. Thus by Lemma 2.15, for any points \( x \) and \( y \) in \( X \) if \( b\text{Ker}_\theta (\{x\}) \neq b\text{Ker}_\theta (\{y\}) \) then \( b\text{Cl}_\theta (\{x\}) \neq b\text{Cl}_\theta (\{y\}) \). Now we prove that \( b\text{Ker}_\theta (\{x\}) \cap b\text{Ker}_\theta (\{y\}) = \emptyset \). Assume that \( z \in b\text{Ker}_\theta (\{x\}) \cap b\text{Ker}_\theta (\{y\}) \). By \( z \in b\text{Ker}_\theta (\{x\}) \) and Lemma 2.14, it follows that \( x \in b\text{Cl}_\theta (\{z\}) \). Since \( x \in b\text{Cl}_\theta (\{x\}) \), by theorem 2.17 \( b\text{Cl}_\theta (\{x\}) = b\text{Cl}_\theta (\{z\}) \). Similarly, we have \( b\text{Cl}_\theta (\{y\}) = b\text{Cl}_\theta (\{z\}) = b\text{Cl}_\theta (\{x\}) \). This is a contradiction. Therefore, we have \( b\text{Ker}_\theta (\{x\}) \cap b\text{Ker}_\theta (\{y\}) = \emptyset \). Conversely, let \((X, \tau)\) be a topological space such that for any points \( x \) and \( y \) in \( X \), \( b\text{Ker}_\theta (\{x\}) \neq b\text{Ker}_\theta (\{y\}) \) implies \( b\text{Ker}_\theta (\{x\}) \cap b\text{Ker}_\theta (\{y\}) = \emptyset \). If \( b\text{Cl}_\theta (\{x\}) \neq b\text{Cl}_\theta (\{y\}) \), then by Lemma 2.15, \( b\text{Ker}_\theta (\{x\}) \neq b\text{Ker}_\theta (\{y\}) \). Hence \( b\text{Ker}_\theta (\{x\}) \cap b\text{Ker}_\theta (\{y\}) = \emptyset \) which implies \( b\text{Cl}_\theta (\{x\}) \cap b\text{Cl}_\theta (\{y\}) = \emptyset \). Because \( z \in b\text{Cl}_\theta (\{x\}) \) implies \( x \in b\text{Ker}_\theta (\{z\}) \) and therefore \( b\text{Ker}_\theta (\{x\}) \cap b\text{Ker}_\theta (\{z\}) \neq \emptyset \). By hypothesis, we have \( b\text{Ker}_\theta (\{x\}) = b\text{Ker}_\theta (\{z\}) \). Then \( z \in b\text{Cl}_\theta (\{x\}) \cap b\text{Cl}_\theta (\{y\}) \) implies that \( b\text{Ker}_\theta (\{x\}) = b\text{Ker}_\theta (\{z\}) = b\text{Ker}_\theta (\{y\}) \). This is a contradiction. Hence \( b\text{Cl}_\theta (\{x\}) \cap b\text{Cl}_\theta (\{y\}) = \emptyset \) and by Theorem 2.17 \((X, \tau)\) is \(b\theta-R_0\) space. \( \square \)

**Theorem 2.19.** For a topological space \((X, \tau)\), the following properties are equivalent:

1. \((X, \tau)\) is a \(b\theta-R_0\) space;
2. For any nonempty sets \( A \), \( G \in B\theta O(X) \) such that \( A \cap G \neq \emptyset \) there exists \( F \in B\theta O(X) \) such that \( A \cap F \neq \emptyset \) and \( F \subset G \);
3. Any \( G \in B\theta O(X) \) such that \( G = \bigcup \{ F \in B\theta C(X) | F \subset G \} \);
4. Any \( F \in B\theta C(X) \), \( F = \bigcap \{ G \in B\theta O(X) | F \subset G \} \);
5. For any \( x \in X \), \( b\text{Cl}_\theta (\{x\}) \subset b\text{Ker}_\theta (\{x\}) \).

**Proof.** (1) \(\Rightarrow\) (2): Let \( A \) be a nonempty set of \( X \) and \( G \in B\theta O(X) \) such that \( A \cap G \neq \emptyset \). There exists \( x \in A \cap G \). Since \( x \in G \in B\theta O(X) \), \( b\text{Cl}_\theta (\{x\}) \subset G \).
Set $F = b\text{Cl}_\theta\{x\}$, then $F \in B\theta C(X)$, $F \subset G$ and $A \cap F \neq \emptyset$.

(2) $\Rightarrow$ (3): Let $G \in B\theta O(X)$, then $G \supset \bigcup\{F \in B\theta C(X) | F \subset G\}$. Let $x$ be any point of $G$. There exists $F \in B\theta C(X)$ such that $x \in F$ and $F \subset G$. Therefore, we have $x \in F \subset \bigcup\{F \in B\theta C(X) | F \subset G\}$ and hence $G = \bigcup\{F \in B\theta C(X) | F \subset G\}$.

(3) $\Rightarrow$ (4): This is obvious.

(4) $\Rightarrow$ (5): Let $x$ be any point of $X$ and $y \notin b\text{Ker}_\theta\{x\}$. There exists $V \in B\theta O(X, x)$ $y \notin V$; hence $b\text{Cl}_\theta\{\{y\}\} \cap V = \emptyset$. By (4) $(\cap\{G \in B\theta O(X) | b\text{Cl}_\theta\{\{y\}\}\subset G\}) \cap V = \emptyset$ and there exists $G \in B\theta O(X)$ such that $x \notin G$ and $b\text{Cl}_\theta\{\{x\}\} \subset G$. Therefore, $b\text{Cl}_\theta\{\{x\}\} \cap G = \emptyset$ and $y \notin b\text{Cl}_\theta\{\{x\}\}$. Consequently, we obtain $b\text{Cl}_\theta\{\{x\}\} \subset b\text{Ker}_\theta\{\{x\}\}$.

(5) $\Rightarrow$ (1): Let $G \subset B\theta O(X, x)$. Let $y \in b\text{Ker}_\theta\{\{x\}\}$, then $x \in b\text{Cl}_\theta\{\{x\}\}$ and $y \in G$. This implies that $b\text{Ker}_\theta\{\{x\}\} \subset G$. Therefore, we obtain $x \in b\text{Cl}_\theta\{\{x\}\} \subset b\text{Ker}_\theta\{\{x\}\} \subset G$. This shows that $(X, \tau)$ is a $b\theta$-R$_0$ space.  

**Corollary 2.20.** For a topological space $(X, \tau)$, the following properties are equivalent:

1. $(X, \tau)$ is a $b\theta$-R$_0$ space;
2. $b\text{Cl}_\theta\{\{x\}\} = b\text{Ker}_\theta\{\{x\}\}$ for all $x \in X$.

**Proof.** (1) $\Rightarrow$ (2): Suppose that $(X, \tau)$ is a $b\theta$-R$_0$ space. By Theorem 2.19, $b\text{Cl}_\theta\{\{x\}\} \subset b\text{Ker}_\theta\{\{x\}\}$ for each $x \in X$. Let $y \in b\text{Ker}_\theta\{\{x\}\}$, then $x \in b\text{Cl}_\theta\{\{y\}\}$ and by theorem 2.17 $b\text{Cl}_\theta\{\{x\}\} = b\text{Cl}_\theta\{\{y\}\}$. Therefore, $y \in b\text{Cl}_\theta\{\{x\}\}$ and hence $b\text{Ker}_\theta\{\{x\}\} \subset b\text{Cl}_\theta\{\{x\}\}$. This shows that $b\text{Cl}_\theta\{\{x\}\} = b\text{Ker}_\theta\{\{x\}\}$.

(2) $\Rightarrow$ (1): This is obvious theorem 2.19.  

**Corollary 2.21.** If for any point $x$ of a $b\theta$-R$_0$ space $(X, \tau)$, $b\text{Cl}_\theta\{\{x\}\} \cap b\text{Ker}_\theta\{\{x\}\} = \{x\}$, then $b\text{Ker}_\theta\{\{x\}\} = \{x\}$.

**Proof.** The proof follows from Theorem 2.19 (5).  

**Theorem 2.22.** For a topological space $(X, \tau)$, the following properties are equivalent:

1. $(X, \tau)$ is a $b\theta$-R$_0$ space;
2. $x \in b\text{Cl}_\theta\{\{y\}\}$ if and only if $y \in b\text{Cl}_\theta\{\{x\}\}$ for any points $x$ and $y$ in $X$.  

Proof. (1) ⇒ (2): Assume that \((X, \tau)\) is \(b\theta\)-R0. Let \(x \in b\text{Cl}_\theta\{\{y\}\}\) and \(A \in B\theta O(X, y)\). Now by hypothesis, \(x \in A\). Therefore, every \(b\theta\)-open set containing \(y\) contains \(x\). Hence \(y \in b\text{Cl}_\theta\{\{x\}\}\).

(2) ⇒ (1): Let \(U \in B\theta O(X, x)\). If \(y \notin U\), then \(x \notin b\text{Cl}_\theta\{\{y\}\}\) and hence \(y \notin b\text{Cl}_\theta\{\{x\}\}\). This implies that \(b\text{Cl}_\theta\{\{x\}\} \subseteq U\). Hence \((X, \tau)\) is \(b\theta\)-R0.

**Theorem 2.23.** For a topological space \((X, \tau)\), the following properties are equivalent:

1. \((X, \tau)\) is a \(b\theta\)-R0 space;
2. If \(F\) is \(b\theta\)-closed subset of \(X\), then \(F = b\text{Ker}_\theta(F)\);
3. If \(F\) is \(b\theta\)-closed subset of \(X\) and \(x \in F\), then \(b\text{Ker}_\theta(\{x\}) \subseteq F\);
4. If \(x \in X\), then \(b\text{Ker}_\theta(\{x\}) \subseteq b\text{Cl}_\theta(\{x\})\).

Proof. (1) ⇒ (2): Let \(F\) be \(b\theta\)-closed subset of \(X\) and \(x \notin F\). Thus \(X \setminus F \in B\theta O(X, x)\). Since \((X, \tau)\) is \(b\theta\)-R0, \(b\text{Cl}_\theta(\{x\}) \subseteq X \setminus F\). Thus \(b\text{Cl}_\theta(\{x\}) \cap F = \emptyset\) and hence \(x \notin b\text{Ker}_\theta(F)\). Therefore, \(b\text{Ker}_\theta(F) = F\).

(2) ⇒ (3): In general, \(A \subseteq B\) implies \(b\text{Ker}_\theta(A) \subseteq b\text{Ker}_\theta(B)\). Therefore, it follows from (2) that \(b\text{Ker}_\theta(\{x\}) \subseteq b\text{Ker}_\theta(F) = F\).

(3) ⇒ (4): Since \(x \in b\text{Cl}_\theta(\{x\})\) and \(b\text{Cl}_\theta(\{x\})\) is \(b\theta\)-closed, by (3) \(b\text{Ker}_\theta(\{x\}) \subseteq b\text{Cl}_\theta(\{x\})\).

(4) ⇒ (1): We show implication by using Theorem 2.22. Let \(x \in b\text{Cl}_\theta(\{y\})\). Then \(y \in b\text{Ker}_\theta(\{x\})\). Since \(x \in b\text{Cl}_\theta(\{x\})\) and \(b\text{Cl}_\theta(\{x\})\) is \(b\theta\)-closed, by (4) we obtain \(y \in b\text{Ker}_\theta(\{x\}) \subseteq b\text{Cl}_\theta(\{x\})\). Therefore, \(x \in b\text{Cl}_\theta(\{x\})\) implies \(y \in b\text{Cl}_\theta(\{x\})\). The converse is obvious and \((X, \tau)\) is \(b\theta\)-R0.

**Definition 11.** A filterbase \(F\) is called \(b\theta\)-convergent to a point \(x\) in \(X\) if for any \(U \in B\theta O(X, x)\), there exists \(B \in F\) such that \(B\) is a subset of \(U\).

**Lemma 2.24.** Let \((X, \tau)\) be a topological space and let \(x\) and \(y\) be any two points in \(X\) such that every net in \(X\) \(b\theta\)-converging to \(y\) \(b\theta\)-converges to \(x\). Then \(x \in b\text{Cl}_\theta(\{y\})\).

Proof. Suppose that \(x_n = y\) for each \(n \in N\). Then \(\{x_n\}_{n \in N}\) is a net in \(b\text{Cl}_\theta(\{y\})\). Since \(\{x_n\}_{n \in N}\) \(b\theta\)-converges to \(y\), then \(\{x_n\}_{n \in N}\) \(b\theta\)-converges to \(x\) and this implies that \(x \in b\text{Cl}_\theta(\{y\})\).
Theorem 2.25. For a topological space \((X, \tau)\), the following statements are equivalent:

(1) \((X, \tau)\) is a \(b\theta\)-\(R_0\) space;

(2) If \(x, y \in X\), then \(y \in b\text{Cl}_0(\{x\})\) if and only if every net in \(X\) \(b\theta\)-converging to \(y\) \(b\theta\)-converges to \(x\).

Proof. (1) \(\Rightarrow\) (2): Let \(x, y \in X\) such that \(y \in b\text{Cl}_0(\{x\})\). Suppose that \(\{x_\alpha\}_{\alpha \in \mathcal{N}}\) be a net in \(X\) such that \(\{x_\alpha\}_{\alpha \in \mathcal{N}}\) \(b\theta\)-converges to \(y\). Since \(y \in b\text{Cl}_0(\{x\})\), \(b\text{Cl}_0(\{x\}) = b\text{Cl}_0(\{y\})\). Therefore \(x \in b\text{Cl}_0(\{y\})\). This means that \(\{x_\alpha\}_{\alpha \in \mathcal{N}}\) \(b\theta\)-converges to \(x\). Conversely, let \(x, y \in X\) such that every net in \(X\) \(b\theta\)-converging to \(y\) \(b\theta\)-converges to \(x\). Then \(x \in b\text{Cl}_0(\{y\})\). Hence \(y \in b\text{Cl}_0(\{x\})\).

(2) \(\Rightarrow\) (1): Assume that \(x\) and \(y\) are any two points of \(X\) such that \(b\text{Cl}_0(\{x\}) \cap b\text{Cl}_0(\{y\}) \neq \emptyset\). Let \(z \in b\text{Cl}_0(\{x\}) \cap b\text{Cl}_0(\{y\})\). So there exists a net \(\{x_\alpha\}_{\alpha \in \mathcal{N}}\) in \(b\text{Cl}_0(\{x\})\) such that \(\{x_\alpha\}_{\alpha \in \mathcal{N}}\) \(b\theta\)-converges to \(z\). Since \(z \in b\text{Cl}_0(\{y\})\), \(\{x_\alpha\}_{\alpha \in \mathcal{N}} \neq b\text{Cl}_0(\{y\})\). Similarly we can obtain \(x \in b\text{Cl}_0(\{y\})\). Therefore \(b\text{Cl}_0(\{x\}) = b\text{Cl}_0(\{y\})\) and hence \((X, \tau)\) is \(b\theta\)-\(R_0\).

3. On \(b\theta\)-\(R_1\) Spaces

Definition 12. A topological space \((X, \tau)\) is said to be \(b\theta\)-\(R_1\) if for \(x, y\) in \(X\) with \(b\text{Cl}_0(\{x\}) \neq b\text{Cl}_0(\{y\})\), there exist disjoint \(b\theta\)-open sets \(U\) and \(V\) such that \(b\text{Cl}_0(\{x\}) \subset U\) and \(b\text{Cl}_0(\{y\}) \subset V\).

Proposition 3.1. If \((X, \tau)\) is \(b\theta\)-\(R_1\), then it is \(b\theta\)-\(R_0\).

Proof. Let \(U \in \theta\text{B}(X, x)\). If \(y \notin U\), then since \(x \notin b\text{Cl}_0(\{y\})\), \(b\text{Cl}_0(\{x\}) \neq b\text{Cl}_0(\{y\})\). Hence there exists a \(b\theta\)-open \(V\) such that \(b\text{Cl}_0(\{y\}) \subset V\) and \(x \notin V\). Thus \(b\text{Cl}_0(\{x\}) \subset U\). Therefore \((X, \tau)\) is \(b\theta\)-\(R_0\).

Theorem 3.2. A topological space \((X, \tau)\) is \(b\theta\)-\(R_1\) if and only if for \(x, y \in X\), \(b\text{Ker}_0(\{x\}) \neq b\text{Ker}_0(\{y\})\), there exists disjoint \(b\theta\)-open sets \(U\) and \(V\) such that \(b\text{Cl}_0(\{x\}) \subset U\) and \(b\text{Cl}_0(\{y\}) \subset V\).
Proof. It follows from Lemma 2.15.

Definition 13. A topological space $(X, \tau)$ is said to be:

(1) $b$-$\theta$-$T_1$ is for each pair of distinct points $x$ and $y$ of $X$, there exist $b$-$\theta$-open sets $U$ and $V$ of $X$ such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$.

(2) $b$-$\theta$-$T_2$ if for each pair of disjoint points $x$ and $y$ in $X$, there exist disjoint $b$-$\theta$-open sets $U$ and $V$ in $X$ such that $x \in U$ and $y \in V$.

Theorem 3.3. The following properties are equivalent:

(1) $(X, \tau)$ is $b$-$\theta$-$T_2$,

(2) $(X, \tau)$ is $b$-$\theta$-$R_1$ and $b$-$\theta$-$T_1$,

(3) $(X, \tau)$ is $b$-$\theta$-$R_1$ and $b$-$\theta$-$T_0$.

Proof. (1) $\Rightarrow$ (2): Since $(X, \tau)$ is $b$-$\theta$-$T_2$, then it is $b$-$\theta$-$T_1$. If $x, y \in X$ such that $b\text{Cl}_\theta(\{x\}) \neq b\text{Cl}_\theta(\{y\})$, then $x \neq y$ and there exists disjoint $b$-$\theta$-open sets $U$ and $V$ such that $x \in U$ and $y \in V$ and $b\text{Cl}_\theta(\{x\}) = \{x\} \subset U$ and $b\text{Cl}_\theta(\{y\}) = \{y\} \subset V$. Hence $(X, \tau)$ is $b$-$\theta$-$R_1$.

(2) $\Rightarrow$ (3): Since $(X, \tau)$ is $b$-$\theta$-$T_1$, then $(X, \tau)$ is $b$-$\theta$-$T_0$.

(3) $\Rightarrow$ (1): Since $(X, \tau)$ is $b$-$\theta$-$R_1$, and $b$-$\theta$-$T_1$, then $(X, \tau)$ is $b$-$\theta$-$R_0$ and $b$-$\theta$-$T_0$. Let $x, y \in X$ such that $x \neq y$. Since $b\text{Cl}_\theta(\{x\}) = \{x\} \neq \{y\} = b\text{Cl}_\theta(\{y\})$, then there exists disjoint $b$-$\theta$-open sets $U$ and $V$ such that $x \in U$ and $y \in V$. Hence, $(X, \tau)$ is $b$-$\theta$-$T_2$.

Theorem 3.4. The following properties are equivalent:

(1) $(X, \tau)$ is $b$-$\theta$-$R_1$,

(2) for each $x, y \in X$ one of the following holds:

(a) If $U$ is $b$-$\theta$-open, then $x \in U$ if and only if $y \in U$.

(b) there exists disjoint $b$-$\theta$-open sets $U$ and $V$ such that $x \in U$ and $y \in V$, and

(3) If $x, y \in X$ such that $b\text{Cl}_\theta(\{x\}) \notin b\text{Cl}_\theta(\{y\})$, then there exists $b$-$\theta$-closed sets $F_1$ and $F_2$ such that $x \in F_1$, $y \notin F_2$, $y \in F_1$, $x \notin F_2$, and $X = F_1 \cup F_2$.

Proof. (1) $\Rightarrow$ (2): Let $x, y \in X$. Then $b\text{Cl}_\theta(\{x\}) = b\text{Cl}_\theta(\{y\})$ or $b\text{Cl}_\theta(\{x\}) \neq b\text{Cl}_\theta(\{y\})$. If $b\text{Cl}_\theta(\{x\}) = b\text{Cl}_\theta(\{y\})$ and $U$ is $b$-$\theta$-open, then $x \in U$ implies
y \in b\mathrm{Cl}_\theta(\{x\}) \subset U \text{ and } y \in U \text{ implies } x \in b\mathrm{Cl}_\theta(\{y\}) \subset U. \text{ Thus consider the case that } b\mathrm{Cl}_\theta(\{x\}) \neq b\mathrm{Cl}_\theta(\{y\}). \text{ Then there exists disjoint } b\theta\text{-open sets } U \text{ and } V \text{ such that } x \in b\mathrm{Cl}_\theta(\{x\}) \subset U \text{ and } y \in b\mathrm{Cl}_\theta(\{y\}) \subset V. 

\text{(2) } \Rightarrow \text{ (3): Let } x, y \in X \text{ such that } b\mathrm{Cl}_\theta(\{x\}) \neq b\mathrm{Cl}_\theta(\{y\}). \text{ Then } x \notin b\mathrm{Cl}_\theta(\{y\}) \text{ or } y \notin b\mathrm{Cl}_\theta(\{x\}), \text{ say } x \notin b\mathrm{Cl}_\theta(\{y\}). \text{ Then there exists a } b\theta\text{-open set } A \text{ such that } x \in A \text{ and } y \notin A, \text{ which implies there exists disjoint } b\theta\text{-open sets } U \text{ and } V \text{ such that } x \in U \text{ and } y \in V. \text{ Then } F_1 = X \setminus V \text{ and } F_2 = X \setminus U \text{ are } b\theta\text{-closed sets such that } x \in F_1, y \notin F_1, y \in F_2, x \notin F_2, X = F_1 \cup F_2. 

\text{(3) } \Rightarrow \text{ (1): Let } x \text{ and } y \text{ be any points in } X \text{ with } b\mathrm{Cl}_\theta(\{x\}) \neq b\mathrm{Cl}_\theta(\{y\}). \text{ Then } b\mathrm{Cl}_\theta(\{x\}) \cap b\mathrm{Cl}_\theta(\{y\}) = \emptyset. \text{ In fact, if } z \in b\mathrm{Cl}_\theta(\{x\}) \cap b\mathrm{Cl}_\theta(\{y\}), \text{ then } b\mathrm{Cl}_\theta(\{z\}) \neq b\mathrm{Cl}_\theta(\{x\}) \text{ or } b\mathrm{Cl}_\theta(\{z\}) \neq b\mathrm{Cl}_\theta(\{y\}). \text{ In case, } b\mathrm{Cl}_\theta(\{z\}) \neq b\mathrm{Cl}_\theta(\{x\}), \text{ by (iii), there exists a } b\theta\text{-closed set } F \text{ such that } x \in F \text{ and } z \notin F. \text{ Then } z \in b\mathrm{Cl}_\theta(\{x\}) \subset F. \text{ This contradicts that } z \notin F. \text{ In case, } b\mathrm{Cl}_\theta(\{z\}) \neq b\mathrm{Cl}_\theta(\{y\}), \text{ similarly, this leads to the contradiction. Hence } b\mathrm{Cl}_\theta(\{x\}) \cap b\mathrm{Cl}_\theta(\{y\}) = \emptyset. \text{ Let } U \text{ be } b\theta\text{-open and let } x \in U. \text{ Then } b\mathrm{Cl}_\theta(\{x\}) \subset U, \text{ for suppose not. Let } y \in b\mathrm{Cl}_\theta(\{x\}) \cap (X \setminus U). \text{ Then } b\mathrm{Cl}_\theta(\{x\}) \neq b\mathrm{Cl}_\theta(\{y\}) \text{ and there exists } b\theta\text{-closed sets } F_1 \text{ and } F_2 \text{ such that } x \in F_1, y \notin F_1, y \in F_2, x \notin F_2, X = F_1 \cup F_2. \text{ Then } y \in X \setminus F_1, \text{ which is } b\theta\text{-open, and } x \notin X \setminus F_1 \text{ which is a contradiction. Hence } (X, \tau) \text{ is } b\theta\text{-R}_0. \text{ Let } a, b \in X \text{ such that } b\mathrm{Cl}_\theta(\{a\}) \neq b\mathrm{Cl}_\theta(\{b\}). \text{ Then there exists } b\theta\text{-closed sets } A_1 \text{ and } A_2 \text{ such that } a \in A_1, b \notin A_1, a \notin A_2, \text{ and } X = A_1 \cup A_2. \text{ Thus } a \in X \setminus A_2 \text{ and } b \in X \setminus A_1, \text{ which are } b\theta\text{-open, which implies } b\mathrm{Cl}_\theta(\{a\}) \subset A_1 \setminus A_2 \text{ and } b\mathrm{Cl}_\theta(\{b\}) \subset A_2 \setminus A_1. \text{ Hence } (X, \tau) \text{ is } b\theta\text{-R}_1. 

\textbf{Theorem 3.5.} \text{ A topological space } (X, \tau) \text{ is } b\theta\text{-T}_2 \text{ if and only if for } x, y \in X \text{ such that } x \neq y, \text{ there exists } b\theta\text{-closed sets } F_1 \text{ and } F_2 \text{ such that } x \in F_1, y \notin F_1, y \in F_2, x \notin F_2, \text{ and } X = F_1 \cup F_2.

\text{Proof.} \text{ The straightforward proof is omitted. } \square

\textbf{Remark 3.6.} \text{ If } \{x_\lambda\}_{\lambda \in A} \text{ is a net in } (X, \tau), \beta\lim(\{x_\lambda\}_{\lambda \in A}) = \{x \in X : \{x_\lambda\}_{\lambda \in A} \text{ } b\theta\text{-converges to } x\}

\textbf{Theorem 3.7.} \text{ For a topological space } (X, \tau), \text{ the following properties are equivalent}

\begin{enumerate}
\item \text{} (X, \tau) \text{ is } b\theta\text{-R}_1;
\item \text{} for } x, y \in X, b\mathrm{Cl}_\theta(\{x\}) = b\mathrm{Cl}_\theta(\{y\}), \text{ whenever there exists a net } \{x_\lambda\}_{\lambda \in A} \text{ such that } x, y \in \beta\lim(\{x_\lambda\}_{\lambda \in A});
\end{enumerate}
(3) \((X, \tau)\) is \(b\theta\)-\(R_0\), and for every \(b\theta\)-convergent net \(\{x_\lambda\}_{\lambda \in A} \) in \(X\), 
\[ \beta \lim(\{x_\lambda\}_{\lambda \in A}) = b \text{Cl}_\theta(\{x\}) \] for some \(x \in X\).

**Proof.** (1) \(\Rightarrow\) (2): Let \(x, y \in X\), such that there exists a net \(\{x_\lambda\}_{\lambda \in A} \) in \(X\) such that \(x, y \in \beta \lim(\{x_\lambda\}_{\lambda \in A})\). Then (a) if \(U\) is \(b\theta\)-open, then \(x \in U\) if and only if \(y \in U\) or (b) there exists disjoint \(b\theta\)-open sets \(U\) and \(V\) such that \(x \in U\) and \(y \in V\). Since \(x, y \in \beta \lim(\{x_\lambda\}_{\lambda \in A})\), then (1) is satisfied, which implies 
\[ b \text{Cl}_\theta(\{x\}) = b \text{Cl}_\theta(\{y\}) \].

(2) \(\Rightarrow\) (3): Let \(U \in B\theta O(X, x)\). Let \(y \notin U\). For each \(n \in \mathbb{N}\) let \(x_n = x\). Then \(\{x_n\}_{n \in \mathbb{N}}\) \(b\theta\)-converges to \(x\) and since \(b \text{Cl}_\theta(\{x\}) \neq b \text{Cl}_\theta(\{y\})\), that \(y \in A\) and \(x \notin A\). Thus, \(y \notin b \text{Cl}_\theta(\{x\})\) and \(b \text{Cl}_\theta(\{y\}) \subset U\). Hence \((X, \tau)\) is \(b\theta\)-\(R_0\). Let \(\{x_\lambda\}_{\lambda \in A} \) be \(b\theta\)-convergent net in \(X\). Let \(x \in X\) such that \(\{x_\lambda\}_{\lambda \in A} \) \(b\theta\)-converges to \(y\). If \(y \notin b \text{Cl}_\theta(\{x\})\), then \(\{x_\lambda\}_{\lambda \in A} \) \(b\theta\)-converges to \(y\), which implies \(b \text{Cl}_\theta(\{x\}) \subset \beta \lim(\{x_\lambda\}_{\lambda \in A})\) and if \(y \in \beta \lim(\{x_\lambda\}_{\lambda \in A})\), then \(x, y \in \beta \lim(\{x_\lambda\}_{\lambda \in A})\), which implies \(y \in b \text{Cl}_\theta(\{y\}) = b \text{Cl}_\theta(\{x\})\). Hence \(\beta \lim(\{x_\lambda\}_{\lambda \in A}) = b \text{Cl}_\theta(\{x\})\).

(3) \(\Rightarrow\) (1): Assume that \((X, \tau)\) is not \(b\theta\)-\(R_1\). Then there exists \(x, y \in X\) such that \(b \text{Cl}_\theta(\{x\}) \neq b \text{Cl}_\theta(\{y\})\) and every \(b\theta\)-open set containing \(b \text{Cl}_\theta(\{x\})\) intersects every \(b\theta\)-open set containing \(b \text{Cl}_\theta(\{y\})\). Since \((X, \tau)\) is \(b\theta\)-\(R_0\), then every \(b\theta\)-open set containing \(x\) contains \(b \text{Cl}_\theta(\{x\})\) and every \(b\theta\)-open set containing \(y\) contains \(b \text{Cl}_\theta(\{y\})\), which implies that every \(b\theta\)-open set containing \(x\) intersects every \(b\theta\)-open set containing \(y\). Let \(D_x = \{U \subset X : U \in B\theta O(X, x)\}\). Let \(\geq_x\) be the binary relation on \(D_x\) defined by 
\[ U_1 \geq_x U_2 \text{ if and only if } U_1 \subset U_2. \]
Then, clearly \((D_x, \geq_x)\) is a directed set. Let \(D_y = \{U \subset X : U \in B\theta O(X, y)\}\) and \(\geq_y\) be binary relation on \(D_y\) defined by \(U_1 \geq_y U_2\) if and only if \(U_1 \subset U_2\). Then, \((D_y, \geq_y)\) is also a directed set. Let \(D = \{(U_1, U_2) : U_1 \in D_x \text{ and } U_2 \in D_y\}\) and let \(\geq\) be the binary relation on \(D\) defined by \((U_1, U_2) \geq (V_1, V_2)\) if and only if \(U_1 \geq_x V_1 \text{ and } U_2 \geq_y V_2\). Then, \((D, \geq)\) is directed set. For each \((U_1, U_2) \in D\), let \(x(U_1, U_2) \in (U_1, U_2)\). Then \(\{x(U_1, U_2)\} \subset D\) is a net in \(X\) that \(b\theta\)-converges to both \(x\) and \(y\). Thus, there exists \(z \in X\) such that 
\[ \beta \lim(\{x(U_1, U_2)\} \subset D) = b \text{Cl}_\theta(\{z\}) \], which implies \(x, y \in b \text{Cl}_\theta(\{z\})\). Since \(\{b \text{Cl}_\theta(\{w\}) : w \in X\}\) is a decomposition of \(X\), then \(b \text{Cl}_\theta(\{x\}) = b \text{Cl}_\theta(\{z\}) = b \text{Cl}_\theta(\{y\})\), which is a contradiction. Hence \((X, \tau)\) is \(b\theta\)-\(R_1\). \(\square\)

**Theorem 3.8.** A topological space \((X, \tau)\) is \(b\theta\)-\(T_2\) if and only if every \(b\theta\)-convergent net in \(X\) \(b\theta\)-converges to a unique point.
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Proof. The proof follows from Theorem 3.7 and 3.3.

Theorem 3.9. For a topological space \((X, \tau)\), the following properties are equivalent

1. \((X, \tau)\) is \(b\)-\(\theta\)-\(R_1\),

2. for each pair \(x, y \in X\), \(b \text{Cl}_\theta(\{x\}) \neq b \text{Cl}_\theta(\{y\})\), there exists a \(b\)-\(\theta\)-regular set \(V\) such that \(y \notin V\);

3. for each pair \(x, y \in X\) \(b \text{Cl}_\theta(\{x\}) \neq b \text{Cl}_\theta(\{y\})\), there exists a \(b\)-\(\theta\)-continuous function \(f : (X, \tau) \to [0, 1]\) such that \(f(x) = 0\) and \(f(y) = 1\).

Proof. (1) \(\Rightarrow\) (2): Let \(x, y \in X\) such that \(b \text{Cl}_\theta(\{x\}) \notin b \text{Cl}_\theta(\{y\})\). Then there exists disjoint \(b\)-\(\theta\)-open sets \(U\) and \(W\) such that \(b \text{Cl}_\theta(\{x\}) \subset U\) and \(b \text{Cl}_\theta(\{y\}) \subset W\) and \(V = b \text{Cl}_\theta(\{U\})\) is \(b\)-\(\theta\)-regular such that \(x \in V\) and \(y \notin V\).

(2) \(\Rightarrow\) (3): Let \(x, y \in X\) such that \(b \text{Cl}_\theta(\{x\}) \neq b \text{Cl}_\theta(\{y\})\). Let \(V\) be a \(b\)-\(\theta\)-open, \(b\)-\(\theta\)-closed set of \(X\) such that \(x \in V\) and \(y \notin V\). Thus, the function \(f : (X, \tau) \to [0, 1]\) defined by \(f(z) = 0\) if \(z \in V\) and \(f(z) = 1\) if \(z \notin V\) satisfies the desired properties.

(3) \(\Rightarrow\) (1): Let \(x, y \in X\) such that \(b \text{Cl}_\theta(\{x\}) \neq b \text{Cl}_\theta(\{y\})\). Let \(f : (X, \tau) \to [0, 1]\) such that \(f\) is \(b\)-\(\theta\)-continuous, \(f(x) = 0\) and \(f(y) = 1\). Then \(U = f^{-1}([0, 0.5])\) and \(V = f^{-1}((0.5, 1])\) are disjoint such that \(b\)-\(\theta\)-open, \(b\)-\(\theta\)-closed set of \(X\) and \(b \text{Cl}_\theta(\{x\}) \subset U\) and \(b \text{Cl}_\theta(\{y\}) \subset V\).

Theorem 3.10. For a topological space \((X, \tau)\), the following properties are equivalent

1. \((X, \tau)\) is \(b\)-\(\theta\)-\(R_1\),

2. for each pair \(x, y \in X\), \(x \neq y\), there exists a \(b\)-\(\theta\)-open, \(b\)-\(\theta\)-closed set \(V\) such that \(y \notin V\);

3. for each pair \(x, y \in X\), \(x \neq y\), there exists a \(b\)-\(\theta\)-continuous function \(f : (X, \tau) \to [0, 1]\) such that \(f(x) = 0\) and \(f(y) = 1\).

Proof. The proof is similar to that of Theorem 3.9.

References


