International Journal of Pure and Applied Mathematics
Volume 103 No. 2 2015, 295-310
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v103i2.14

ijpam.eu

ON θ-b-GENERALIZED CLOSED SETS IN TOPOLOGY

R. Balaji[1 §], G. Shanmugam[2], N. Rajesh[3]

[1]Department of Mathematics
Agni College of Technology
Chennai, 600130, TamilNadu, INDIA

[2]Department of Mathematics
Jeppiaar Engineering College
Chennai 600119, TamilNadu, INDIA

[3]Department of Mathematics
Rajah Serfoji Govt. College
Thanjavur, 613005, Tamilnadu, INDIA

Abstract: The aim of this paper is to introduce and study a new class of generalized closed sets called θ-b-generalized closed sets in topological space.

AMS Subject Classification: 54C10, 54D10
Key Words: b-open sets, θ-b-generalized closed sets, Λ_b-sets, θ-b-closure

1. Introduction and Preliminaries

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the variously modified forms of continuity, separation axioms etc. by utilizing generalized open sets. For a subset A of a topological space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A and the interior of A, respectively. A set A is called $b-$open [1](=γ-open

Received: May 8, 2015

§Correspondence author

[4]) if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$. The complement of a b-open sets is called a b-closed set. The intersection of all b-closed sets of (X, τ) containing A is called the b-closure [1] of A and is denoted by $b\,\mathrm{Cl}(A)$. A set A is b-closed if and only if A $= b\,\mathrm{Cl}(A)$. The family of all b-open sets of (X, τ) is denoted by $BO(X, \tau)$. For each $x \in X$, the family of all b-open sets of (X, τ) containing a point x is denoted by $BO(X, x)$. The b-interior of A is the union of all b-open sets contained in A and is denoted by $b\,\mathrm{Int}(A)$. A subset A is called b-regular [5] if it is both b-open and b-closed. The b-θ-clouser [5], denoted by $b\,\mathrm{Cl}_\theta(A)$, is the set of all $x \in X$ such that $b\mathrm{Cl}(U) \cap A \neq \varnothing$ for every $U \in BO(X, x)$. A subset A is called b-θ-closed [5] if $A = b\,\mathrm{Cl}_\theta(A)$. The set $\{x \in X | b\mathrm{Cl}(U) \subset A$ for some $U \in BO(X, x)\}$ is called the b-θ-interior [5] of A and is denoted by $b\,\mathrm{Int}_\theta(A)$. By [5], it is proved that, for a subset A, $b\,\mathrm{Cl}_\theta(A)$ is the intersection of all b-θ-closed sets containing A. The aim of this paper is to introduce and study a new class of generalized closed sets called θ- b-generalized closed sets in topological space.

2. On θ-b-Generalized Closed Sets

Definition 1. A subset A of a topological space (X, τ) is called θ-b-generalized closed (briefly θ-bg-closed) if $b\,\mathrm{Cl}_\theta(A) \subset U$ whenever $A \subset U$ and $U \in BO(X, \tau)$.

The complement of a θ-b-generalized closed set is called a θ-b-generalized open set (briefly θ-bg-open).

Lemma 2.1. *Every b-θ-closed set is θ-bg-closed but not conversely.*

Proof. Let $A \subset X$ be b-θ-closed. Then $A = b\,\mathrm{Cl}_\theta(A)$. Let $A \subset U$ and $U \in BO(X, \tau)$. It follows that $b\,\mathrm{Cl}_\theta(A) \subset U$. This means that A is θ-bg-closed. □

Example 2.2. Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, \{a\}, X\}$. Then the set $\{b, c\}$ is θ-bg-closed but not b-θ-closed in (X, τ).

Theorem 2.3. *A set $A \subset (X, \tau)$ is θ-bg-open if and only if $F \subset b\,\mathrm{Int}_\theta(A)$ whenever F is b-closed in X and $F \subset A$.*

Proof. Let A be θ-bg-open and $F \subset A$, where F is b-closed. It is obvious that $X \backslash A$ is contained in $X \backslash F$. This implies that $b\,\mathrm{Cl}_\theta(X \backslash A) \subset X \backslash F$. Hence

$b\,\mathrm{Cl}_\theta(X\backslash A) = X\backslash(b\,\mathrm{Int}_\theta(A)) \subset X\backslash F$, that is, $F \subset b\,\mathrm{Int}_\theta(A)$. Conversely, if F is a b-closed set with $F \subset b\,\mathrm{Int}_\theta(A)$ whenever $F \subset A$, then it follows that $X\backslash A \subset X\backslash F$ and $X\backslash(b\,\mathrm{Int}_\theta(A)) \subset X\backslash F$, that is, $b\,\mathrm{Cl}_\theta(X\backslash A) \subset X\backslash F$. Therefore, $X\backslash A$ is θ-bg-closed and hence A is θ-bg-open. □

Lemma 2.4. *Let A be θ-bg-closed subset of (X, τ). Then:* (1) $b\,\mathrm{Cl}_\theta(A)\backslash A$ *does not contain a nonempty b-closed set.*

(2) $b\,\mathrm{Cl}_\theta(A)\backslash A$ *is θ-bg-open.*

Proof. (1). Let F be a b-closed set such that $F \subset b\,\mathrm{Cl}_\theta(A)\backslash A$. Since F^c is b-open and $A \subset X\backslash F$, $b\,\mathrm{Cl}_\theta(A) \subset X\backslash F$, that is $F \subset X\backslash(b\,\mathrm{Cl}_\theta(A))$. This implies that $F \subset X\backslash(b\,\mathrm{Cl}_\theta(A)) \cap b\,\mathrm{Cl}_\theta(A) = \varnothing$.

(2). If A is θ-bg-closed and F is a b-closed set such that $F \subset b\,\mathrm{Cl}_\theta(A)\backslash A$, then by (1), F is empty and therefore $F \subset b\,\mathrm{Int}_\theta(b\,\mathrm{Cl}_\theta(A)\backslash A)$. By Theorem 2.3, $b\,\mathrm{Cl}_\theta(A)\backslash A$ is θ-bg-open. □

Definition 2. A topological space (X, τ) is called b-T_1 **[2]** if for distinct points $x, y \in X$, there exists a b-open set containing x but not y and a b-open set containing y but not x,or equivalently (X, τ) is b-T_1 if and only if every singleton is b-closed **[2]**.

Theorem 2.5. *A topological space (X, τ) is b-T_1 if and only if every θ-bg-closed set is b-θ-closed.*

Proof. Let $A \subset X$ be θ-bg-closed and $x \in b\,\mathrm{Cl}_\theta(A)$. Since X is b-T_1, $\{x\}$ is b-closed and thus by Lemma 2.4, $x \notin b\,\mathrm{Cl}_\theta(A)\backslash A$. Since $x \in b\,\mathrm{Cl}_\theta(A)$, then $x \in A$. This shows that $b\,\mathrm{Cl}_\theta(A) \subset A$ or equivalently A is b-θ-closed. Conversely, let $x \in X$. Assume that $\{x\}$ is not b-closed. Then $X\backslash\{x\}$ is not b-open but is θ-bg-closed since the only b-open superset of $X\backslash\{x\}$ is X. By hypothesis, $X\backslash\{x\}$ is b-θ-closed and thus $\{x\}$ is b-θ-open. Since a singleton is b-θ-open if and only if it is b-regular, $\{x\}$ is b-regular. □

Definition 3. A subset A of a topological space (X, τ) is called a Λ_b-set **[2]** if $A = A^{\Lambda_b}$, where $A^{\Lambda_b} = \cap\{U | A \subset U, U \in BO(X, \tau)\}$.

Definition 4. A subset A of a topological space (X, τ) is called a generalized Λ_b-set (briefly $g.\Lambda_b$-set)**[2]** if $A^{\Lambda_b} \subset F$ whenever $A \subset F$ and F is a b-closed set of X.

Definition 5. (1) For a subset A of a topological space (X, τ), we define $b\ker_\theta(A)$ as follows $bKer_\theta(A) = \{x \in X | b\,\mathrm{Cl}_\theta(\{x\}) \cap A \neq \varnothing\}$.(2) A subset A of (X, τ) is called θ-generalized Λ_b-set (briefly θ-$g.\Lambda_b$-set) if $bKer_\theta(A) \subset F$, whenever $A \subset F$ and F is a b-closed set of (X, τ).

Lemma 2.6. [5] *For any subset A of a topological space (X, τ), $b\,\mathrm{Cl}_\theta(A)$ is b-θ-closed.*

Lemma 2.7. *If A is a θ-bg-closed set of a topological space (X, τ) such that $A \subset B \subset b\,\mathrm{Cl}_\theta(A)$, then B is also a θ-bg-closed set of (X, τ).*

Proof. Let U be a b-open set of (X, τ) such that $B \subset U$. Then $A \subset U$. Since A is θ-bg-closed, $b\,\mathrm{Cl}_\theta(A) \subset U$. By using Lemma2.6, $b\,\mathrm{Cl}_\theta(B) \subset b\,\mathrm{Cl}_\theta(b\,\mathrm{Cl}_\theta(A)) = b\,\mathrm{Cl}_\theta(A) \subset U$. Therefore, B is also a θ-bg-closed set of (X, τ). □

Proposition 2.8. *Let (X, τ) be a topological space and A be a subset of X. Then, $bKer_\theta(A) = \cap\{U | A \subset U, U$ is b-θ-open$\}$ for any subset $A \subset (X, \tau)$.*

Proof. Let $H = \cap\{U | A \subset U, U$ is b-θ-open$\}$ and $x \in H$. Suppose that $x \notin bKer_\theta(A)$ which means $b\,\mathrm{Cl}_\theta(\{x\}) \cap A = \varnothing$. Hence $x \notin X \backslash b\,\mathrm{Cl}_\theta(\{x\})$, where $X \backslash b\,\mathrm{Cl}_\theta(\{x\})$ is b-θ-open set containing A by Lemma 2.6. But this is impossible since $x \in H$. Consequently $x \in bKer_\theta(A)$. If $x \in bKer_\theta(A)$ and $x \notin H$, then there exists a b-θ-open set U containing A such that $x \notin U$. Assume that $y \in b\,\mathrm{Cl}_\theta(\{x\}) \cap A$. Thus $y \in U$ and $x \notin U$. But this is a contradiction and hence the claim. □

Proposition 2.9. (1) *For any set $A \subset X$, $A \subset A^{\Lambda_b} \subset bKer_\theta(A) \subset b\,\mathrm{Cl}_\theta(A)$.*

(2) *Every b-θ-closed set is a Λ_b-set.*

(3) *Every θ-$g.\Lambda_b$-set is a $g.\Lambda_b$-set.*

Proof. (1) Let A be a subset of (X, τ). It is shown in [2] that $A \subset A^{\Lambda_b}$. Now we prove that $A^{\Lambda_b} \subset bKer_\theta(A)$. Suppose that x is not a point of $bKer_\theta(A)$. It follows that $A \subset X \backslash b\,\mathrm{Cl}_\theta(\{x\})(= U$, say). Since $b\,\mathrm{Cl}_\theta(\{x\})$ is b-θ-closed by Lemma 2.6, so U is b-θ-open. Hence U is b-open since $U \subset b\,\mathrm{Cl}(U) \subset b\,\mathrm{Cl}_\theta(U)$.

Hence there exists a b-open set U containing A but not X, i.e. $x \notin A^{\Lambda_b}$. This shows that $A^{\Lambda_b} \subset bKer_\theta(A)$. proving $bKer_\theta(A) \subset b\,\mathrm{Cl}_\theta(A)$, let $x \in bKer_\theta(A)$. Suppose that $x \notin b\,\mathrm{Cl}_\theta(A)$. Then, there exists a b-open set U containing x such that $b\,\mathrm{Cl}(U) \cap A = \varnothing$. Since U is a b-open subset of X, it follows that $b\,\mathrm{Cl}_\theta(U) = b\,\mathrm{Cl}(U)$ [5]. Hence $b\,\mathrm{Cl}_\theta(U) \cap A = \varnothing$. This implies that $A \subset X \backslash b\,\mathrm{Cl}_\theta(U)$. Therefore $x \notin X \backslash b\,\mathrm{Cl}_\theta(U)$, where $X \backslash b\,\mathrm{Cl}_\theta(U)$ b-θ-open set containing A by Lemma 2.6. But this is impossible since $x \in bKer_\theta(A)$ by proposition 2.8. Consequently $x \in b\,\mathrm{Cl}_\theta(A)$.

(2). Let A be b-θ-closed set. Since $A^{\Lambda_b} \subset b\,\mathrm{Cl}_\theta(A)$ and $A = b\,\mathrm{Cl}_\theta(A)$, then $A^{\Lambda_b} = A$. Thus A is a Λ_b-set.

(3). Let $A \subset F$, where F is b-closed. Then we have $A^{\Lambda_b} \subset bKer_\theta(A) \subset F$. This implies that A is a $g.\Lambda_b$-set. □

Definition 6. A topological space (X, τ) is called b-R_1 [**3**] if for $x, y \in X$ with $b\,\mathrm{Cl}(\{x\}) \neq b\,\mathrm{Cl}(\{y\})$, there exists disjoint b-open sets U and V such that $b\,\mathrm{Cl}(\{x\}) \subset U$ and $b\,\mathrm{Cl}(\{y\}) \subset V$, or equivalently (X, τ) is b-R_1 [**3**] if and only if for each $x \in X$, $b\,\mathrm{Cl}(\{x\}) = b\,\mathrm{Cl}_\theta(\{x\})$.

Theorem 2.10. *Let (X, τ) is b-R_1. A subset A of (X, τ) is a $g.\Lambda_b$-set if and only if A is a θ-$g.\Lambda_b$-set.*

Proof. Sufficiency. It is an immediate consequence of Proposition 2.9 (3).

Necessity. By the fact X is b-R_1 if and only if $b\,\mathrm{Cl}(\{x\}) = b\,\mathrm{Cl}_\theta(\{x\})$ for each $x \in X$, the proof follows from the observation that $A^{\Lambda_b} = bKer_\theta(A)$. □

Definition 7. A subset A of a topological space (X, τ) is called a θ-Λ_b-set if $A = bKer_\theta(A)$.

Definition 8. A subset A of a topological space (X, τ) is called Λ_b-b-θ-closed if $A = L \cap S$, where L is a θ-Λ_b-set and S is b-θ-closed.

Lemma 2.11. *For a subset A of a topological space (X, τ) the following conditions are equivalent:*

(1) *A is Λ_b-b-θ-closed;*

(2) *$A = L \cap b\,\mathrm{Cl}_\theta(A)$, where L is a θ-Λ_b-set;*

(3) *$A = bKer_\theta(A)$, that is A is θ-Λ_b-set:*

Proof. (1) $\Rightarrow$ (2): Suppose that $A = L \cap S$, where L is a θ-Λ_b-set and S is b-θ-closed. Then $A \subset L$ and $A \subset b\,\mathrm{Cl}_\theta(A) \subset S$. Now, we have $A \subset L \cap b\,\mathrm{Cl}_\theta(A) \subset L \cap S = A$. This means that $A = L \cap b\,\mathrm{Cl}_\theta(A)$.

(2) $\Rightarrow$ (3): Suppose that $A = L \cap b\,\mathrm{Cl}_\theta(A)$ where L is θ-Λ_b-set. We have $A \subset bKer_\theta(A) \subset L$ and $A \subset b\,\mathrm{Cl}_\theta(A)$. So $A = bKer_\theta(A) \cap b\,\mathrm{Cl}_\theta(A)$ and Hence $A = bKer_\theta(A)$ by Proposition 2.9.

(3) $\Rightarrow$ (1): It is clear that $bKer_\theta(bKer_\theta(A)) = bKer_\theta(A)$ for any set A. Therefore $bKer_\theta(A)$ is a θ-Λ_b-set. Suppose that $A = bKer_\theta(A)$. By Proposition 2.9, $A = bKer_\theta(A) \cap b\,\mathrm{Cl}_\theta(A)$. Clearly A is the intersection of a θ-Λ_b-set and a b-θ-closed set and hence the result. □

Definition 9. A subset A of a topological space (X, τ) is called quasi b-θ-closed (briefly *qbt*-closed) if $b\,\mathrm{Cl}_\theta(A) \subset U$ whenever $A \subset U$ and U is b-θ-open in (X, τ).

Lemma 2.12. *A set A of a topological space (X, τ) is qbt-closed if and only if $b\,\mathrm{Cl}_\theta(A) \subset bKer_\theta(A)$.*

Proof. Necessity. Let $x \in X$ such that $x \notin bKer_\theta(A)$. So there exists a b-θ-open subset U such that $A \subset U$ with $x \notin U$. This means that $x \notin b\,\mathrm{Cl}_\theta(A)$ since A is *qbt*-closed.

Sufficiently. Obvious. □

Theorem 2.13. *For a subset A of a topological space (X, τ), the following are equivalent:*

(1) *A is b-θ-closed;*

(2) *A is qbt-closed and Λ_b-b-θ-closed.*

Proof. (1) $\Rightarrow$ (2): It is obvious that every b-θ-closed set is both *qbt*-closed and Λ_b-b-θ-closed.

(2) $\Rightarrow$ (1): Since A is *qbt*-closed, then $b\,\mathrm{Cl}_\theta(A) \subset bKer_\theta(A)$. Now $A = bKer_\theta(A) \cap b\,\mathrm{Cl}_\theta(A) = b\,\mathrm{Cl}_\theta(A)$. □

Lemma 2.14. *Let (X, τ) be a topological space and $x, y \in X$. Then the following two statement are equivalent:*

(1) *$y \in bKer_\theta(\{x\})$;*

(2) $x \in b\,\mathrm{Cl}_\theta(\{y\})$.

Proof. Let $y \notin bKer_\theta(\{x\})$. It follows that there exists a b-θ-open set U containing x such that $y \notin U$. This means that $x \notin b\,\mathrm{Cl}_\theta(\{y\})$. The converse is similar. □

Lemma 2.15. *The following statements are equivalent for any two points x and y in a topological space (X, τ):*

(1) $bKer_\theta(\{x\}) \neq bKer_\theta(\{y\})$;

(2) $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$.

Proof. (1) $\Rightarrow$ (2): Let $bKer_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$. Then there exists a point z in X such that $z \in bKer_\theta(\{x\})$ and $z \notin bKer_\theta(\{y\})$. By $z \in bKer_\theta(\{x\})$, it follows that $\{x\} \cap b\,\mathrm{Cl}_\theta(\{z\}) \neq \varnothing$. This implies $x \in b\,\mathrm{Cl}_\theta(\{z\})$. By $z \notin bKer_\theta(\{y\})$, we obtain $\{y\} \cap b\,\mathrm{Cl}_\theta(\{z\}) = \varnothing$. Since $x \in b\,\mathrm{Cl}_\theta(\{z\})$, $b\,\mathrm{Cl}_\theta(\{x\}) \subset b\,\mathrm{Cl}_\theta(\{z\})$ and $\{y\} \cap b\,\mathrm{Cl}_\theta(\{x\}) = \varnothing$. Hence it follows that $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$. Now $bKer_\theta(\{x\}) \neq bKer_\theta(\{y\})$ implies that $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$.

(2) $\Rightarrow$ (1): Let $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$. Then there exists a point z in X such that $z \in b\,\mathrm{Cl}_\theta(\{x\})$ and $z \notin b\,\mathrm{Cl}_\theta(\{y\})$. This means that there exists a b-θ-open set containing z and therefore x but not y, that is, $y \notin bKer_\theta(\{x\})$. Hence $bKer_\theta(\{x\}) \neq bKer_\theta(\{y\})$. □

Definition 10. A topological space (X, τ) is a b-θ-R_0 space if every b-θ-open set contains the b-θ-closure of each of its singletons.

Example 2.16. Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, \{a\}, X\}$. Then (X, τ) is a b-θ-R_0 topological space.

Theorem 2.17. *A topological space (X, τ) is b-θ-R_0 if and only if for x and y in X, $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$ implies $b\,\mathrm{Cl}_\theta(\{x\}) \cap b\,\mathrm{Cl}_\theta(\{y\}) = \varnothing$.*

Proof. Suppose that (X, τ) is b-θ-R_0 and $x, y \in X$ such that $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$. Then, there exists $z \in b\,\mathrm{Cl}_\theta(\{x\})$ such that $z \notin b\,\mathrm{Cl}_\theta(\{y\})$ ($or z \in b\,\mathrm{Cl}_\theta(\{y\})$ such that$Z \notin b\,\mathrm{Cl}_\theta(\{x\})$). There exists $V \in BO(X, \tau)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin b\,\mathrm{Cl}_\theta(\{y\})$. Thus $x \in X \backslash b\,\mathrm{Cl}_\theta(\{y\})$, which implies $b\,\mathrm{Cl}_\theta(\{x\}) \subset X \backslash b\,\mathrm{Cl}_\theta(\{y\})$ and $b\,\mathrm{Cl}_\theta(\{x\}) \cap b\,\mathrm{Cl}_\theta(\{y\}) = \varnothing$. The proof for otherwise is similar. Conversely, Let V be b-θ-

open and let $x \in V$. we will show that $b\,\mathrm{Cl}_\theta(\{x\}) \subset V$. Let $y \notin V$ that is, $y \in X \backslash V$. Then $x \neq y$ and $x \notin b\,\mathrm{Cl}_\theta(\{y\})$. This shows that $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$. By assumption, $b\,\mathrm{Cl}_\theta(\{x\}) \cap b\,\mathrm{Cl}_\theta(\{y\}) = \varnothing$. Hence $y \notin b\,\mathrm{Cl}_\theta(\{x\})$. Therefore, $b\,\mathrm{Cl}_\theta(\{x\}) \subset V$. □

Theorem 2.18. *A topological space (X, τ) is b-θ-R_0 if and only if for any points x and y in X, $bKer_\theta(\{x\}) \neq bKer_\theta(\{y\})$ implies $bKer_\theta(\{x\}) \cap bKer_\theta(\{y\}) = \varnothing$.*

Proof. Suppose that (X, τ) is b-θ-R_0 space. Thus by Lemma 2.15, for any points x and y in X if $bKer_\theta(\{x\}) \neq bKer_\theta(\{y\})$ then $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$. Now we prove that $bKer_\theta(\{x\}) \cap bKer_\theta(\{y\}) = \varnothing$. Assume that $z \in bKer_\theta(\{x\}) \cap bKer_\theta(\{y\})$. By $z \in bKer_\theta(\{x\})$ and Lemma 2.14, it follows that $x \in b\,\mathrm{Cl}_\theta(\{z\})$. Since $x \in b\,\mathrm{Cl}_\theta(\{x\})$, by theorem 2.17 $b\,\mathrm{Cl}_\theta(\{x\}) = b\,\mathrm{Cl}_\theta(\{z\})$. Similarly, we have $b\,\mathrm{Cl}_\theta(\{y\}) = b\,\mathrm{Cl}_\theta(\{z\}) = b\,\mathrm{Cl}_\theta(\{x\})$. This is a contradiction. Therefore, we have $bKer_\theta(\{x\}) \cap bKer_\theta(\{y\}) = \varnothing$. Conversely, let (X, τ) be a topological space such that for any points x and y in X, $bKer_\theta(\{x\}) \neq bKer_\theta(\{y\})$ implies $bKer_\theta(\{x\}) \cap bKer_\theta(\{y\}) = \varnothing$. If $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$, then by Lemma 2.15, $bKer_\theta(\{x\}) \neq bKer_\theta(\{y\})$. Hence $bKer_\theta(\{x\}) \cap bKer_\theta(\{y\}) = \varnothing$ which implies $b\,\mathrm{Cl}_\theta(\{x\}) \cap b\,\mathrm{Cl}_\theta(\{y\}) = \varnothing$. Because $z \in b\,\mathrm{Cl}_\theta(\{x\})$ implies that $x \in bKer_\theta(\{z\})$ and therefore $bKer_\theta(\{x\}) \cap bKer_\theta(\{z\}) \neq \varnothing$. By hypothesis, we have $bKer_\theta(\{x\}) = bKer_\theta(\{z\})$. Then $z \in b\,\mathrm{Cl}_\theta(\{x\}) \cap b\,\mathrm{Cl}_\theta(\{y\})$ implies that $bKer_\theta(\{x\}) = bKer_\theta(\{z\}) = bKer_\theta(\{y\})$. This is a contradiction. Hence $b\,\mathrm{Cl}_\theta(\{x\}) \cap b\,\mathrm{Cl}_\theta(\{y\}) = \varnothing$ and by Theorem 2.17 (X, τ) is b-θ-R_0 space. □

Theorem 2.19. *For a topological space (X, τ), the following properties are equivalent:*

(1) *(X, τ) is a b-θ-R_0 space;*

(2) *For any nonempty sets A, $G \in B\theta O(X)$ such that $A \cap G \neq \varnothing$ there exists $F \in B\theta O(X)$ such that $A \cap F \neq \varnothing$ and $F \subset G$;*

(3) *Any $G \in B\theta O(X)$,$G = \cup\{F \in B\theta C(X) | F \subset G\}$;*

(4) *Any $F \in B\theta C(X)$, $F = \cap\{G \in B\theta O(X) | F \subset G\}$;*

(5) *For any $x \in X$, $b\,\mathrm{Cl}_\theta(\{x\}) \subset bKer_\theta(\{x\})$.*

Proof. (1) ⇒ (2): Let A be a nonempty set of X and $G \in B\theta O(X)$ such that $A \cap G \neq \varnothing$. There exists $x \in A \cap G$. Since $x \in G \in B\theta O(X)$, $b\,\mathrm{Cl}_\theta(\{x\}) \subset G$.

Set $F = b\,\mathrm{Cl}_\theta(\{x\})$, then $F \in B\theta C(X)$, $F \subset G$ and $A \cap F \neq \varnothing$.

(2) $\Rightarrow$ (3): Let $G \in B\theta O(X)$, then $G \supset \bigcup\{F \in B\theta C(X)|F \subset G\}$. Let x be any point of G. There exists $F \in B\theta C(X)$ such that $x \in F$ and $F \subset G$. Therefore, we have $x \in F \subset \bigcup\{F \in B\theta C(X)|F \subset G\}$ and hence $G = \bigcup\{F \in B\theta C(X)|F \subset G\}$.

(3) $\Rightarrow$ (4): This is obvious.

(4) $\Rightarrow$ (5): Let x be any point of X and $y \notin bKer_\theta(\{x\})$. There exists $V \in B\theta O(X,x)$ $y \notin V$; hence $b\,\mathrm{Cl}_\theta(\{y\}) \cap V = \varnothing$. By (4) $(\cap\{G \in B\theta O(X)|b\,\mathrm{Cl}_\theta(\{y\}) \subset G\}) \cap V = \varnothing$ and there exists $G \in B\theta O(X)$ such that $x \notin G$ and $b\,\mathrm{Cl}_\theta(\{x\}) \subset G$. Therefore, $b\,\mathrm{Cl}_\theta(\{x\}) \cap G = \varnothing$ and $y \notin b\,\mathrm{Cl}_\theta(\{x\})$. Consequently, we obtain $b\,\mathrm{Cl}_\theta(\{x\}) \subset bKer_\theta(\{x\})$.

(5) $\Rightarrow$ (1): Let $G \subset B\theta O(X,x)$. Let $y \in bKer_\theta(\{x\})$, then $x \in b\,\mathrm{Cl}_\theta(\{x\})$ and $y \in G$. This implies that $bKer_\theta(\{x\}) \subset G$. Therefore, we obtain $x \in b\,\mathrm{Cl}_\theta(\{x\}) \subset bKer_\theta(\{x\}) \subset G$. This shows that (X,τ) is a b-θ-R_0 space. □

Corollary 2.20. *For a topological space (X,τ), the following properties are equivalent:*

(1) *(X,τ) is a b-θ-R_0 space;*

(2) *$b\,\mathrm{Cl}_\theta(\{x\}) = bKer_\theta(\{x\})$ for all $x \in X$.*

Proof. (1) $\Rightarrow$ (2): Suppose that (X,τ) is a b-θ-R_0 space. By Theorem 2.19, $b\,\mathrm{Cl}_\theta(\{x\}) \subset bKer_\theta(\{x\})$ for each $x \in X$. Let $y \in bKer_\theta(\{x\})$, then $x \in b\,\mathrm{Cl}_\theta(\{y\})$ and by theorem 2.17 $b\,\mathrm{Cl}_\theta(\{x\}) = b\,\mathrm{Cl}_\theta(\{y\})$. Therefore, $y \in b\,\mathrm{Cl}_\theta(\{x\})$ and hence $bKer_\theta(\{x\}) \subset b\,\mathrm{Cl}_\theta(\{x\})$.This shows that $b\,\mathrm{Cl}_\theta(\{x\}) = bKer_\theta(\{x\})$.

(2) $\Rightarrow$ (1): This is obvious theorem 2.19. □

Corollary 2.21. *If for any point x of a b-θ-R_0 space (X,τ), $b\,\mathrm{Cl}_\theta(\{x\}) \cap bKer_\theta(\{x\}) = \{x\}$, then $bKer_\theta(\{x\}) = \{x\}$.*

Proof. The proof follows from Theorem 2.19 (5). □

Theorem 2.22. *For a topological space (X,τ), the following properties are equivalent:*

(1) *(X,τ) is a b-θ-R_0 space;*

(2) *$x \in b\,\mathrm{Cl}_\theta(\{y\})$ if and only if $y \in b\,\mathrm{Cl}_\theta(\{x\})$ for any points x and y in X.*

Proof. (1) $\Rightarrow$ (2): Assume that (X, τ) is b-θ-R_0. Let $x \in b\,\mathrm{Cl}_\theta(\{y\})$ and $A \in B\theta O(X, y)$. Now by hypothesis, $x \in A$. Therefore, every b-θ-open set containing y contains x. Hence $y \in b\,\mathrm{Cl}_\theta(\{x\})$.

(2) $\Rightarrow$ (1): Let $U \in B\theta O(X, x)$. If $y \notin U$, then $x \notin b\,\mathrm{Cl}_\theta(\{y\})$ and hence $y \notin b\,\mathrm{Cl}_\theta(\{x\})$. This implies that $b\,\mathrm{Cl}_\theta(\{x\}) \subset U$. Hence (X, τ) is b-θ-R_0. □

Theorem 2.23. *For a topological space (X, τ), the following properties are equivalent:*

(1) *(X, τ) is a b-θ-R_0 space;*

(2) *If F is b-θ-closed subset of X, then $F = bKer_\theta(F)$;*

(3) *If F is b-θ-closed subset of X and $x \in F$, then $bKer_\theta(\{x\}) \subset F$;*

(4) *If $x \in X$, then $bKer_\theta(\{x\}) \subset b\,\mathrm{Cl}_\theta(\{x\})$.*

Proof. (1) $\Rightarrow$ (2): Let F be b-θ-closed subset of X and $x \notin F$. Thus $X \backslash F \in B\theta O(X, x)$. Since (X, τ) is b-θ-R_0, $b\,\mathrm{Cl}_\theta(\{x\}) \subset X \backslash F$. Thus $b\,\mathrm{Cl}_\theta(\{x\}) \cap F = \varnothing$ and hence $x \notin bKer_\theta(F)$. Therefore, $bKer_\theta(F) = F$.

(2) $\Rightarrow$ (3): In general, $A \subset B$ implies $bKer_\theta(A) \subset bKer_\theta(B)$. Therefore, it follows from (2) that $bKer_\theta(\{x\}) \subset bKer_\theta(F) = F$.

(3) $\Rightarrow$ (4): Since $x \in b\,\mathrm{Cl}_\theta(\{x\})$ and $b\,\mathrm{Cl}_\theta(\{x\})$ is b-θ-closed, by (3) $bKer_\theta(\{x\}) \subset b\,\mathrm{Cl}_\theta(\{x\})$.

(4) $\Rightarrow$ (1): We show implication by using Theorem2.22. Let $x \in b\,\mathrm{Cl}_\theta(\{y\})$. Then $y \in bKer_\theta(\{x\})$. Since $x \in b\,\mathrm{Cl}_\theta(\{x\})$ and $b\,\mathrm{Cl}_\theta(\{x\})$ is b-θ-closed, by (4) we obtain $y \in bKer_\theta(\{x\}) \subset b\,\mathrm{Cl}_\theta(\{x\})$. Therefore, $x \in b\,\mathrm{Cl}_\theta(\{x\})$ implies $y \in b\,\mathrm{Cl}_\theta(\{x\})$. The converse is obvious and (X, τ) is b-θ-R_0. □

Definition 11. A filterbase $\mathcal{F}$ is called b-θ-convergent to a point x in X if for any $U \in B\theta O(X, x)$, there exists $B \in \mathcal{F}$ such that B is a subset of U.

Lemma 2.24. *Let (X, τ) be a topological space and let x and y be any two points in X such that every net in X b-θ-converging to y b-θ-converges to x. Then $x \in b\,\mathrm{Cl}_\theta(\{y\})$.*

Proof. Suppose that $x_n = y$ for each $n \in N$. Then $\{x_n\}_{n \in N}$ is a net in $b\,\mathrm{Cl}_\theta(\{y\})$. Since $\{x_n\}_{n \in N}$ b-θ-converges to y, then $\{x_n\}_{n \in N}$ b-θ-converges to x and this implies that $x \in b\,\mathrm{Cl}_\theta(\{y\})$. □

Theorem 2.25. *For a topological space (X, τ), the following statements are equivalent:*

(1) (X, τ) is a b-θ-R_0 space;

(2) If $x, y \in X$, then $y \in b\,\mathrm{Cl}_\theta(\{x\})$ if and only if every net in X b-θ-converging to y b-θ-converges to x.

Proof. (1) $\Rightarrow$ (2): Let $x, y \in X$ such that $y \in b\,\mathrm{Cl}_\theta(\{x\})$. Suppose that $\{x_\alpha\}_{\alpha \in N}$ be a net in X such that $\{x_\alpha\}_{\alpha \in N}$ b-θ-converges to y. Since $y \in \mathrm{Cl}_\theta(\{x\})$, $b\,\mathrm{Cl}_\theta(\{x\}) = b\,\mathrm{Cl}_\theta(\{y\})$. Therefore $x \in b\,\mathrm{Cl}_\theta(\{y\})$. This means that $\{x_\alpha\}_{\alpha \in N}$ b-θ-converges to x. Conversely, let $x, y \in X$ such that every net in X b-θ-converging to y b-θ-converges to x. Then $x \in b\,\mathrm{Cl}_\theta(\{y\})$. Then we have $b\,\mathrm{Cl}_\theta(\{x\}) = b\,\mathrm{Cl}_\theta(\{y\})$. Therefore $y \in b\,\mathrm{Cl}_\theta(\{x\})$.

(2) $\Rightarrow$ (1): Assume that x and y are any two points of X such that $b\,\mathrm{Cl}_\theta(\{x\}) \cap b\,\mathrm{Cl}_\theta(\{y\}) \neq \varnothing$. Let $z \in b\,\mathrm{Cl}_\theta(\{x\}) \cap b\,\mathrm{Cl}_\theta(\{y\})$. So there exists a net $\{x_\alpha\}_{\alpha \in N}$ in $b\,\mathrm{Cl}_\theta(\{x\})$, such that $\{x_\alpha\}_{\alpha \in N}$ b-θ-converges to z. Since $z \in b\,\mathrm{Cl}_\theta(\{y\})$, then $\{x_\alpha\}_{\alpha \in N}$ b-θ-converges to y. It follows that $y \in b\,\mathrm{Cl}_\theta(\{x\})$. Similarly we can obtain $x \in b\,\mathrm{Cl}_\theta(\{y\})$. Therefore $b\,\mathrm{Cl}_\theta(\{x\}) = b\,\mathrm{Cl}_\theta(\{y\})$ and hence (X, τ) is b-θ-R_0. □

3. On b-θ-R_1 Spaces

Definition 12. A topological space (X, τ) is said to be b-θ-R_1 if for x, y in X with $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$, there exist disjoint b-θ-open sets U and V such that $b\,\mathrm{Cl}_\theta(\{x\}) \subset U$ and $b\,\mathrm{Cl}_\theta(\{y\}) \subset V$.

Proposition 3.1. *If (X, τ) is b-θ-R_1, then it is b-θ-R_0.*

Proof. Let $U \in B\theta O(X, x)$. If $y \notin U$, then since $x \notin b\,\mathrm{Cl}_\theta(\{y\})$, $b\,\mathrm{Cl}_\theta(\{x\}) \notin b\,\mathrm{Cl}_\theta(\{y\})$. Hence there exists a b-θ-open V_y such that $b\,\mathrm{Cl}_\theta(\{y\}) \subset V_y$ and $x \notin V_y$, which implies $y \notin b\,\mathrm{Cl}_\theta(\{x\})$. Thus $b\,\mathrm{Cl}_\theta(\{x\}) \subset U$. Therefore (X, τ) is b-θ-R_0. □

Theorem 3.2. *A topological space (X, τ) is b-θ-R_1 if and only if for $x, y \in X$, $bKer_\theta(\{x\}) \neq bKer_\theta(\{y\})$, there exists disjoint b-θ-open sets U and V such that $b\,\mathrm{Cl}_\theta(\{x\}) \subset U$ and $b\,\mathrm{Cl}_\theta(\{y\}) \subset V$.*

Proof. It follows from Lemma 2.15 □

Definition 13. *A* topological space (X,τ) is said to be:

(1) b-θ-T_1 is for each pair of distinct points x and y of X, there exist b-θ-open sets U and V of X such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$.

(2) b-θ-T_2 if for each pair of disjoint points x and y in X, there exist disjoint b-θ-open sets U and V in X such that $x \in U$ and $y \in V$.

Theorem 3.3. *The following properties are equivalent:*

(1) (X,τ) is b-θ-T_2,

(2)(X,τ) is b-θ-R_1 and b-θ-T_1,

(3) (X,τ) is b-θ-R_1 and b-θ-T_0.

Proof. (1) ⇒ (2): Since (X,τ) is b-θ-T_2, then it is b-θ-T_1. If $x, y \in X$ such that $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$, then $x \neq y$ and there exists disjoint b-θ-open sets U and V such that $x \in U$ and $y \in V$ and $b\,\mathrm{Cl}_\theta(\{x\}) = \{x\} \subset U$ and $b\,\mathrm{Cl}_\theta(\{y\}) = \{y\} \subset V$. Hence (X,τ) is b-θ-R_1.

(2) ⇒ (3): Since (X,τ) is b-θ-T_1, then(X,τ) is b-θ-T_0.

(3) ⇒ (1): Since (X,τ) is b-θ-R_1, and b-θ-T_1, then (X,τ) is b-θ-R_0 and b-θ-T_0. Let $x, y \in X$ such that $x \neq y$. Since $b\,\mathrm{Cl}_\theta(\{x\} = \{x\} \neq \{y\} = b\,\mathrm{Cl}_\theta(\{y\})$, then there exists disjoint b-θ-open sets U and V such that $x \in U$ and $y \in V$. Hence, (X,τ) is b-θ-T_2. □

Theorem 3.4. *The following properties are equivalent:*

(1) (X,τ) is b-θ-R_1,

(2) for each $x, y \in X$ one of the following holds:

(a) If U is b-θ-open, then $x \in U$ if and only if $y \in U$.

(b) there exists disjoint b-θ-open sets U and V such that $x \in U$ and $y \in V$, and

(3) If $x, y \in X$ such that $b\,\mathrm{Cl}_\theta(\{x\}) \notin b\,\mathrm{Cl}_\theta(\{y\})$, then there exists b-θ-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_2$, $y \in F_1$, $x \notin F_2$, and $X = F_1 \cup F_2$.

Proof. (1) ⇒ (2): Let $x, y \in X$. Then $b\,\mathrm{Cl}_\theta(\{x\}) = b\,\mathrm{Cl}_\theta(\{y\})$ or $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$. If $b\,\mathrm{Cl}_\theta(\{x\}) = b\,\mathrm{Cl}_\theta(\{y\})$ and U is b-θ-open, then $x \in U$ implies

$y \in b\,\mathrm{Cl}_\theta(\{x\}) \subset U$ and $y \in U$ implies $x \in b\,\mathrm{Cl}_\theta(\{y\}) \subset U$. Thus consider the case that $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$. Then there exists disjoint b-θ-open sets U and V such that $x \in b\,\mathrm{Cl}_\theta(\{x\}) \subset U$ and $y \in b\,\mathrm{Cl}_\theta(\{y\}) \subset V$.

(2) $\Rightarrow$ (3): Let $x, y \in X$ such that $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$. Then $x \notin b\,\mathrm{Cl}_\theta(\{y\})$ or $y \notin b\,\mathrm{Cl}_\theta(\{x\})$, say $x \notin b\,\mathrm{Cl}_\theta(\{y\})$. Then there exists a b-θ-open set A such that $x \in A$ and $y \notin A$, which implies there exists disjoint b-θ-open sets U and V such that $x \in U$ and $y \in V$. Then $F_1 = X \backslash V$ and $F_2 = X \backslash U$ are b-θ-closed sets such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$, $X = F_1 \cup F_2$.

(3) $\Rightarrow$ (1): Let x and y be any points in X with $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$. Then $b\,\mathrm{Cl}_\theta(\{x\}) \cap b\,\mathrm{Cl}_\theta(\{y\}) = \emptyset$. In fact, if $z \in b\,\mathrm{Cl}_\theta(\{x\}) \cap b\,\mathrm{Cl}_\theta(\{y\})$, then $b\,\mathrm{Cl}_\theta(\{z\}) \neq b\,\mathrm{Cl}_\theta(\{x\})$ or $b\,\mathrm{Cl}_\theta(\{z\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$. In case, $b\,\mathrm{Cl}_\theta(\{z\}) \neq b\,\mathrm{Cl}_\theta(\{x\})$, by (*iii*), there exists a b-θ-closed set F such that $x \in F$ and $z \notin F$. Then $z \in b\,\mathrm{Cl}_\theta(\{x\}) \subset F$. This contradicts that $z \notin F$. In case, $b\,\mathrm{Cl}_\theta(\{z\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$, similarly, this leads to the contradiction. Hence $b\,\mathrm{Cl}_\theta(\{x\}) \cap b\,\mathrm{Cl}_\theta(\{y\}) = \emptyset$. Let U be b-θ-open and let $x \in U$. Then $b\,\mathrm{Cl}_\theta(\{x\}) \subset U$, for suppose not. Let $y \in b\,\mathrm{Cl}_\theta(\{x\}) \cap (X \backslash U)$. Then $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$ and there exists b-θ-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$,$y \in F_2$, $x \notin F_2$, $X = F_1 \cup F_2$. Then $y \in X \backslash F_1$, which is b-θ-open, and $x \notin X \backslash F_1$ which is a contradiction. Hence (X, τ) is b-θ-R_0. Let $a, b \in X$ such that $b\,\mathrm{Cl}_\theta(\{a\}) \neq b\,\mathrm{Cl}_\theta(\{b\})$. Then there exists b-θ-closed sets A_1 and A_2 such that $a \in A_1, b \notin A_1, a \notin A_2$, and $X = A_1 \cup A_2$. Thus $a \in X \backslash A_2$ and $b \in X \backslash A_1$, which are b-θ-open, which implies $b\,\mathrm{Cl}_\theta(\{a\}) \subset A_1 \backslash A_2$ and $b\,\mathrm{Cl}_\theta(\{b\}) \subset A_2 \backslash A_1$. Hence (X, τ) is b-θ-R_1. □

Theorem 3.5. *A topological space(X, τ) is b-θ-T_2 if and only if for x, $y \in X$ such that $x \neq y$, there exists b-θ-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$,$y \notin F_2$, $x \notin F_2$,and $X = F_1 \cup F_2$.*

Proof. The straightforward proof is omitted. □

Remark 3.6. If$\{x_\lambda\}_{\lambda \in A}$ is a net in (X, τ), ${}_\beta\mathrm{lim}(\{x_\lambda\}_{\lambda \in A}) = \{x \in X : \{x_\lambda\}_{\lambda \in A}$ b-θ-converges to $x\}$

Theorem 3.7. *For a topological space (X, τ), the following properties are equivalent*

(1) *(X, τ) is b-θ-R_1;*

(2) *for $x, y \in X$, $b\,\mathrm{Cl}_\theta(\{x\}) = b\,\mathrm{Cl}_\theta(\{y\})$, whenever there exists a net $\{x_\lambda\}_{\lambda \in A}$ such that $x, y \in_\beta \lim(\{x_\lambda\}_{\lambda \in A})$;*

(3) (X,τ) *is* b-θ-R_0, *and for every* b-θ-*convergent net* $\{x_\lambda\}_{\lambda\in A}$ *in* X, ${}_\beta\lim(\{x_\lambda\}_{\lambda\in A}) = b\,\mathrm{Cl}_\theta(\{x\})$ *for some* $x \in X$.

Proof. (1) ⇒ (2): Let $x, y \in X$. such that there exists a net $\{x_\lambda\}_{\lambda\in A}$ in X such that $x, y \in_\beta \lim(\{x_\lambda\}_{\lambda\in A})$. Then (a) if U is b-θ-open, then $x \in U$ if and only if $y \in U$ or (b) there exists disjoint b-θ-open sets U and V such that $x \in U$ and $y \in V$. Since $x, y \in_\beta \lim(\{x_\lambda\}_{\lambda\in A})$, then (1) is satisfied, which implies $b\,\mathrm{Cl}_\theta(\{x\}) = b\,\mathrm{Cl}_\theta(\{y\})$.

(2) ⇒ (3): Let $U \in B\theta O(X,x)$. Let $y \notin U$. For each $n \in N$ let $x_n = x$. Then $\{x_n\}_{n\in N}$ b-θ-converges to x and since $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$, that $y \in A$ and $x \notin A$. Thus, $y \notin b\,\mathrm{Cl}_\theta(\{x\})$ and $b\,\mathrm{Cl}_\theta(\{y\}) \subset U$. Hence (X,τ) is b-θ-R_0. Let $\{x_\lambda\}_{\lambda\in A}$ be b-θ-convergent net in X. Let $x \in X$ such that $\{x_\lambda\}_{\lambda\in A}$ b-θ-converges to x. If $y \in b\,\mathrm{Cl}_\theta(\{x\})$,then $\{x_\lambda\}_{\lambda\in A}$ b-θ-converges to y, which implies $b\,\mathrm{Cl}_\theta(\{x\}) \subset_\beta \lim(\{x_\lambda\}_{\lambda\in A})$ and if $y \in_\beta \lim(\{x_\lambda\}_{\lambda\in A})$, then $x, y \in_\beta \lim(\{x_\lambda\}_{\lambda\in A})$, which implies $y \in b\,\mathrm{Cl}_\theta(\{y\}) = b\,\mathrm{Cl}_\theta(\{x\})$. Hence ${}_\beta\lim(\{x_\lambda\}_{\lambda\in A}) = b\,\mathrm{Cl}_\theta(\{x\})$.

(3) ⇒ (1): Assume that (X,τ) is not b-θ-R_1. Then there exists $x, y \in X$ such that $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$ and every b-θ-open set containing $b\,\mathrm{Cl}_\theta(\{x\})$ intersects every b-θ-open set containing $b\,\mathrm{Cl}_\theta(\{y\})$. Since (X,τ) is b-θ-R_0, then every b-θ-open set containing x contains $b\,\mathrm{Cl}_\theta(\{x\})$ and every b-θ-open set containing y contains $b\,\mathrm{Cl}_\theta(\{y\})$, which implies that every b-θ-open set containing x intersects every b-θ-open set containing y. Let $D_x = \{U \subset X : U \in B\theta O(X,x)\}$. Let $\geq_x$ be the binary relation on D_x defined by $U_1 \geq_x U_2$ if and only if $U_1 \subset U_2$. Then, clearly $(D_x, \geq_x)$ is a directed set. Let $D_y = \{U \subset X : U \in B\theta O(X,y)\}$ and let $\geq_y$ be binary relation on D_y defined by $U_1 \geq_y U_2$ if and only if $U_1 \subset U_2$. Then, $(D_y, \geq_y)$ is also a directed set. Let $D = \{(U_1,U_2) : U_1 \in D_x and U_2 \in D_y\}$ and let $\geq$ be the binary relation on D defined by $(U_1,U_2) \geq (V_1,V_2)$ if and only if $U_1 \geq_x V_1$ and $U_2 \geq_y V_2$. Then, $(D, \geq)$ is directed set. For each $(U_1,U_2) \in D$, let $x_{(U_1,U_2)} \in (U_1,U_2)$. Then $\{x_{(U_1,U_2)}\}_{(U_1,U_2)\in D}$ is a net in X that b-θ-converges to both x and y . Thus, there exists $z \in X$ such that ${}_\beta\lim(\{x_{(U_1,U_2)}\}_{(U_1,U_2)\in D}) = b\,\mathrm{Cl}_\theta(\{z\})$, which implies $x, y \in b\,\mathrm{Cl}_\theta(\{z\})$. Since $\{b\,\mathrm{Cl}_\theta(\{w\}) : w \in X\}$ is a decomposition of X, then $b\,\mathrm{Cl}_\theta(\{x\}) = b\,\mathrm{Cl}_\theta(\{z\}) = b\,\mathrm{Cl}_\theta(\{y\})$, which is a contradiction. Hence (X,τ) is b-θ-R_1. □

Theorem 3.8. *A topological space* (X,τ) *is* b-θ-T_2 *if and only if every* b-θ-*convergent net in* X b-θ-*converges to a unique point.*

Proof. The proof follows from Theorem 3.7 and 3.3. □

Theorem 3.9. *For a topological space (X, τ), the following properties are equivalent*

(1) (X, τ) *is b-θ-R_1,*

(2) *for each pair $x, y \in X$, $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$, there exists a b-θ-regular set V such that $y \notin V$;*

(3) *for each pair $x, y \in X$ $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$, there exists a b-θ-continuous function $f : (X, \tau) \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$.*

Proof. (1) $\Rightarrow$ (2):Let $x, y \in X$ such that $b\,\mathrm{Cl}_\theta(\{x\}) \notin b\,\mathrm{Cl}_\theta(\{y\})$. Then there exists disjoint b-θ-open sets U and W such that $b\,\mathrm{Cl}_\theta(\{x\}) \subset U$ and $b\,\mathrm{Cl}_\theta(\{y\}) \subset W$ and $V = b\,\mathrm{Cl}_\theta(\{U\})$ is b-θ-regular such that $x \in V$ and $y \notin V$.

(2) $\Rightarrow$ (3): Let $x, y \in X$ such that $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$. Let V be a b-θ-open, a b-θ-closed set of X such that $x \in V$ and $y \notin V$. Thus, the function $f : (X, \tau) \to [0, 1]$ defined by $f(z) = 0$ if $z \in V$ and $f(z) = 1$ if $z \notin V$ satisfies the desired properties.

(3) $\Rightarrow$ (1): Let $x, y \in X$ such that $b\,\mathrm{Cl}_\theta(\{x\}) \neq b\,\mathrm{Cl}_\theta(\{y\})$. Let $f : (X, \tau) \to [0, 1]$ such that f is b-θ-continuous, $f(x) = 0$ and $f(y) = 1$. Then $U = f^{-1}([0, 0.5))$ and $V = f^{-1}((0.5, 1])$ are disjoint such that b-θ-open, b-θ-closed set of X and $b\,\mathrm{Cl}_\theta(\{x\}) \subset U$ and $b\,\mathrm{Cl}_\theta(\{y\}) \subset V$. □

Theorem 3.10. *For a topological space (X, τ), the following properties are equivalent*

(1) (X, τ) *is b-θ-R_1,*

(2) *for each pair $x, y \in X$, $x \neq y$, there exists a b-θ-open, b-θ-closed set V such that $y \notin V$;*

(3) *for each pair $x, y \in X$, $x \neq y$, there exists a b-θ-continuous function $f : (X, \tau) \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$.*

Proof. The proof is similar to that of Theorem 3.9. □

References

[1] D. Andrijevic, On b-open sets, *Math. Vesnik*, **48** (1996), 59-64.

[2] M. Caldas, S. Jafari, T. Noiri, On Λ_b-sets and the associated τ^{Λ_b}, *Acta Math. Hungar.*, **110**, No. 4 (2006), 337-345.

[3] E. Ekici, On R-spaces, *International J. Pure. Appl. Math.*, **25**, No. 2 (2005), 163-172.

[4] A.A. El-Atik, *A Study of Some Types of Mappings on Topological Spaces*, Master's Thesis, Faculty of Science, Tanta University, Tanta, Egypt (1997).

[5] J.H. Park, Strongly θ-b-continuous functions, *Acta. Math. Hungar.*, **110**, No. 4 (2006), 347-359.

[6] N. Rajesh, On maps and generalized Λ_b-sets, *Vietnam J. Math.*, **36**, No. 4 (2008), 395-403.

[7] N. Rajesh, On weakly b-irresolute functions, *Acta Universitatis Apulensis*, **30** (2012), 153-160.