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ANOTE ON THE GEODESIC-RAY PROPERTY IN MANIFOLDS WITHOUT CONJUGATE POINTS

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Abstract: Let W^n be a C^∞ complete, simply connected n-dimensional Riemannian manifold without conjugate points. The relationship between geodesic-ray property, starshapedness and convexity are established in W^n.Furthermore, some examples are given.

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Key Words: convex set, manifolds without conjugate points, starshaped sets

1. Introduction

The behavior of geodesics in Riemannian manifolds without conjugate points as well as focal points has been studied by many geometers auch as [1], [3], [4], [6], [11], [13] and[14]. Hopf in a celebrated paper [12] proved that 2-dimensional tori without conjugate points are flat. So their universal coverings are flat planes where geodesics are straight lines. In 1994, Burago and Ivanov [5] generalized this result for n-dimensional tori. Green[10] in the late 1950_s, proved the divergence of Jacobi fields and geodesic rays in the universal covering of compact surfaces without conjugate points. Green also proved the same result for com-

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pact manifolds without conjugate points of any dimension. Eberlein [7]maked a complete proof of the divergence in manifolds N regardless of dimension of N. On the other hand observes that the divergence might not be uniform, it could depend on the geodesic. Recently, Ruggiero[17] proved that geodesic rays in universal covering which meet an axis of a covering isometry diverge from this axis.

Motivated by earlier research work works [15, 16] and by the importance of the concept of geodesic rays, we establish the relationship between geodesic rays, starshapedness and convexity in Riemannian manifolds.

2. Preliminaries

In this section, we introduce some defintions and known results of Riemannian manifolds, which help us throughout the article. We refer to [21] for the standard material on differential geometry.
Let N be a C^∞ n-dimensional Riemannian manifold, and T_zN be the tangent space to N at z. Also, assume that $\mu_z(x_1, x_2)$ is a positive inner product on the tangent space T_zN ($x_1, x_2 \in T_zN$), which is given for each point of N. Then, a C^∞ map $\mu\colon z \longrightarrow \mu_z$, which assigns a positive inner product μ_z to T_zN for each point z of N is called a Riemannian metric.
The length of a piecewise C^1 curve $\eta\colon [a_1, a_2] \longrightarrow N$ which is defined as follows:

$$L(\eta) = \int_{a_1}^{a_2} \|\acute{\eta}(x)\| dx.$$

We define $d(z_1, z_2) = inf\,\{L(\eta)\colon \eta$ is a piecewise C^1 curve joining z_1 to $z_2\}$ for any points $z_1, z_2 \in N.\bigtriangledown_X Y, X, Y \in N$ is a unique determined Riemannian connection which called Levi-Civita connection on every Riemannian manifolds. Furthermore, a smooth path η is a geodesic if and only if its tangent vector is a prallel vector field along the path η, i.e , η satisfies the equation $\bigtriangledown_{\acute{\eta}(t)}\acute{\eta}(t) = 0$. Every path η is joining $z_1, z_2 \in N$ where $L(\eta) = d(z_1, z_2)$ is a minimal geodesic. Finally, assume that (N, η) is a complete n-dimensional Riemannian manifold with Riemannian connection $\bigtriangledown$. Let $x_1, x_2 \in N$ and $\eta\colon [0, 1] \longrightarrow N$ be a geodesic joining the points x_1 and x_2 , which means that $\eta_{x_1,x_2}(0) = x_2$ and $\eta_{x_1,x_2}(1) = x_1$.

Definition 1 (see[19]). A subset B in a Riemannian manifold N is convex if for each pair points $p, q \in N$, there is a unique minimal geodesic segment from p to q and this segment is in B.

When dealing with a subset $B \subset W^n$, where W^n is a C^∞ complete, simply connected n- dimensional Riemannian manifold without conjugate points, the word " a unique minimal geodesic segment" should be replaced by " the geodesic segment".

Definition 2 (see[18]). A subset S in a Riemannian manifold N is starshaped if there is a point $p \in S$ such that for all $q \in S$ there is a unique minimal geodesic segment γ_{pq} from p to q and this segment is in S. In such a case, the set S is starshaped with respect to p or p sees S via S.

Remark 1. The subset of S consisting of all points like p is called the kernel of S ($kerS$). In W^n, a subset S is starshaped if there is a point $p \in S$such that for all $q \in S$, the geodesic segment γ_{pq} joining p and q is contained in S .

Let $W_1$and W_2 be complete, simply connected C^∞ Riemannian manifolds without conjugate points. Assume that $W_1 \times W_2$ is a complete, simply connected C^∞ Riemannian manifold without conjugate points. Notice that $dim(W_1 \times W_2) = dim(W_1) + dim(W_2)$. Consequently, each pair of different points $p = (p_1, p_2)$ and $q = (q_1, q_2)$ in $W_1 \times W_2$ are joined by a unique geodesic segment γ .This segment when naturally projected on W_1 and W_2 yields two geodesic segments joining p_i and q_i , $i = 1, 2$ each one is unique in its own manifold. The natural projection will be denoted by $\varphi_i \colon W_1 \times W_2 \longrightarrow W_i$ where $\varphi_i(p_1, p_2) = p_i, i = 1, 2$ (see [2]).

In 1974, Stavrakas [20] defined the half-ray property in Euclidean space as follows:

Definition 3. let $S \subset \mathbb{R}^n$, and let S^c be the complement of S. S has the half-ray property if and if for every $x \in S^c$ there exists a half line L with x as vertex x such that $L \cap S = \phi$.

In the same paper there are some geometric properties of starshaped in Euclidean space were considered as follows: let $S \subset \mathbb{R}^n, n \geq 2$, be compact and suppose that $\cap_{x \in E(x)} S(x) \neq \phi$. Then, the following are equivalent:

1. S has the half- ray property.
2. $kerS = \cap_{x \in E(x)} S(x)$.

Also, Goodey in [8] used the half-ray property to prove the following theorem:

Theorem 4. *If $S \subset \mathbb{R}^n$ is a nonseparating (its complement is connected) compact set and $\cap \{S(y) \colon y \in E(S)\} \neq \phi$, where $E(S)$ is the totality of $(n-2)$-extreme points of S, then S is starshaped set.*

3. The Geodesic-Ray Property

Let us start by defining the geodesic-ray property in a C^∞ complete Riemannian manifold N. Let $\gamma\colon (-\infty,\infty) \longrightarrow N$be a maximal geodesic. Let $m \in N$be a point at γ such that $\gamma(0) = m$ and $\gamma(0) = X \in T_m N$. The geodesic γ can be divided at two half geodesic rays $\gamma = \gamma_+ \cup \gamma_-$ where $\gamma_+\colon\ [0,\infty)$such that $\acute{\gamma_+}(0) = X$ and $\gamma_-\colon\ (-\infty, 0]$ such that $\acute{\gamma_-}(0) = -X$

Definition 5. Let $B \subset N$and let B^c be the complement of B. We say that B has the geodesic-ray property if for all $x \in B^c$ and for any geodesic $\gamma = \gamma_+ \cup \gamma_-$ such that $\gamma(0) = x$, then one or both of two half geodesic rays γ_+ or γ_- has empty intersection of B.

In the light of definition (5), we can prove that the geodesic-ray property is an intersection property in the following proposition:

Proposition 1. *The intersection of two sets have the geodesic-ray property has the geodesic-ray property*

Proof. Let $B_1, B_2 \subset N$ have the geodesic-ray property. Let us assume , on the contrary, that $B_1 \cap B_2$ has no the geodesic-ray property. Hence, there exists a point $x \in (B_1 \cap B_2)^c = B_1^c \cup B_2^c$ and a geodesic $\gamma = \gamma_+ \cup \gamma_-$, where $\gamma(0) = x$, such that the two rays γ_+ and γ_- have no empty intersection with $B_1 \cap B_2$. Since $x \in B_1^c$ or $x \in B_2^c$, then either B_1 or B_2 has no geodesic-ray property which is a contradiction. □

Remark 2. In general, the union of geodesic-ray property is not geodesic-ray property.

Example 1. Let A be the union of two closed balls $D_1 = B(a,r)$ and $D_2 = B(b,r)$in $\mathbb{R}^n$ of equal radii r and center a, b in $\mathbb{R}^n$ such that $D_1 \cap D_2 = \phi$. It is clear that A is the union of a two line property. Let $c = \frac{(a+b)}{2} \in \mathbb{R}^n$, then the half line $\overrightarrow{(ca)}$ starting from $c \in \mathbb{R}^n$ and passing throw $a \in \mathbb{R}^n$ has no empty intersection with $D_1 \subset A$. Also, the half line $\overrightarrow{(cb)}$ has no empty intersection with $D_2 \subset A$. Since $c \in A^c$, then A has no line property.

Theorem 6. *If p is an interior point in a closed subset $B \subset W^n$ with smooth boundary ∂B and B has the geodesic-ray property, then each geodesic ray from p intersects the hypersurface ∂B exactly at one point and the intersection is transversal.*

Proof. Let us consider an arbitrary point $q \in \partial B$. Firstly, we show that the geodesic ray η through p and q should intersect ∂B at q transversally. Assume that the intersection is tangential as indicated in Figure 1. Draw a thin geodesic cone C with vertex at q and axis η whose base is included in $U(p) \subset B$, where $U(p)$ is an open neighborhood about p. From the figure it becomes clear that there exists a geodesic $\gamma = \gamma_+ \cup \gamma_-$ through a point $x \in W^n \backslash B$ such that $\gamma_{xq} \subset \gamma_+$and $\gamma_{\acute{p}x} \subset \gamma_-$ where $\acute{p} \in U(p)$. Consequently, $\gamma_+ \cap B \neq \phi$ and $\gamma_- \cap B \neq \phi$ which contradicts B has the geodesic-ray property.

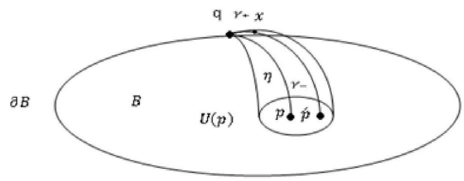

Figure 1: The geodesic-ray η intersects ∂B at q tangentially

To complete the proof assume that there exists a geodesic ray η from p which intersects ∂B twice, then we shall arrive at Figure 2. Hence, there exists a geodesic $\gamma = \gamma_+ \cup \gamma_-$ through a point $x \in W^n \backslash B$ such that $\gamma_{qx} \subset \gamma_+$ where $q \in \partial B$ and $\gamma_{xs} \subset \gamma_-$ where $s \in \partial B$. Consequently, $\gamma_+ \cap B \neq \phi$ and $\gamma_- \cap B \neq \phi$ which contradicts B has the geodesic-ray property.

□

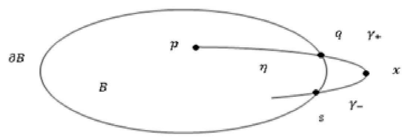

Figure 2: The geodesic η intersects ∂B twice

Corollary 7. *Let B be a closed subset in W^n with smooth boundary ∂B. If B has the geodesic-ray property, then each tangent geodesic η to ∂B has the property $\eta \cap Int(B) = \phi$.*

Corollary 8. *Let B be a closed subset in W^n with smooth boundary ∂B and B has the geodesic-ray property. If $p \in \partial B$, then B lies on one side of the tangent geodesic hypersurface of ∂B at p.*

4. Starshapedness and Geodesic - ray Property

In this section, we study the relationship between starshapedness in a C^∞ complete, simply connected Riemannian manifold without conjugate points and geodesic-ray property.

Theorem 9. *Let $S \subset W^n$ starshaped set . S has the geodesic- ray property if and if $S = kerS$.*

Proof. Let $S = kerS$. Now, we want to prove that S has the geodesic-ray property. Suppose that S does not have the geodesic- ray property. So, there is $x \in S^c$ and $x = \gamma_+(0) = \gamma_-(0)$ such that $\gamma_+ \cap S \neq \phi$ and $\gamma_- \cap S \neq \phi$. Then, there exists two different points $p, q \in S$ such that $p \in \gamma_+ \cap S$ and $q \in \gamma_- \cap S$. Consequently, $\gamma_{px} \subset \gamma_+$ and $\gamma_{xq} \subset \gamma_-$ which means that the geodesic segment γ_{pq} joining p and q is not included in S and hence p does not see q via S. This is a contradiction to the assumption that $S = kerS$. Assume on the contrary that $S \neq kerS$,then there is two points $p, q \in S$such that p does not see q via S. Thus, the geodesic segment γ_{pq} joining p and q is not contained in S. Then, there is a point $x \in \gamma_{pq}$ such that $x \notin S$. This implies that there exists a geodesic$\gamma = \gamma_+ \cup \gamma_-$ through a point $x \in S^c$ and $x = \gamma_+(0) = \gamma_-(0)$ such that $\gamma_{px} \subset \gamma_+$ and $\gamma_{xq} \subset \gamma_-$. Then, $\gamma_+ \cap S \neq \phi$ and $\gamma_- \cap S \neq \phi$. This argument shows that S does not have the geodesic-ray property. □

Theorem 10. *A closed starshaped set $S \in W^n$ has the geodesic-ray property if and only if $\partial S \subset kerS$.*

Proof. If S has the geodesic-ray property ,then $kerS = S \supset \partial S$. Now, let $\partial S \subset kerS$, we want to prove that S has the geodesic-ray property. Suppose that S is not ,then there is $x \in S^c$ and $x = \gamma_+(0) = \gamma_-(0)$ such that $\gamma_+ \cap \partial S \neq \phi$ and $\gamma_- \cap \partial S \neq \phi$.This implies that there is $p \in \gamma_+ \cap \partial S$ and $q \in \gamma_- \cap \partial S$ such that $\gamma_{pq} \subset \gamma$ and $\gamma_{pq} \cap S = \phi$ and hence p does not see q via S which contradicts the fact that $\partial S \subset kerS$ and so S has geodesic-ray property. □

Theorem 11. *Assume that S is an open connected subset of W^2. Then, $kerS$ is the set of all points of maximal visibility.*

Proof. Let us consider that Z is the set of all points of maximal visibility in S. We want to prove that $Z = kerS$. It is clear that $kerS \subset Z$, so we will show that $Z \subset kerS$. Let $x \notin kerS$ and $x \in Z$. Then, there exists a geodesic $\gamma = \gamma_+ \cup \gamma_-$ through a point $x \in Z$ and $x = \gamma_+(0) = \gamma_-(0)$ such that $\gamma_+ \cap kerS \neq \phi$ and $\gamma_- \cap kerS \neq \phi$.Then, there exists two different points

$p, q \in kerS$ such that $p \in \gamma_+ \cap kerS$ and $q \in \gamma_- \cap kerS$. Consequently, $\gamma_{px} \subset \gamma_+$and $\gamma_{xq} \subset \gamma_-$.This implies that the geodesic segment γ_{pq} joining p and q is not included in $kerS$.This argument shows that $kerS$ is non-convex which contradiction. Then, $kerS \subset Z$ and the proof is complete. □

5. Convexity and the Geodesic-Ray Property

In this section, we aim to give the relationship between convex and the geodesic-ray property in a Riemannian manifold.

Theorem 12. *Let B be an open subset in a C^∞ complete n-dimensional Riemannian manifold N. If B has the geodesic-ray property, ,then B is convex.*

Proof. Assume on the contrary that B is not convex (see Figure 3).Then, there exists a pair of points $p, q \in B$ such that the geodesic segment γ_{pq} joining p and q is not contained in B. Thus, there is a point $x \in \gamma_{pq}$ such that $x \notin B$. So, there exists a geodesic $\gamma = \gamma_+ \cup \gamma_-$ through a point $x \in B^c$ and $x = \gamma_+(0) = \gamma_-(0)$ such that $\gamma_{px} \subset \gamma_+$ and $\gamma_{xq} \subset \gamma_-$. Then, $\gamma_+ \cap B \neq \phi$ and $\gamma_- \cap B \neq \phi$. This argument shows that B does not have the geodesic-ray property contradicting the hypothesis. □

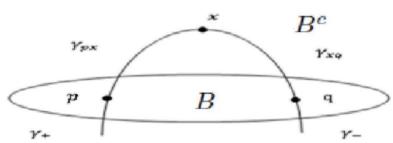

Figure 3: B is not convex

The converse of the last Theorem 12 is not generally true even in genaral Riemannian manifold. The following example indicates this claim.

Example 2. Let $B(p, r), 0 < r < \frac{\pi}{2}$ be an open geodesic ball in the unit sphere $S^n \subset \mathbb{R}^{n+1}$ centered at the north pole p with radius $0 < r < \frac{\pi}{2}$ (see Figure 4). Let q_1 and q_2 be two arbitrary points in B ,then there exists a geodesic $\gamma = \gamma_+ \cup \gamma_-$ through a point $x \in B^c$ and $x = \gamma_+(0) = \gamma_-(0)$ such that $\gamma_{q_1x} \subset \gamma_+$ and $\gamma_{xq_2} \subset \gamma_-$. Thus, $\gamma_+ \cap B \neq \phi$ and $\gamma_- \cap B \neq \phi$. This implies that B does not have the geodesic-ray property while $B(p, r)$ is convex subset.

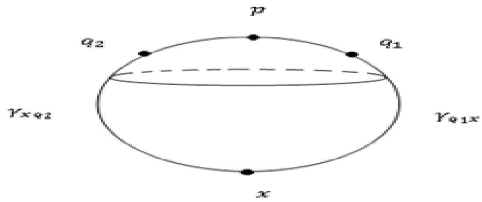

Figure 4: $B(p, r), 0 < r < \frac{\pi}{2}$

Theorem 13. *Let $B \subset W^n$ be an open subset. B has the geodesic-ray property if and only if B is convex.*

Proof. Let $B \subset W^n$ be a convex subset which does not have the geodesic-ray property. Then, there exists a geodesic$\gamma = \gamma_+ \cup \gamma_-$ through a point $x \in B^c$ and $x = \gamma_+(0) = \gamma_-(0)$ such that $\gamma_+ \cap B \neq \phi$ and $\gamma_- \cap B \neq \phi$. There exists two different points $p, q \in B$ such that $p \in \gamma_+ \cap B$ and $q \in \gamma_- \cap B$. Consequently, $\gamma_{px} \subset \gamma_+$ and $\gamma_{xq} \subset \gamma_-$ such that the geodesic segment γ_{pq} joining p and q is not included in B (see Figure 5). This argument shows that B is non-convex contradicting the hypothesis.

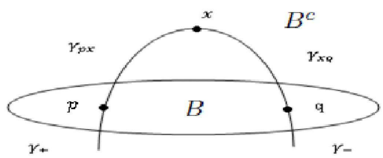

Figure 5: B does not have the geodesic-ray property

The proof of another side can be carried out similar to that of Theorem 12. □

Remark 3. Let B be a closed subset in a C^∞ complete, simply connected n- dimensional Riemannian manifold without conjugate points W^n.If B is a strictly convex, then B has the geodesic-ray property.

Now, we give the following example of set which has the geodesic-ray property but it is not strictly convex.

Example 3. Let $B \subset \mathbb{R}^2$ be a closed subset. Then, B has the line property but it is not strictly convex (see Figure 6).

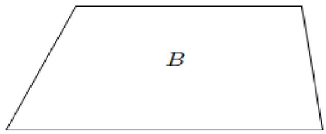

Figure 6: B has the line property but it is not strictly convex

6. Geodesic -ray Property in Riemannian Manifolds Product

Geodesic-ray property in the Cartesian product of two complete, simply connected Riemannian manifolds without conjugate points is given by the following theorem:

Theorem 14. *Let $B_1 \subset W_1$ and $B_2 \subset W_2$ where W_1 and W_2 are C^∞ complete, simply connected Riemannian manifolds without conjugate points. $B_1 \times B_2$ has geodesic-ray property if and only if B_1 and B_2 have the geodesic-ray property.*

Proof. Firstly, assume that $B_1 \times B_2$ has the geodesic-ray property of $W_1 \times W_2$. Suppose that B_1 and B_2 do not have the geodesic-ray property. Then, there exist a geodesic segment $\gamma^1 = \gamma^1_+ \cup \gamma^1_-$ through a point $x_1 \in B_1^c$ and $x_1 = \gamma^1_+(0) = \gamma^1_-(0)$ such that $\gamma^1_+ \cap B_1 \neq \phi$ and $\gamma^1_- \cap B_1 \neq \phi$. So, there exists two different points $p_1, q_1 \in B_1$ such that $p_1 \in \gamma^1_+ \cap B_1$ and $q_1 \in \gamma^1_- \cap B_1$. Consequently, $\gamma^1_{p_1x_1} \subset \gamma^1_+$ and $\gamma^1_{x_1q_1} \subset \gamma^1_+$,then the geodesic segment $\gamma^1_{p_1q_1} \not\subset B_1$. Since B_2 does not have the geodesic-ray property. Thus, there exists a geodesic $\gamma^2 = \gamma^2_+ \cup \gamma^2_-$ through a point $x_2 \in B_2^c$ and $x_2 = \gamma^2_+(0) = \gamma^2_-(0)$ such that $\gamma^2_+ \cap B_2 \neq \phi$ and $\gamma^2_- \cap B_2 \neq \phi$. Then, there exist two different points $p_2, q_2 \in B_2$ such that $p_2 \in \gamma^2_+ \cap B_2$ and $q_2 \in \gamma^2_- \cap B_2$. Consequently, $\gamma^2_{p_2x_2} \subset \gamma^2_+$ and $\gamma^2_{x_2q_2} \subset \gamma^2_+$,then the geodesic segment $\gamma^2_{p_2q_2} \not\subset B_2$. Let $p = (p_1, p_2), q = (q_1, q_2) \in B_1 \times B_2$. We can claim that $\gamma_{pq} \not\subset B_1 \times B_2$. Let $x = (x_1, x_2) = \gamma_+(0) = \gamma_-(0)$, where $\gamma_{px} \subset \gamma_+$ and $\gamma_{xq} \subset \gamma_-$,but $\gamma_+ \cap (B_1 \times B_2) \neq \phi$ and $\gamma_- \cap (B_1 \times B_2) \neq \phi$. This argument shows that $B_1 \times B_2$ has the geodesic-ray property.

Now, let $B_1 \times B_2$ does not have the geodesic-ray property ,then there exist a geodesic $\gamma = \gamma_+ \cup \gamma_-$ through a point $x = (x_1, x_2) \in (B_1 \times B_2)^c$ and $x = \gamma_+(0) = \gamma_-(0)$ such that $\gamma_+ \cap (B_1 \times B_2) \neq \phi$ and $\gamma_- \cap (B_1 \times B_2) \neq \phi$. Then, there exist two different points $p = (p_1, p_2), q = (q_1, q_2) \in B_1 \times B_2$ such that $p \in \gamma_+ \cap (B_1 \times B_2)$ and $q \in \gamma_- \cap (B_1 \times B_2)$. Consequently, $\gamma_{px} \subset \gamma_+$ and $\gamma_{xq} \subset \gamma_-$.

Then, the geodesic segment $\gamma_{pq} \not\subset B_1 \times B_2$. Since $(x_1, x_2) \notin (B_1 \times B_2)$, we have that one - at least- of the following statements $x_1 \notin B_1$,$x_2 \notin B_2$. Thus, $x_1 \in B_1^c$, $\gamma_{p_1 x_1} \subset \gamma_+^1$ and $\gamma_{x_1 q_1} \subset \gamma_-^1$ this implies that $\gamma_+^1 \cap B_1 \neq \phi$ and $\gamma_-^1 \cap B_1 \neq \phi$. This argument shows that B_1 does not have the geodesic-ray property. □

The following example may be considered as an application of Theorem 14.

Example 4. Consider $B_1 \subset \mathbb{R}^1$ and $B_2 \subset \mathbb{R}^2$ where $B_1 = [0, 1]$ and $B_2 = \{(x_1, x_2) \colon x_1^2 + x_2^2 \leq 1\}$ is the unit disc. It is clear that B_1 and B_2 have the line property. Also, $B_1 \times B_2 \subset \mathbb{R}^1 \times \mathbb{R}^2 = \mathbb{R}^3$ which represents a truncated cylinder has the line property.

In the light of Theorem 14 we can prove the following corollary:

Corollary 15. *Assume that $B \subset W_1 \times W_2$ has the geodesic-ray property . Then, the natural projections $B_i = \varphi_i B \subset W_i$ of B onto $W_i, i = 1, 2$ have the geodesic-ray property.*

It is worth mentioning that the converse of the above corollary is not necessarily true, i.e., for a subset $B \subset W_1 \times W_2$ the natural projections $\varphi_i B \subset W_i, i = 1, 2$ might have the geodesic-ray property although B does not have the geodesic-ray property. Figure 7 shows this fact. Notice that in this case, $\varphi_1 B \times \varphi_2 B \neq B$.

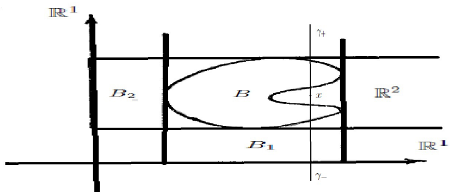

Figure 7: B_1 and B_2 have the line property but B does not have

7. Conclusion

All results of the present work are valid in Euclidean space $\mathbb{R}^n$ as a manifolds without conjugate points [9]. On the other hand, the generalization of Theorem11 to $W^n, n \geq 3$ is more difficult and is left as open problem.

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