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THE SOLUTION OF THE DARBOUX PROBLEM FOR THE TELEGRATH EQUATION WITH DEVIATION FROM THE CHARACTERISTIC

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Abstract: The article deals with the problem of constructing solution of the Darboux problem for telegraph equation for the case with deviation from the characteristic. Preliminarily was constructed Riemann-Hadamard function. Then using the Riemann-Hadamard was constructed a solution of the Darboux problem explicitly.

AMS Subject Classification: 35L20, 35L99
Key Words: Darboux problem, the Riemann-Hadamard function, telegraph equation

1. Introduction and Preliminary

Consider a problem of the Darboux type for the telegrath equation

$$L_0 v = v_{xx} - v_{yy} + cv = 0, \tag{1}$$

where c is an arbitrary complex number in the domain D bounded by the characteristic CB ($x + y = 1$) of equation (1), line AC ($kx - y = 0$), and by the segment AB of the axis y = 0.

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Problem D**.** In the domain D find a function $v(x,y)$ which satisfies the conditions:

$$v(x,y) \in C(\overline{D}) \wedge C^1(D \cup AB) \wedge C^2(D);$$

$$Lv(x,y) \equiv 0, \quad (x,y) \in D;$$

$$v(x,0) = \tau(x), \quad 0 < x < 1;$$

$$v(x,kx) = \varphi(x), \quad 0 < x < \frac{1}{1+k},$$

where $\tau(x)$ and $\phi(x)$ are given sufficiently smooth functions. On the plane (x,y) we pass to the characteristic coordinates $\xi = x+y$, $\eta = x-y$. Then equation (1) takes the form

$$Lu = u_{\xi\eta} + \frac{c}{4}u = 0$$

where

$$u(\xi,\eta) = u\left(\frac{1}{2}(\xi+\eta), \frac{1}{2}(\xi-\eta)\right)$$

and the domain D is mapped to the domain

$$\Delta = \left\{ (\xi,\eta) \mid 0 < \xi < \eta < \alpha\xi < 1, \ \alpha = \frac{1-k}{1+k} > 1 \right\}$$

and, respectively, Problem D is posed as follows:

Problem D'**.** In the domain Δ find a function $v(x,y)$ which satisfies the conditions

$$u(\xi,\eta) \in C(\overline{\Delta}) \wedge C^1(\Delta \cup \{\eta = \alpha\xi\}) \wedge u_{\xi\eta} \in C\left(\Delta\right);$$

$$Lu(\xi,\eta) \equiv 0, \ (\xi,\eta) \in \Delta,$$

$$u(\xi,\alpha\xi) = \tau(\xi), \ 0 \le \xi \le \frac{1}{\alpha};$$

$$u(\xi,\xi) = \psi(\xi), \ 0 \le \xi \le 1;$$

$$\psi(0) = \varphi(0).$$

It is well known that the Riemann-Hadamard function plays an important role in the study of problem D'; this function was defined and constructed in [1-6] for some special cases of Eq. (1).In this section, we present an in a sense modified (as compared with the approaches used in the above-mentioned papers) approach to defining the Riemann-Hadamard function of problem D' for Eq. (1) in case if boundary values is defined on non-characteristic.

2. Construction Function Riemann-Hadamard

Let domain Δ is divided into following subdomains

$$\sigma_0 = \left\{ (\xi,\eta) \middle| \eta < \alpha\xi, \xi < \frac{\eta_0}{\alpha}, \eta > \xi_0 \right\},$$

$$\sigma_{2k} = \left\{ (\xi,\eta) \middle| \eta < \alpha\xi, \xi < \frac{\eta_0}{\alpha^{k+1}}, \eta > \frac{\xi_0}{\alpha^k} \right\},$$

$$\sigma_{2k+1} = \left\{ (\xi,\eta) \middle| \eta < \alpha\xi, \xi < \frac{\xi_0}{\alpha^{k+1}}, \eta > \frac{\eta_0}{\alpha^{k+1}} \right\},$$

$$\omega_0 = \left\{ (\xi,\eta) \middle| \eta < \eta_0, \xi < \xi_0, \eta > \xi_0, \xi > \frac{\eta_0}{\alpha} \right\},$$

$$\omega_{2k} = \left\{ (\xi,\eta) \middle| \xi > \frac{\eta_0}{\alpha^{k+1}}, \xi < \frac{\xi_0}{\alpha^k}, \eta < \frac{\eta_0}{\alpha^k}, \eta > \frac{\xi_0}{\alpha^k} \right\},$$

$$\omega_{2k+1} = \left\{ (\xi,\eta) \middle| \xi < \frac{\eta_0}{\alpha^{k+1}}, \xi > \frac{\xi_0}{\alpha^{k+1}}, \eta > \frac{\eta_0}{\alpha^{k+1}}, \eta < \frac{\xi_0}{\alpha^k} \right\},$$

$$\Delta_{2k+1} = \left\{ (\xi,\eta) \middle| \eta > \xi, \xi > \frac{\eta_0}{\alpha^{k+1}}, \eta < \frac{\xi_0}{\alpha^k} \right\},$$

$$\Delta_{2k} = \left\{ (\xi,\eta) \middle| \eta > \xi, \xi > \frac{\xi_0}{\alpha^k}, \eta < \frac{\eta_0}{\alpha^k} \right\},$$

$$k = 1, 2, ...$$

In what follows, we assume that function is known as the RiemannHadamard function $R(\xi,\eta;\xi_0,\eta_0)$ satisfies conditions

$$1.\ LR(\xi,\eta;\xi_0,\eta_0) = R_{\xi\eta} + cR = 0.$$

$$2.\ R_\xi \Big|_{\eta=\eta_0} = 0, \quad R_\eta \Big|_{\xi=\xi_0} = 0, \quad R \Big|_{\eta=\xi \cup \eta=\alpha\xi} = 0.$$

$$3.\ \frac{\partial\,[R_1]}{\partial\xi} = 0, [R_1] = \lim_{\varepsilon\to 0} \left[R\left(\xi; \frac{\xi_0}{\alpha^{k-1}} + \varepsilon; \xi_0; \eta_0\right) - R\left(\xi; \frac{\xi_0}{\alpha^{k-1}} - \varepsilon; \xi_0; \eta_0\right) \right]$$

$$\frac{\partial\,[R_2]}{\partial\xi} = 0, [R_2] = \lim_{\varepsilon\to 0} \left[R\left(\xi; \frac{\eta_0}{\alpha^k} + \varepsilon; \xi_0; \eta_0\right) - R\left(\xi; \frac{\eta_0}{\alpha^k} - \varepsilon; \xi_0; \eta_0\right) \right]$$

$$\frac{\partial\,[R_3]}{\partial\eta} = 0, [R_3] = \lim_{\varepsilon\to 0} \left[R\left(\frac{\xi_0}{\alpha^k} + \varepsilon; \eta; \xi_0; \eta_0\right) - R\left(\frac{\xi_0}{\alpha^k} - \varepsilon; \eta; \xi_0; \eta_0\right) \right]$$

$$\frac{\partial\left[R_4\right]}{\partial\xi}=0,\left[R_4\right]=\lim_{\varepsilon\to 0}\left[R\left(\frac{\eta_0}{\alpha^k}+\varepsilon;\xi\xi_0;\eta_0\right)-R\left(\frac{\eta_0}{\alpha^k}-\varepsilon;\xi;\xi_0;\eta_0\right)\right]$$

$$4.\ R(\xi,\eta,\xi_0,\eta_0)=1$$

$$k=1,2,....$$

Then the function of the Riemann-Hadamard is determined by the recurrent formulas as follows

$$R_{\sigma_{2k}}=R_{\omega_{2k}}-J_0\left(\sqrt{c\left(\eta\alpha^{-k-1}-\xi_0\right)\left(\alpha^{k+1}\xi-\eta_0\right)}\right),\ (\xi,\eta)\in\sigma_{2k},$$

$$R_{\sigma_{2k+1}}=R_{\omega_{2k+1}}+J_0\left(\sqrt{c(\alpha^{k+1}\xi-\xi_0)\left(\eta\alpha^{-k-1}-\eta_0\right)}\right),\ (\xi,\eta)\in\sigma_{2k+1},$$

$$R_{\Delta_{2k+1}}=R_{\omega_{2k}}-J_0\left(\sqrt{c(\alpha^k\eta-\xi_0)\left(\xi\alpha^{-k}-\eta_0\right)}\right),\ (\xi,\eta)\in\Delta_{2k+1},$$

$$R_{\Delta_{2k+2}}=R_{\omega_{2k+1}}+J_0\left(\sqrt{c(\alpha^{k+1}\eta-\eta_0)\left(\xi\alpha^{-k-1}-\xi_0\right)}\right),\ (\xi,\eta)\in\Delta_{2k+2},$$

$$R_{\omega_{2k+1}}=R_{\Delta_{2k+1}}-J_0\left(\sqrt{c(\alpha^{k+1}\xi-\eta_0)\left(\eta\alpha^{-k-1}-\xi_0\right)}\right),\ (\xi,\eta)\in\omega_{2k+1},$$

$$R_{\omega_{2k+2}}=R_{\Delta_{2k+2}}-J_0\left(\sqrt{c(\alpha^{k+1}\xi-\xi_0)\left(\eta\alpha^{-k-1}-\eta_0\right)}\right),\ (\xi,\eta)\in\omega_{2k+2},$$

where $J_0(\cdot)$ is Bessel function of zero order.

3. Construction Solution of Darboux Problem

One can readily see that

$$u\cdot LR-R\cdot Lu=\frac{1}{2}\left(uR_\eta-Ru_\eta\right)_\xi+\frac{1}{2}\left(uR_\xi-Ru_\xi\right)_\eta \tag{2}$$

By using relation (2), where u is a regular solution of Eq. (1) in the domain Δ and R is function of Riemann-Hadamard, and by applying the Green formula to the above-mentioned subdomains ω_k, σ_k, Δ_k of the domain Δ, one can readily justify the relations

$$0=\int\limits_{(\cup\partial\Delta_i\cup\partial\omega_k\cup\partial\sigma_m)}\left(uR_\xi-Ru_\xi\right)d\xi-\left(uR_\eta-Ru_\eta\right)d\eta=$$

$$= I_{ED} + I_{DC} + I_{CA} + I_{AE},$$

where $D = (\xi_0, \eta_0)$, $C = \left(\frac{\eta_0}{\alpha}, \eta_0\right)$, $B = \left(\frac{\xi_0}{\alpha}, \xi_0\right)$, $A = (0,0)$, $E = (\xi_0, \xi_0)$.

Calculating the integrals I_{ED}, I_{DC}, I_{CA}, I_{AE}, one obtains:

$$I_{ED} = uR_1|_E^D = uR_1(D) - uR_1(E) = u(\xi_0, \eta_0) - \tau(\xi_0)$$

$$I_{DC} = -\, uR_1|_D^C = -uR_1(C) + uR_1(D) = -\psi\left(\frac{\eta_0}{\alpha}\right) + u(\xi_0, \eta_0)$$

Since $d\eta = \alpha d\xi$ on AC then

$$I_{AC} = \int\limits_0^{\frac{\eta_0}{\alpha}} u\left[(R_\xi - \alpha(R_\eta\right] d\xi =$$

$$= \sum_{n=1}^{\infty} \left(\int\limits_0^{\frac{\eta_0}{\alpha^n}} u\left[(R_{\sigma_{2n-2}})_\xi - \alpha(R_{\sigma_{2n-2}})_\eta\right] d\xi + \int\limits_0^{\frac{\xi_0}{\alpha^n}} u\left[(R_{\sigma_{2n-1}})_\xi - \alpha(R_{\sigma_{2n-1}})_\eta\right] d\xi \right) =$$

$$= \sum_{n=1}^{\infty} A\left(\eta_0 - \alpha^{2n-1}\xi_0\right) \int\limits_0^{\frac{\eta_0}{\alpha^n}} \frac{J_1\left(\sqrt{c(\alpha^n\xi - \eta_0)\left(\xi\alpha^{1-n} - \xi_0\right)}\right)}{\sqrt{\left(\alpha^{n-1}\xi - \alpha^{2n-2}\xi_0\right)\left(\alpha^n\xi - \eta_0\right)}} \tau(\xi) d\xi +$$

$$+ \sum_{n=1}^{\infty} A\left(\alpha^{2n-1}\eta_0 - \xi_0\right) \int\limits_0^{\frac{\xi_0}{\alpha^n}} \frac{J_1\left(\sqrt{c(\alpha^n\xi - \xi_0)\left(\xi\alpha^{1-n} - \eta_0\right)}\right)}{\sqrt{\left(\alpha^{n-1}\xi - \alpha^{2n-2}\eta_0\right)\left(\alpha^n\xi - \xi_0\right)}} \tau(\xi) d\xi$$

. Again, on AE, $d\eta = d\xi$. Hence

$$I_{AE} = \int\limits_0^{\xi_0} u\left(R_\xi - R_\eta\right) d\xi = \int\limits_0^{\xi_0} u\left[(R_{\Delta_1})_\xi - (R_{\Delta_1})_\eta\right] d\xi +$$

$$+ \sum_{n=1}^{\infty} \left(\int\limits_0^{\frac{\eta_0}{\alpha^n}} u\left[(R_{\Delta_{2n}})_\xi - \alpha(R_{\Delta_{2n}})_\eta\right] d\xi + \int\limits_0^{\frac{\xi_0}{\alpha^n}} u\left[(R_{\Delta_{2n+1}})_\xi - \alpha(R_{\Delta_{2n+1}})_\eta\right] d\xi \right) =$$

$$= A\left(\eta - \xi\right) \int_0^{\xi} \frac{J_1\left(\sqrt{c(t-\xi)(t-\eta)}\right)}{\sqrt{(t-\xi)\,(t-\eta)}} \psi(t) dt +$$

$$+\sum_{n=1}^{\infty} A\left(\alpha^{2n}\xi_0-\eta_0\right)\int\limits_0^{\frac{\eta_0}{\alpha^n}} \frac{J_1\left(\sqrt{c(\alpha^n\xi-\eta_0)\left(\xi\alpha^{-n}-\xi_0\right)}\right)}{\sqrt{\left(\alpha^n\xi-\alpha^{2n}\xi_0\right)\left(\alpha^n\xi-\eta_0\right)}}\psi(\xi)d\xi+$$

$$+\sum_{n=1}^{\infty} A\left(\xi_0-\alpha^{2n}\eta_0\right)\int\limits_0^{\frac{\xi_0}{\alpha^n}} \frac{J_1\left(\sqrt{c(\alpha^n\xi-\xi_0)\left(\xi\alpha^{-n}-\eta_0\right)}\right)}{\sqrt{\left(\alpha^n\xi-\alpha^{2n}\eta_0\right)\left(\alpha^n\xi-\xi_0\right)}}\psi(\xi)d\xi$$

.

Therefore, one can readily show that the solution of the problem D' can be represented at the point $(\xi,\eta)\in\Delta$ in the form

$$u(\xi,\eta)=\tau(\xi)+\psi\left(\frac{\eta}{\alpha}\right)+A\left(\eta-\xi\right)\int_0^{\xi}\frac{J_1\left(\sqrt{c(t-\xi)(t-\eta)}\right)}{\sqrt{(t-\xi)\left(t-\eta\right)}}\psi(t)dt+$$

$$+\sum_{n=1}^{\infty} A\left(\eta-\alpha^{2n-1}\xi\right)\int\limits_0^{\frac{\eta}{\alpha^n}} \frac{J_1\left(\sqrt{c(\alpha^n t-\eta)\left(t\alpha^{1-n}-\xi\right)}\right)}{\sqrt{\left(\alpha^{n-1}t-\alpha^{2n-2}\xi\right)\left(\alpha^n t-\eta\right)}}\tau(t)dt+$$

$$+\sum_{n=1}^{\infty} A\left(\alpha^{2n-1}\eta-\xi\right)\int\limits_0^{\frac{\xi}{\alpha^n}} \frac{J_1\left(\sqrt{c(\alpha^n t-\xi)\left(t\alpha^{1-n}-\eta\right)}\right)}{\sqrt{\left(\alpha^{n-1}t-\alpha^{2n-2}\eta\right)\left(\alpha^n t-\xi\right)}}\tau(t)dt+$$

$$+\sum_{n=1}^{\infty} A\left(\alpha^{2n}\xi-\eta\right)\int\limits_0^{\frac{\eta}{\alpha^n}} \frac{J_1\left(\sqrt{c(\alpha^n t-\eta)\left(t\alpha^{-n}-\xi\right)}\right)}{\sqrt{\left(\alpha^n t-\alpha^{2n}\xi\right)\left(\alpha^n t-\eta\right)}}\psi(t)dt+$$

$$+\sum_{n=1}^{\infty} A\left(\xi-\alpha^{2n}\eta\right)\int\limits_0^{\frac{\xi}{\alpha^n}} \frac{J_1\left(\sqrt{c(\alpha^n t-\xi)\left(t\alpha^{-n}-\eta\right)}\right)}{\sqrt{\left(\alpha^n\xi-\alpha^{2n}\eta\right)\left(\alpha^n t-\xi\right)}}\psi(t)dt \qquad (3)$$

Theorem 1. *If functions* $\tau(\xi)\in C^1\left[0,\frac{1}{\alpha}\right]$, $\alpha>1$, $\psi(\xi)\in C^2[0,1]$, *there exist a solution to the problem* D' *of the form* (3).

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