EPIMORPHISM OF RINGS AND ABSOLUTELY FLAT MODULES

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Abstract: Let $R \subset S$ be a commutative ring extension. We prove that if $\phi : R \to S$ is an epimorphism of rings with $\phi$ injective homomorphism and $S$ is $R$-projective, then an $R$-module $M$ is $R$-injective and $S$ is absolutely flat over $R$ if and only if $\text{Hom}_R(S, M)$ is $S$-injective and $S$ is an absolutely flat ring.

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1. Introduction

Throughout this article, $R$ denotes a commutative ring with identity and all $R$-modules are unitary. For standard terminology, the references are [2], [10] and [11].

A ring $R$ is said to be an absolutely flat ring if every $R$-module is flat. The concept of absolutely flat modules was introduced and studied in [4]. An $R$-module $M$ is said to be absolutely flat module if for every $R$-module $N$, $M \otimes_R N$ is $R$-flat. Many properties and characterizations of absolutely flat modules are proved in [5], [6], [7] and [8]. In [8] it is proved that if $\phi : R \to S$ is an epimorphism of rings and $S$ a noetherian ring, then $S$ is absolutely flat over $R$ and self injective if and only if $S$ is an absolutely flat ring and $S$ is $R$-injective. In this article we prove that if $\phi : R \to S$ is an epimorphism of rings where $\phi$ is injective and $S$ is $R$-projective then the $R$-module $M$ is $R$-injective and $S$ is
absolutely flat over $R$ if and only if $S$ is an absolutely flat ring and $\text{Hom}_R(S, M)$ is $S$-injective.

We need the following definition and proposition.

**Definition 1.** A ring homomorphism $\phi : R \to S$ is an epimorphism, if for any two ring homomorphism $f : S \to L$ and $g : S \to L$, for some ring $L$ with $f \circ \phi = g \circ \phi$, we have $f = g$.

In that case we have the isomorphism $S \otimes_R S \cong S$ [9]. Also for any $S$—modules $M$ and $N$, we have $M \otimes_R N \cong M \otimes_S N$. Hence for $N = S$, $S \otimes_R M \cong S \otimes_S M \cong M$.

**Proposition 1.** Let $A$ be an $R$-module, $B$ an $(R, S)$-module and $C$ an $S$-module where $R$ and $S$ are rings. Then there is a natural isomorphism

$$\text{Hom}_S(B \otimes_R A, C) \cong \text{Hom}_R(A, \text{Hom}_S(B, C)).$$

Now we prove the main theorem.

**Theorem 2.** Let $\phi : R \to S$ be an epimorphism of rings where $\phi$ is injective and $S$, $R$-projective. Then for an $R$-module $M$ the following are equivalent:

1. $M$ is $R$-injective and $S$ is absolutely flat over $R$
2. $\text{Hom}_R(S, M)$ is $S$-injective and $S$ is an absolutely flat ring.

**Proof:** Let $M$ be $R$-injective and $S$ absolutely flat over $R$. Then for any $S$-module $N$, we show that $N$ is $S$-flat. Since $S$ is absolutely flat over $R$, $N \otimes_R S$ is $R$-flat. Then for every exact sequence of $S$-modules

$$0 \to L' \to L \to L'' \to 0,$$

we get the exact sequence

$$0 \to L' \otimes_R (N \otimes_R S) \to L \otimes_R (N \otimes_R S) \to L'' \otimes_R (N \otimes_R S) \to 0.$$

This implies the sequence

$$0 \to (L' \otimes_R S) \otimes_R N \to (L \otimes_R S) \otimes_R N \to (L'' \otimes_R S) \otimes_R N \to 0$$

is exact. Since $\phi$ is an epimorphism, for any $S$-module $L'$, $L' \otimes_R S \cong L'$. Hence we have the exact sequence

$$0 \to L' \otimes_R N \to L \otimes_R N \to L'' \otimes_R N \to 0$$

and therefore the sequence
$0 \to (L' \otimes_S S) \otimes_R N \to (L \otimes_S S) \otimes_R N \to (L'' \otimes_S S) \otimes_R N \to 0$

is exact which implies the exactness of the sequence

$0 \to L' \otimes_S (N \otimes_R S) \to L \otimes_S (N \otimes_R S) \to L'' \otimes_S (N \otimes_R S) \to 0$.

That is the sequence

$0 \to L' \otimes_S N \to L \otimes_S N \to L'' \otimes_S N \to 0$

is exact and hence $N$ is $S$-flat. So, $S$ is an absolutely flat ring.

Next we prove that $\text{Hom}_R(S, M)$ is $S$-injective. Let

$0 \to L' \to L \to L'' \to 0$

be an exact sequence of $S$-modules. Now $M$ being $R$-injective, the sequence

$0 \to \text{Hom}_R(L'', M) \to \text{Hom}_R(L, M) \to \text{Hom}_R(L', M) \to 0$

is exact. That is the sequence

$0 \to \text{Hom}_R(L'' \otimes_R S, M) \to \text{Hom}_R(L \otimes_R S, M) \to \text{Hom}_R(L' \otimes_R S, M) \to 0$

is exact.

Then by the isomorphism

$\text{Hom}_R(M, \text{Hom}_R(N, K)) \cong \text{Hom}_R(M \otimes_R N, K)$,

we get the exact sequence

$0 \to \text{Hom}_R(L'', \text{Hom}_R(S, M)) \to \text{Hom}_R(L, \text{Hom}_R(S, M))$

$\quad \to \text{Hom}_R(L', \text{Hom}_R(S, M)) \to 0.$

That is,

$0 \to \text{Hom}_R(L'', \text{Hom}_S(S, \text{Hom}_R(S, M)))$

$\to \text{Hom}_R(L, \text{Hom}_S(S, \text{Hom}_R(S, M)))$

$\to \text{Hom}_R(L', \text{Hom}_S(S, \text{Hom}_R(S, M))) \to 0$
is exact. By Proposition 1,

\[ 0 \to \text{Hom}_S(L'' \otimes_R S, \text{Hom}_R(S, M)) \to \text{Hom}_S(L \otimes_R S, \text{Hom}_R(S, M)) \]
\[ \quad \to \text{Hom}_S(L' \otimes_R S, \text{Hom}_R(S, M)) \to 0 \]

is exact. Since \( \phi \) is an epimorphism of rings, \( S \otimes_R L \cong L \) for every \( S \)-module \( L \) and therefore we have the exact sequence

\[ 0 \to \text{Hom}_S(L'', \text{Hom}_R(S, M)) \to \text{Hom}_S(L, \text{Hom}_R(S, M)) \]
\[ \quad \to \text{Hom}_S(L', \text{Hom}_R(S, M)) \to 0. \]

So we proved that \( \text{Hom}_R(S, M) \) is \( S \)-injective.

Next we assume \( \text{Hom}_R(S, M) \) is \( S \)-injective and \( S \) is an absolutely flat ring and we prove that \( M \) is \( R \)-injective and \( S \) is absolutely flat over \( R \). Consider exact sequence of \( R \)-modules

\[ 0 \to N' \to N \to N'' \to 0. \]

Since \( S \) is \( R \)-flat,

\[ 0 \to N' \otimes_R S \to N \otimes_R S \to N'' \otimes_R S \to 0 \]

is exact. Let \( L \) be any \( R \)-module. Now the \( S \)-module \( L \otimes_R S \) is \( S \)-flat. Hence the sequence

\[ 0 \to (N' \otimes_R S) \otimes_S (L \otimes_R S) \to (N \otimes_R S) \otimes_S (L \otimes_R S) \]
\[ \quad \to (N'' \otimes_R S) \otimes_S (L \otimes_R S) \to 0 \]

is exact. That is

\[ 0 \to N' \otimes_R (L \otimes_R S) \to N \otimes_R (L \otimes_R S) \to N'' \otimes_R (L \otimes_R S) \to 0 \]

is exact. Hence \( L \otimes_R S \) is flat over \( R \). Since \( L \) is an arbitrary \( R \)-module, \( S \) is an absolutely flat \( R \)-module.

Since \( \text{Hom}_R(S, M) \) is \( S \)-injective, we get the exact sequence

\[ 0 \to \text{Hom}_S(N'' \otimes_R S, \text{Hom}_R(S, M)) \to \text{Hom}_S(N \otimes_R S, \text{Hom}_R(S, M)) \]
\[ \quad \to \text{Hom}_S(N' \otimes_R S, \text{Hom}_R(S, M)) \to 0. \]

That is
$0 \to \text{Hom}_R(N'', \text{Hom}_R(S, M)) \to \text{Hom}_R(N, \text{Hom}_R(S, M))$

$\to \text{Hom}_R(N', \text{Hom}_R(S, M)) \to 0$

is exact. This implies that $\text{Hom}_R(S, M)$ is $R$-injective.

Since $S$ is $R$-projective and $R \to S$ is a monomorphism, the sequence

$0 \to R \to S \to S/R \to 0$

splits [1]. Hence

$0 \to \text{Hom}_R(S/R, M) \to \text{Hom}_R(S, M) \to \text{Hom}_R(R, M) \to 0$

splits. That is $\text{Hom}_R(R, M) \cong M$ is a direct summand of $\text{Hom}_R(S, M)$ which is $R$-injective. Therefore $M$ is $R$-injective. This completes the theorem.

References


