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A FOURTH ORDER TRIGONOMETRICALLY FITTED METHOD WITH THE BLOCK UNIFICATION IMPLEMENTATION APPROACH FOR OSCILLATORY INITIAL VALUE PROBLEMS

P.L. Ndukum[1 §], T.A. Biala[2], S.N. Jator[3], R.B. Adeniyi[4]

[1]Department of Mathematics
University of Dschang
CAMEROON

[2]Department of Mathematics and Computer Science
Sule Lamido University
P.M.B. 048 Kafin Hausa, NIGERIA

[3]Department of Mathematics
University of Ilorin
Ilorin, NIGERIA

[4]Department of Mathematics and Statistics
Austin Peay State University
Clarksville, TN 37044, USA

Abstract: This paper is concerned with the construction and implementation of a continuous fourth order trigonometrically fitted method on oscillatory initial value problems (IVPs). The continuous scheme is used to generate four discrete methods as by products. The four discrete schemes are weighted thesame and used via the Block Unification Approach to obtain approximate solutions to first order IVPs with oscillating solutions. The convergence of the method is established and two test problems are given to show the accuracy and computational efficiency of the scheme.

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§Correspondence author

1. Introduction

Oscillatory IVPs frequently arise in areas such as quantum mechanics, classical mechanics, celestial mechanics, astrophysics, theoretical physics and chemistry, nuclear physics and biological sciences. A number of numerical methods based on the use of polynomial basis functions have been developed for solving this class of important problems (Gear [4], Cash [1], Ngwane and Jator [8] among others).

Other methods based on exponential fitting techniques which take advantage of the special properties of the solution that may be known in advance have also been proposed (Simos [6], Vanden Berghe *et al* [5], Jator *et al* [2]). For a periodic IVP whose frequency (or a reasonable estimate of it) is known, it is advantangeous to tune a method to take this estimate, thus the motivation for exponentially fitted methods.

In what follows, we consider the system of first order IVP

$$y' = f(x, y), \quad y(a) = y_0, \qquad x \in [a, b] \tag{1}$$

with periodic or oscillatory solutions where $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ satisfies a Lipschitz condition (Lambert [9]) and y, $y_0 \in \mathbb{R}^m$. In this paper, we construct a continuous scheme which provides methods that are combined and applied via the Block Unification Approach (BUA) which takes the frequency of the solution as a priori knowledge. The coefficients of the methods are functions of the frequency and the stepsize.

The paper is organized as follows: In Section 2, we explain the construction of continuous scheme based on a trigonometric basis representation in $U(x)$ for the exact solution $y(x)$ which is used to generate four discrete schemes which are used via the BUA for solving (1). Section 3 details the convergence and implementation of the methods. Numerical test are given to show the accuracy (small errors) and computational efficiency (number of steps and function evaluation) of the BTFEBDM in section 4. Finally, we give some concluding remarks in section 5.

2. The Fourth Order Trigonometrically Fitted Method

In this section, we shall construct a continuous scheme which produces four discrete methods as by-products. The continuous scheme has the general form

$$U(x) = \sum_{r=0}^{2} \alpha_r(w, h, x) y_{n+r} + (\beta_3(w, h, x) f_{n+3} + \beta_4(w, h, x) f_{n+4}) \tag{2}$$

where w is the frequency, h is the stepsize, $\alpha_r(w,h,x)$, $\beta_3(w,h,x)$ and $\beta_4(w,h,x)$, $r = 0(1)(2)$, are coefficients that depend on the frequency and stepsize. We note that y_{n+j} are the numerical solutions to the analytical solutions $y(x_{n+j})$, $j = 1, \cdots, k$ and $f_{n+3} = f(x_{n+3}, y_{n+3})$, $f_{n+4} = f(x_{n+4}, y_{n+4})$.

To obtain (2), we seek an approximation to the exact solution $y(x)$ on the interval $[x_n, x_{n+3}]$ by the interpolating function of the form

$$U(x) = \sum_{r=0}^{2} b_r x^r + b_3 \sin(wx) + b_4 \cos(wx) \approx y(x) \tag{3}$$

with the first derivative given by

$$U'(x) = \sum_{r=0}^{2} r b_r x^{r-1} + w b_3 \cos(wx) - w b_4 \sin(wx) \approx f(x,y) \tag{4}$$

where $b_0, b_1, \cdots, b_4$, are coefficients to be uniquely determined. We then impose that the interpolating function (3) coincides with the analytical solution at the points x_0, x_1, x_2 to obtain the following set of equations:

$$U(x_0) = y_0, \quad U(x_1) = y_1, \quad U(x_2) = y_2 \tag{5}$$

We also require the function (3) to satisfy the differential equation (1) at the points x_3 and x_4 to obtain

$$U'(x_3) = f_3 \qquad U'(x_4) = f_4 \tag{6}$$

Equations (5) and (6) lead to a system of five equations which is solved for the values b_r. The continuous scheme is developed by substituting the values of b_j, $j = 0(1)4$ into (3). After some algebraic manipulations, the continuous scheme is expressed in the form (2).

Letting $u = wh$ and evaluating (2) at x_3, x_4 and also evaluating $U'(x)$ at x_1 and x_2 to obtain the formulas

$$\begin{aligned} y_{n+3} &= \sum_{r=0}^{2} \alpha_{1r}(u) y_{n+r} + h(\beta_{13}(u) f_{n+3} + \beta_{14}(u) f_{n+4}), \\ y_{n+4} &= \sum_{r=0}^{2} \alpha_{2r}(u) y_{n+r} + h(\beta_{23}(u) f_{n+3} + \beta_{24}(u) f_{n+4}), \\ h y'_{n+1} &= \sum_{r=0}^{2} \alpha_{3r}(u) y_{n+r} + h(\beta_{33}(u) f_{n+3} + \beta_{34}(u) f_{n+4}), \end{aligned} \tag{7}$$

$$hy'_{n+2} = \sum_{r=0}^{2} \alpha_{4r}(u) y_{n+r} + h(\beta_{43}(u) f_{n+3} + \beta_{44}(u) f_{n+4}),$$
$$n = 0(4)(N-4).$$

We note that the first two methods in (7) are of $O(h^5)$ and the last two are of $O(h^4)$.

The coefficients of the the methods in (7) are given as

$$\alpha_{10}(u) = \frac{5u - 11u\cos(u) + 7u\cos(2u) - u\cos(3u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}$$

$$\alpha_{11}(u) = \frac{-12u + 23u\cos(u) - 9u\cos(2u) - 3u\cos(3u) + u\cos(4u) - 2\sin(u) - 6u^2\sin(u) - 2\sin(2u) + 2\sin(3u)}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}$$

$$\alpha_{12}(u) = \frac{7u - 5u\cos(u) - 15u\cos(2u) + 17u\cos(3u) - 4u\cos(4u) + 2\sin(u) + 6u^2\sin(u) + 2\sin(2u) - 2\sin(3u)}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}$$

$$\alpha_{20}(u) = \frac{-3u - 4u\cos(u) + 7u\cos(2u) + 2\sin(u) + 6u^2\sin(u) + 2\sin(2u) - 2\sin(3u)}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}$$

$$\alpha_{21}(u) = \frac{8u + 4u\cos(u) - 8u\cos(2u) - 4u\cos(3u)2\sin(u) - 16u^2\sin(u) - 4\sin(2u) + 2\sin(4u)}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}$$

$$\alpha_{22}(u) = \frac{-5u + 7u\cos(u) - 16u\cos(2u) + 17u\cos(3u) - 3u\cos(4u) + 2\sin(u) + 12u^2\sin(u) + 2\sin(3u) - 2\sin(4u)}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}$$

$$\alpha_{30}(u) = \frac{-2u - 4u\cos(u) + 10u\cos(2u) - 4u\cos(3u) - u^2\sin(u) + 5u^2\sin(2u) - 3u^2\sin(3u)}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}$$

$$\alpha_{31}(u) = \frac{6u\cos(u) - 4u\cos(2u) - 6u\cos(3u) + 4u\cos(4u) - 12u^2\sin(2u) + 8u^2\sin(3u)}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}$$

$$\alpha_{32}(u) = \frac{2u - 2u\cos(u) - 6u\cos(2u) + 10u\cos(3u) - 4u\cos(4u) + u^2\sin(u) + 7u^2\sin(2u) - 5u^2\sin(3u)}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}$$

$$\alpha_{40}(u) = \frac{2u - 6u\cos(u) + 6u\cos(2u) - 2u\cos(3u) + 6u^2\sin(u) - 3u^2\sin(2u))}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}$$

$$\alpha_{41}(u) = \frac{-2u + 4u\cos(u) - 4u\cos(3u) + 2u\cos(4u) - 16u^2\sin(u) + 8u^2\sin(2u))}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}$$

$$\alpha_{42}(u) = \frac{2u\cos(u) - 6u\cos(2u) + 6u\cos(3u) - 2u\cos(4u) + 10u^2\sin(u) - 5u^2\sin(2u))}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}$$

$$\beta_{13}(u) = \frac{2u\cos(u) - 6u\cos(2u) + 6u\cos(3u) - 2u\cos(4u) + 25\sin(u) - 20\sin(2u) + 5\sin(3u)}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}$$

$$\beta_{14}(u) = \frac{-2u + 6u\cos(u) - 6u\cos(2u) + 2u\cos(3u) - 15\sin(u) + 12\sin(2u) - 3\sin(3u)}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}$$

$$\beta_{23}(u) = \frac{2u - 12u\cos(2u) + 16u\cos(3u) - 6u\cos(4u) + 20\sin(u) - 2\sin(2u) - 12\sin(3u) + 5\sin(4u)}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}$$

$$\beta_{24}(u) = \frac{10u\cos(u) - 16u\cos(2u) + 6u\cos(3u) - 8\sin(u) - 2\sin(2u) + 8\sin(3u) - 3\sin(4u)}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}$$

$$\beta_{33}(u) = \frac{-12u + 12u\cos(u) - u\cos(2u) + u\cos(4u) + 12\sin(u) - 6\sin(2u) + 2u^2\sin(3u)}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}$$

$$\beta_{34}(u) = \frac{8u - 7u\cos(u) - u\cos(3u) - 8\sin(u) + 4\sin(2u) - 2u^2\sin(2u)}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}$$

$$\beta_{43}(u) = \frac{7u - 12u\cos(u) + 2u\cos(2u) + 4u\cos(3u) - u\cos(4u) + 8\sin(u) - 4\sin(2u) + 2u^2\sin(2u)}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}$$

$$\beta_{44}(u) = \frac{-5u + 11u\cos(u) - 7u\cos(2u) + u\cos(3u) - 4\sin(u) - 2u^2\sin(u) + 2\sin(2u)}{7u\cos(u) - 17u\cos(2u) + 13u\cos(3u) - 3u\cos(4u) + 4\sin(u) + 2u^2\sin(u) - 2\sin(2u)}.$$

Remark. While Simos [6] has noted that for small values of u, the coefficients of (7) are subject to heavy cancellations, we remark here that there is also an overflow in computations for values of u near the singularity point u_0 of the coefficients of (7). For example, the coefficients $\alpha_{ij}(u)$ and $\beta_{ij}(u)$ have a singularity at the point $u_0 = 2 : 51530574522367...$ and so for values of u near u_0, the computations will be inaccurate. In such cases, the Truncated series expansion s of these coefficients are used for better accuracy.

The Taylor's expansions of the coefficients are:

$$\begin{aligned}
\alpha_{10}(u) =& \frac{17}{197} + \frac{894u^2}{38809} + \frac{461345u^4}{107035222} + \frac{2284188931u^6}{3162890810100} + \frac{223594132156099u^8}{1919115627936276000} \\
&+ \frac{8205133165743716353u^{10}}{442336961083032255240000} + \cdots \\
\alpha_{11}(u) =& -\frac{99}{197} - \frac{9066u^2}{194045} - \frac{3407179u^4}{382268650} - \frac{23842789861u^6}{15814454050500} - \frac{2343699811589527u^8}{9595578139681380000} \\
&- \frac{86154395143019747611u^{10}}{2211684805415161276200000} - \cdots \\
\alpha_{12}(u) =& \frac{279}{197} + \frac{4596u^2}{194045} + \frac{6158314u^4}{1337940275} + \frac{6210922603u^6}{7907227025250} + \frac{153216143851129u^8}{1199447267460172500} \\
&+ \frac{22564364657150582923u^{10}}{1105842402707580638100000} + \cdots \\
\alpha_{20}(u) =& \frac{9}{197} + \frac{2436u^2}{194045} + \frac{3105994u^4}{1337940275} + \frac{1500403064u^6}{3953613512625} + \frac{143788253445923u^8}{2398894534920345000} \\
&+ \frac{2602841628198678457u^{10}}{276460600676895159525000} + \cdots \\
\alpha_{21}(u) =& -\frac{64}{197} - \frac{6816u^2}{194045} - \frac{7096244u^4}{1337940275} - \frac{3248110594u^6}{3953613512625} - \frac{152940191264459u^8}{1199447267460172500} \\
&- \frac{5499751705515288527u^{10}}{276460600676895159525000} - \cdots \\
\alpha_{22}(u) =& \frac{252}{197} + \frac{876u^2}{38809} + \frac{159610u^4}{53517611} + \frac{349541506u^6}{790722702525} + \frac{32418425816599u^8}{479778906984069000} \\
&+ \frac{289691007731661007u^{10}}{27646060067689515952500} + \cdots
\end{aligned}$$

$$\begin{aligned}
\alpha_{30}(u) =& -\frac{579}{197} - \frac{17263u^2}{388090} - \frac{176682599u^4}{32110566600} - \frac{19817185739u^6}{27110492658000} - \frac{1224531014847388 1u^8}{115146937676176560000} \\
& - \frac{4313538486508233571 43u^{10}}{26540217664981935314400000} - \cdots \\
\alpha_{31}(u) =& -\frac{120}{197} + \frac{4142u^2}{38809} + \frac{2875061u^4}{229361190} + \frac{2166046811u^6}{1355524632900} + \frac{261744187891057u^8}{1151469376761765600} \\
& + \frac{456919281742677574 3u^{10}}{13270108832490967657 20000} + \cdots \\
\alpha_{32}(u) =& \frac{177}{197} - \frac{24157u^2}{388090} - \frac{225825941u^4}{32110566600} - \frac{23503750481u^6}{27110492658000} - \frac{13929108640631819u^8}{115146937676176560000} \\
& - \frac{481630007697712157717u^{10}}{26540217664981935314400000} - \cdots \\
\alpha_{40}(u) =& \frac{27}{197} + \frac{12843u^2}{388090} + \frac{60990453u^4}{10703522200} + \frac{19056022037u^6}{21085938734000} + \frac{1809410675362809u^8}{12794104186241840000} \\
& + \frac{72680181712166287 43u^{10}}{3276570082096535224 00000} + \cdots \\
\alpha_{41}(u) =& -\frac{192}{197} - \frac{14144u^2}{194045} - \frac{49456828u^4}{4013820825} - \frac{22873560728u^6}{11860840537875} - \frac{1077565341326983u^8}{3598341802380517500} \\
& - \frac{2771621871323757041u^{10}}{59241557287906105612500} - \cdots \\
\alpha_{42}(u) =& \frac{165}{197} + \frac{3089u^2}{77618} + \frac{42536653u^4}{6422113320} + \frac{38894554663u^6}{3795489721200} + \frac{727895793767927u^8}{4605877507047062400} \\
& + \frac{1305954252968992452 37u^{10}}{53080435329963870628 80000} + \cdots \\
\beta_{13}(u) =& \frac{150}{197} + \frac{831u^2}{38809} + \frac{3830447u^4}{1070352220} + \frac{3567477311u^6}{6325781620200} + \frac{339429774283661u^8}{38382312558725520 00} \\
& + \frac{98457622767238789u^{10}}{70773913773285160838400} + \cdots \\
\beta_{14}(u) =& -\frac{18}{197} - \frac{4281u^2}{194045} - \frac{20718241u^4}{5351761100} - \frac{19839187657u^6}{31628908101000} - \frac{24839816252927u^8}{249235795835880000} \\
& - \frac{697421412006894112 87u^{10}}{442336910830322552400000} - \cdots \\
\beta_{23}(u) =& \frac{288}{197} + \frac{2424u^2}{194045} + \frac{895354u^4}{1337940275} + \frac{882141899u^6}{3953613512625} + \frac{98028564353243u^8}{2398894534920345000} \\
& + \frac{3735341010807698849u^{10}}{552921201353790319050000} + \cdots \\
\beta_{24}(u) =& \frac{60}{197} + \frac{96u^2}{38809} - \frac{50846u^4}{38226865} - \frac{225889273u^6}{790722702525} - \frac{23266487998063u^8}{479778906984069000} \\
& - \frac{17293911636174248 3u^{10}}{2211684805415127620 00} + \cdots \\
\beta_{33}(u) =& -\frac{51}{197} - \frac{14u^2}{194045} - \frac{13456061u^4}{8027641650} - \frac{2650193471u^6}{6777623164500} - \frac{2008953479519759u^8}{28786734419044140000} \\
& - \frac{5878143289107718379u^{10}}{510388801249652602200000} - \cdots \\
\beta_{34}(u) =& \frac{14}{197} + \frac{3461u^2}{194045} + \frac{51483793u^4}{16055283300} + \frac{7143669313u^6}{13555246329000} + \frac{694258029302641u^8}{8224781262584040000}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{17796980504024507814 1u^{10}}{13270108832490967657200000}+\cdots\\
\beta_{43}(u) =&\frac{76}{197}+\frac{3169u^2}{194045}+\frac{54345797u^4}{16055283300}+\frac{57041380439u^6}{94886724303000}+\frac{522861325610593u^8}{5233951712553480000}\\
&+\frac{214020167664336841589u^{10}}{13201088324909676572000000}+\cdots\\
\beta_{44}(u) =&-\frac{17}{197}-\frac{894u^2}{38809}-\frac{461345u^4}{107035222}-\frac{2284188931u^6}{3162890810100}-\frac{223594132156099u^8}{1919115627936276000}\\
&-\frac{820513316574371635 3u^{10}}{442336961083032255240000}-\cdots
\end{aligned}
$$

3. Convergence Analysis and Implementation

In this section, we discuss the convergence and implementation of the methods in section 2. Equation (7) can be compactly written in matrix form by introducing the following matrix notations. Let P be an $N \times N$ matrix defined by

$$
P=\begin{pmatrix}
\alpha_{11} & \alpha_{12} & 1 & 0 \\
\alpha_{21} & \alpha_{22} & 0 & 1 \\
\alpha_{31} & \alpha_{32} & 0 & 0 \\
\alpha_{41} & \alpha_{42} & 0 & 0 \\
 & & & & \alpha_{10} & \alpha_{11} & \alpha_{12} & 1 & 0 \\
 & & & & \alpha_{20} & \alpha_{21} & \alpha_{2} & 0 & 1 \\
 & & & & \alpha_{30} & \alpha_{31} & \alpha_{32} & 0 & 0 \\
 & & & & \alpha_{40} & \alpha_{41} & \alpha_{42} & 0 & 0 \\
 & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \\
 & & & & & & & & & & \alpha_{10} & \alpha_{11} & \alpha_{12} & 1 & 0 \\
 & & & & & & & & & & \alpha_{20} & \alpha_{21} & \alpha_{22} & 0 & 1 \\
 & & & & & & & & & & \alpha_{30} & \alpha_{31} & \alpha_{32} & 0 & 0 \\
 & & & & & & & & & & \alpha_{40} & \alpha_{41} & \alpha_{42} & 0 & 0
\end{pmatrix},
$$

Similarly, let Q be an $N \times N$ matrix defined by

$$Q = h\begin{pmatrix}
0 & 0 & \beta_{13} & \beta_{14} & & & & & & & & \\
0 & 0 & \beta_{23} & \beta_{24} & & & & & & & & \\
1 & 0 & \beta_{33} & \beta_{34} & & & & & & & & \\
0 & 1 & \beta_{43} & \beta_{44} & & & & & & & & \\
 & & & & 0 & 0 & \beta_{13} & \beta_{14} & & & & \\
 & & & & 0 & 0 & \beta_{23} & \beta_{24} & & & & \\
 & & & & 1 & 0 & \beta_{33} & \beta_{34} & & & & \\
 & & & & 0 & 1 & \beta_{43} & \beta_{44} & & & & \\
 & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & & & & & & 0 & 0 & \beta_{13} & \beta_{14} \\
 & & & & & & & & 0 & 0 & \beta_{23} & \beta_{24} \\
 & & & & & & & & 1 & 0 & \beta_{33} & \beta_{34} \\
 & & & & & & & & 0 & 1 & \beta_{43} & \beta_{44}
\end{pmatrix},$$

We define further the following vectors

$$\begin{aligned}
\overline{Y} &= (y_1, \cdots, y_N)^T, \\
Y &= (y(x_1), \cdots, y(x_N))^T, \\
F &= (f_1, \cdots, f_N)^T, \\
L(h) &= \left(\tau_1, \tau_2, h\tau'_3, h\tau'_4, \cdots, \tau_{N-3}, \tau_{N-2}, h\tau'_{N-1}, h\tau'_N\right)^T, \\
C &= (\alpha_{10}y_0, \alpha_{20}y_0, \alpha_{30}y_0, \alpha_{40}y_0, 0 \cdots, 0)^T
\end{aligned}$$

The exact form of the system formed by (7) is given by

$$PY - QF(Y) + C + L(h) = 0, \tag{8}$$

where $L(h)$ is the truncation error vector of the formulas in (7). The approximate form of the system is given by

$$P\overline{Y} - QF(\overline{Y}) + C = 0, \tag{9}$$

where $\overline{Y}$ is the approximate solution of vector Y.
Subtracting (8) from (9) and letting $E = \overline{Y} - Y = (e_1, e_2, \cdots, e_N)^T$ and using the Mean value theorem, we have the error system

$$(P - QB)E = L(h), \tag{10}$$

where B is the Jacobian matrix whose entries are $\dfrac{\partial f_i}{\partial y_i}$, $i = 1(1)N$.
Let $M = -QB$ be a matrix of dimension N so that (10) becomes

$$(P + M)E = L(h), \tag{11}$$

and for sufficiently small h, $P + M$ is a monotone matrix and thus nonsingular (Jain and Aziz [3]). Therefore

$$(P + M)^{-1} = D = (d_{ij}) \geq 0 \quad and \sum_{j=1}^{N} d_{ij} = O(h^{-1}),$$

and

$$\begin{aligned} E &= DL(h), \\ ||E|| &= ||DL(h)||, \\ &= O(h^{-1})O(h^5), \\ &= O(h^4). \end{aligned}$$

which shows that the our method is fourth order convergent.

The BUA makes use of each of the methods in (7) in steps of 4, that is $n = 0, 4, \cdots, N - 4$ and this results in a system of N equations in N unknowns which can be easily solved for the unknowns. This approach has the advantage of simultaneously generating approximate solutions $(y_1, \cdots, y_N)^T$ to the exact-solution $(y(x_1), \cdots, y(x_N))^T$ of (1) on the entire interval integration in just one block.

4. Test Examples

In this section, we give two numerical examples to illustrate the accuracy and efficiency of the our method. We give the errors at the endpoints calculated as Error=$|y_N - y(x_N)|$. We note that the methods require only one function evaluation (FEs) per step and in general require $(N+1)$ FEs on the entire interval. All computations were carried out using a written code in Mathematica 9.0.

Example 4.1. We consider the following inhomogeneous IVP by Simos [6].

$$y'' = -100y + 99\sin(x), \quad y(0) = 1, \quad y'(0) = 11, \quad x \in [0, 1000]$$

where the analytic solution is given by

$$\text{Exact} : y(x) = \cos(10x) + \sin(10x) + \sin(x).$$

Table 1: Results, with $\omega = 10$, for Example 4.1

	Our method		Simos [6]	
N	$\|Error\|$	NFEs	$\|Error\|$	NFEs
1000	2.3×10^{-2}	2002	1.4×10^{-1}	8000
2000	4.3×10^{-4}	4002	3.5×10^{-2}	16000
4000	1.7×10^{-5}	8002	1.1×10^{-3}	32000
8000	1.1×10^{-6}	16002	8.4×10^{-5}	64000
16000	6.3×10^{-7}	32002	5.5×10^{-6}	128000

Example 4.2. We consider the nonlinear Duffing equation which was also solved by Simos [6] using a fourth order method.

$$y'' + y + y^3 = B\cos(\Omega x),\ y(0) = C_0,\ y'(0) = 0,\ x \in [0, 300].$$

The analytic solution is given by

$$\text{Exact} : y(x) = C_1\cos(\Omega x) + C_2\cos(3\Omega x) + C_3\cos(5\Omega x) + C_4\cos(7\Omega x),$$

where $\Omega = 1.01,\ B = 0.002,\ C_0 = 0.200426728069,\ C_1 = 0.200179477536,\ C_2 = 0.246946143 \times 10^{-3},\ C_3 = 0.304016 \times 10^{-6},\ C_4 = 0.374 \times 10^{-9}$. We choose $\omega = 1.01$

Example 4.3. Linear Kramarz problem

We consider the following second-order IVP, (see Nguyen *et al.* [7])

$$y''(t) = \begin{pmatrix} 2498 & 4998 \\ -2499 & -4999 \end{pmatrix} y(t), \quad y(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad y'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad 0 \leq t \leq 100.$$

$$\text{Exact} : y(t) = (2\cos(t), -\cos(t))^T.$$

Table 2: Results, with $\omega = 1.01$, for Example 4.2.

Our Method			Simos [6]		
N	$\|Error\|$	NFEs	N	$\|Error\|$	NFEs
150	2.3×10^{-4}	302	600	5.4×10^{-4}	2400
300	4.5×10^{-5}	602	1200	3.1×10^{-5}	4800
600	2.7×10^{-6}	1202	2400	1.8×10^{-6}	9600
1200	1.7×10^{-7}	2402	4800	1.1×10^{-7}	19200

We compared the end-point global errors for our method with the fourth order exponentially fitted method in Simos [6] and the ”trigonometric implicit Runge-Kutta”, TIRK3 of Nguyen *et al* [7]. From Table 1, 2 and 3, we observe thatour method produces high accuracy with lesser computational effort (Numer of steps and function evaluation) than the other two methods.

5. Conclusion

In this paper we have constructed and implemented a fourth order trigonometrically fitted method on oscillatory initial value problems. The method is applied as a Block unification method to obtain the approximate solutions on the entire interval of integration. We established the convergence of the method. We have also shown that the method is competitive with existing methods cited in the literature.

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Table 3: Results, with $\omega = 1$, for Example 4.3

BTFEBDM 3			Nguyen *et al.* [7]		
N	$\|Error\|$	NFEs	N	$\|Error\|$	NFEs
3	9.2×10^{-13}	16	73	3.3×10^{-12}	327
6	7.0×10^{-13}	28	142	0.9×10^{-11}	707
25	2.4×10^{-13}	104	170	3.7×10^{-12}	811

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