BONDAGE AND NON-BONDAGE NUMBER OF A FUZZY GRAPH

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Abstract: In this paper, bondage and non-bondage set of a fuzzy graph are discussed. The bondage number $b(G)$ and non-bondage number $b_n(G)$ of a fuzzy graph G are defined. The upper bound for both $b(G)$ and $b_n(G)$ are given. Also some results on $b(G)$ and $b_n(G)$ are discussed. The exact values of $b(G)$ and $b_n(G)$ are determined for several classes of fuzzy graphs.

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1. Introduction

Cockayne and Hedetniemi [2] introduced the domination number and the independent domination number of graphs but the concept of dominating sets in graphs was introduced by Ore and Berge [1,10]. In 1990, the concept of the bondage number in graphs was introduced by Fink, Jacobson, Kinch and Roberts [3]. Later in 1994, Hartnell and Douglas. F. Rall [4] discussed about the bounds on the bondage number. Kulli and Janakiram [5] introduced the non-bondage number in graphs. The concept of fuzzy relation was introduced

by Zadeh [14] in his classical paper in 1965. Rosenfeld [11] introduced the notion of fuzzy graph and several fuzzy analogs of graph theoretic concepts such as paths, cycles and connectedness. In 1998, A. Somasundram and S. Somasundram [12] discussed domination in fuzzy graphs using effective edges. Nagoor Gani and Chandrasekaran [6] discussed domination in fuzzy graph using strong arcs. Nagoor Gani and Vadivel [9] discussed domination, independent domination and irredundance in fuzzy graphs using strong arcs. Nagoor Gani and Prasanna Devi [7] discussed edge dominating set and fuzzy edge independent set in fuzzy graphs.

2. Preliminaries

The *bondage number* of a graph G is the minimum cardinality of a set of edges of G whose removal from G results in a graph with domination number larger than that of G . The *non-bondage number* of a graph is the maximum cardinality among all sets of edges $X \subseteq E$ such that the domination number of $G-X$ is same as the domination number of G .

A *fuzzy graph* $G = \langle \sigma, \mu \rangle$ is a pair of functions $\sigma : V \rightarrow [0, 1]$ and $\mu : V \times V \rightarrow [0, 1]$, where for all $x, y \in V$, we have $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$. The *underlying crisp graph* of a fuzzy graph $G = \langle \sigma, \mu \rangle$ is denoted by $G^* = \langle \sigma^*, \mu^* \rangle$, where $\sigma^* = \{v_i \in V / \sigma(v_i) > 0\}$ and $\mu^* = \{(v_i, v_j) \in V \times V / \mu(v_i, v_j) > 0\}$. An edge in G is called an *isolated edge* if it is not adjacent to any edge in G . A fuzzy graph $G = \langle \sigma, \mu \rangle$ is a *complete fuzzy graph* if $\mu(v_i, v_j) = \sigma(v_i) \wedge \sigma(v_j)$ for all $v_i, v_j \in \sigma^*$. An arc (x, y) in a fuzzy graph $G = \langle \sigma, \mu \rangle$ is said to be *strong* if $\mu^\infty(x, y) = \mu(x, y)$. A subset D of V is called a *dominating set* of a fuzzy graph G if for every $v \in V - D$, there exist $u \in D$ such that u dominates v . The *domination number*, $\gamma(G)$, is the smallest number of nodes in any dominating set of G . A subset D of $E(G)$ is said to be an *edge dominating set* of G if for every $e_j \in E(G) - D$ there exist $e_i \in D$ such that e_i dominates e_j . The smallest number of edges in any edge dominating set of G is called its *edge domination number* and it is denoted by $\gamma'(G)$.

3. Bondage Number

In this section we define bondage set and bondage number of fuzzy graphs. The upper bound for the bondage number is also discussed and some results are given.

Definition 3.1. Let G be a fuzzy graph. If there exist a set $X \subseteq S$ such that $\gamma(G - X) > \gamma(G)$ then X is said to be a *bondage set* of G , where S is the set of all strong arcs in G .

Definition 3.2. The *bondage number* $b(G)$ of a fuzzy graph G is the minimum cardinality among all bondage sets of G .

Example 3.3.

1. Here $S = \{e_1, e_2, e_3\}$, $\gamma(G) = 2$ and $X = \{e_1, e_2, e_3\} \implies \gamma(G - X) = 3$.

Fig. 3.1 (G)

Thus X is the only bondage set of G and therefore $b(G) = 3$.

2. Here $S = \{e_2, e_3, e_6, e_7\}$ and $\gamma(G) = 1$.

Fig. 3.2 (G)

$\{e_2\}, \{e_3\}, \{e_6\}, \{e_7\}, \{e_2, e_3\}, \{e_2, e_6\}, \{e_2, e_7\}, \{e_3, e_6\}, \{e_3, e_7\}, \{e_6, e_7\}$ are some bondage set of G and $b(G) = 1$.

Remark 3.4. For some fuzzy graph G , $\gamma(G - X) \not> \gamma(G)$ for any set $X \subseteq S$ then we say that G does not have a bondage set.

Fig. 3.3 (G)

Here $S = \{e_1, e_3, e_5\}$ and $\gamma(G) = 2$.

Thus $\gamma(G - X) \not\geq \gamma(G)$ for any set $X \subseteq S$. Therefore G does not have a bondage set.

Theorem 3.5. *If a fuzzy graph G has an isolated edge then $b(G) = 1$.*

Proof. Let G be a fuzzy graph with an isolated edge e . Let u and v be the end nodes of the isolated edge e . Clearly e is a strong arc and so u or v belongs to the minimum dominating set of G , but not both. Thus deletion of e results u and v as isolated nodes. Therefore u and v both belongs to every dominating set of $G - e$. Thus $\gamma(G - e) > \gamma(G)$ and $\{e\}$ is a bondage set of G . Therefore $b(G) = 1$.

Theorem 3.6. *If a fuzzy graph G has a bondage set then*

$$b(G) \leq \min\{|N_S(u)| + |N_S(v)| - 1 + |S'| : u \text{ and } v \text{ are strong neighbours}\},$$

where $S' = E - S$, set of all non-strong arcs of G .

Proof. Let G be a fuzzy graph which has a bondage set. Let us prove the theorem in two cases as below.

Case (i) Let $|S'| = 0$ i.e., G has only strong arcs and no non-strong arcs. Let $\lambda = \min\{|N_S(x)| + |N_S(y)| - 1 + |S'| : x \text{ and } y \text{ are strong neighbours}\}$. Let u and v be strong neighbours of G such that $|N_S(u)| + |N_S(v)| - 1 + |S'| = \lambda$. Assume that $b(G) > \lambda$. Let S_1 denote the set of all strong arcs that are incident with atleast one of u or v then $|S_1| = \lambda$ and so $\gamma(G - S_1) = \gamma(G)$. Since u and v have no (strong) arcs incident on it, in $G - S_1$ then $\gamma(G - u - v) = \gamma(G) - 2$. If D is a minimum dominating set of the fuzzy graph $G - u - v$, then the set $D - u$ is a dominating set of G with cardinality $\gamma(G) - 1$, a contradiction. Therefore $b(G) \leq \lambda(G)$.

Case (ii) Let $|S'| \neq 0$. Now let us prove this by induction on $|S'|$. If $|S'| = 1$. Then there is exactly one non-strong arc e in G i.e., $G - e$ has only strong arc. Then by Case (i), $b(G - e) \leq \min\{|N_S(u)| + |N_S(v)| - 1 : u \text{ and } v \text{ are strong neighbours}\}$. Let X be a bondage set of $G - e$ such that $|X| = b(G - e)$. Thus e may be a strong or non-strong arc in the fuzzy graph $G - X$. Suppose e is a non-strong arc in $G - X$ then X is also a bondage set of G . Therefore $b(G) = b(G - e)$.

Suppose e is a strong arc in $G - X$ then $X \cup e_1$ is a bondage set of G for some $e_1 \in S - X$. Therefore $b(G) = b(G - e) + 1$. Thus

$$b(G) \leq b(G - e) + 1 = \min\{|N_S(u)| + |N_S(v)| - 1 /$$

$$\begin{aligned}
& u \text{ and } v \text{ are strong neighbours} \} + 1 \\
& = \text{Min}\{|N_S(u)| + |N_S(v)| - 1 + 1/ \\
& \quad u \text{ and } v \text{ are strong neighbours} \} \\
b(G) & \leq \text{Min}\{|N_S(u)| + |N_S(v)| - 1 + |S'|/ \\
& \quad u \text{ and } v \text{ are strong neighbours} \}.
\end{aligned}$$

The result is true for $|S'| = 1$. Assume that this result is true for $|S'| < k$. We have to show that $b(G) \leq \text{Min}\{|N_S(u)| + |N_S(v)| - 1 + |S'|/u \text{ and } v \text{ are strong neighbours} \}$ is true for $|S'| = k$.

If $|S'| = k$ then G has k non-strong arcs. Let e be one of its non-strong arc. Then $G - e$ has $k-1 (< k)$ non-strong arcs. By our assumption $b(G - e) \leq \text{Min}\{|N_S(u)| + |N_S(v)| - 2 + |S'|/u \text{ and } v \text{ are strong neighbours} \}$, because non-strong arcs in $G - e$ is equal to $|S'| - 1$. e may be a strong or a non-strong arc in $G - X_1$, where X_1 is a bondage set of $G - e$ such that $|X_1| = b(G - e)$.

Suppose e does not becomes a strong arc in $G - X_1$, then X_1 is also a bondage set of G . Therefore $b(G) = b(G - e)$. Suppose e becomes a strong arc in $G - X_1$, then $X_1 \cup e_1$ is a bondage set of G for some $e_1 \in S - X_1$. Therefore $b(G) = b(G - e) + 1$. $b(G) \leq b(G - e) + 1 \leq \text{Min}\{|N_S(u)| + |N_S(v)| - 2 + |S'|/u \text{ and } v \text{ are strong neighbours} \} + 1$ $b(G) \leq \text{Min}\{|N_S(u)| + |N_S(v)| - 1 + |S'|/u \text{ and } v \text{ are strong neighbours} \}$. Hence the result.

Corollary 3.7. *If a fuzzy graph G has a bondage set then $b(G) \leq \delta_S(G) + \Delta_S(G) - 1 + |S'|$.*

Proof. By the above theorem,

$$\begin{aligned}
b(G) & \leq \text{Min}\{|N_S(u)| + |N_S(v)| - 1 + |S'|/u \text{ and } v \text{ are strong neighbours} \} \\
& \leq |N_S(u)| + |N_S(v)| - 1 + |S'|.
\end{aligned}$$

Let u be a node of G such that $|N_S(u)| = \delta_S(G)$.

Thus

$$\begin{aligned}
b(G) & \leq \delta_S(G) + |N_S(v)| - 1 + |S'| \\
b(G) & \leq \delta_S(G) + \Delta_S(G) - 1 + |S'|
\end{aligned}$$

(since $|N_S(v)| \leq \Delta_S(G)$).

4. Bondage number for specific fuzzy graphs

The bondage number for a complete fuzzy graph is given. And the bondage number for some specific fuzzy graphs are stated in this section.

Theorem 4.1. *For a complete fuzzy graph G ,*

$$b(G) = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ (n+1)/2, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let G be a complete fuzzy graph with n nodes namely $v_1, v_2, \dots, v_n$. In G , each node dominates all other $n-1$ nodes. Thus $\{v_i\}, i = 1, 2, \dots, n$ are all minimum dominating sets of G and $\gamma(G) = 1$.

Now delete the arc (v_1, v_2) then v_1 and v_2 dominates all $n-2$ nodes other than v_2 and v_1 respectively. Similarly we delete the arcs $(v_3, v_4), (v_5, v_6)$ and so on.

If n is even then delete the arcs $(v_1, v_2), (v_3, v_4), \dots, (v_{n-3}, v_{n-2})$ and (v_{n-1}, v_n) . Thus we get $n/2$ such arcs and these forms a bondage set of G . And so $b(G) = n/2$. If n is odd then delete the arcs $(v_1, v_2), (v_3, v_4), \dots, (v_{n-2}, v_{n-1})$ and (v_n, v_1) . Thus we get $(n+1)/2$ such arcs and these forms a bondage set of G . And so $b(G) = (n+1)/2$. Therefore

$$b(G) = \begin{cases} n/2, & \text{if } n \text{ is even} \\ (n+1)/2, & \text{if } n \text{ is odd} \end{cases}$$

Theorem 4.2. *If G is a fuzzy graph and G^* is a star then $b(G) = 1$.*

Proof. Let G be a fuzzy graph and G^* be a star.

In G , all arcs are strong arcs and the node in the centre dominates all other nodes in G . Therefore $\gamma(G) = 1$. Deletion of any one arc e from G results $\gamma(G-e) = 2 > \gamma(G)$. Thus each edge will form a bondage set and the bondage number, $b(G) = 1$.

Theorem 4.3. *If G is a fuzzy graph and G^* is a cycle with n nodes. Then*

$$b(G) = \begin{cases} 3, & \text{if } n = 3m + 1, m = 1, 2, \dots \\ 2, & \text{otherwise} \end{cases}$$

Proof. Let G be a fuzzy graph and G^* be a cycle with n nodes.

Case (i) Suppose G has more than one weakest arc then all the n arcs of G are strong arcs.

If $n = 3m + 1$ then $\gamma(G) = m + 1$. The domination number increases only if we delete minimum 3 strong arcs. Therefore $b(G) = 3$.

If $n \neq 3m + 1$ then the domination number increases when we delete minimum 2 strong arcs adjacent to the same node. Therefore $b(G) = 2$.

Case (ii) Suppose G has only one weakest arc, say e , then G has $n-1$ strong arcs and deletion of any one strong arc makes the weakest arc as a strong arc in $G - e_1, e_1 (\neq e) \in S$. Clearly $b(G) = 3$ if $n = 3m + 1$ and $b(G) = 2$ if $n \neq 3m + 1$ but the weakest arc does not belongs to any bondage set of G .

Corollary 4.4. Let P_n be a fuzzy graph and it is a path with $n(\geq 2)$ nodes then

$$b(P_n) = \begin{cases} 2, & \text{if } n = 3m + 1, m = 1, 2, \dots \\ 1, & \text{otherwise} \end{cases}$$

5. Non-Bondage Number

In this section, non-bondage set and non-bondage number of a fuzzy graph are defined. Some results on non-bondage number are given. The upper bound for non-bondage number is also defined. And non-bondage number for complete fuzzy graph is given.

Definition 5.1. The set of strong arcs $X \subseteq S$ is called a *non-bondage set* if $\gamma(G - X) = \gamma(G)$ where S is the set of all strong arcs in G .

Example 5.2.

Fig. 5.1 (G)

In this fuzzy graph G , $S = \{e_1, e_2, e_4\}$ and $\gamma(G) = 2$,

$$\{e_1, e_2\}, \{e_1, e_4\}, \{e_2, e_4\}, \{e_1\}, \{e_2\}, \{e_4\}$$

are all non-bondage sets of G .

Definition 5.3. The *non-bondage number*, $b_n(G)$, is the maximum cardinality among all set of strong arcs $X \subseteq S$ such that $\gamma(G - X) = \gamma(G)$, where S is the set of all strong arcs of G .

Example 5.4.

1. Here $S = \{e_1, e_2, e_3, e_4\}$ and $\gamma(G) = 2$.

Fig. 5.2 (G)

$\{e_1, e_3\}, \{e_1, e_4\}, \{e_2, e_4\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_2, e_3, e_4\}$ are all non-bondage sets of G and $b_n(G) = 3$.

2. Here $S = \{e_1, e_2, e_3, e_5\}$ and $\gamma(G) = 1$

Fig. 5.3 (G)

$\{e_1\}, \{e_3\}, \{e_1, e_2\}$ are all non-bondage sets of G and $b_n(G) = 2$.

Remark 5.5. The non-bondage set of a fuzzy graph G need not be an edge dominating set of G . Here $S = \{e_1, e_2, e_3, e_4, e_5\}$ and $\gamma(G) = 2$.

Fig. 5.4 (G)

- $\{e_1, e_2\}$ is a non-bondage set of the fuzzy graph G . This set is not an edge dominating set of G because the edge e_5 is not dominated by any edge in $\{e_1, e_2\}$.
- $\{e_3, e_4\}$ is also a non-bondage set of G . But this set is an edge dominating set of G because each edge e_1, e_2 and e_5 are dominated by the set $\{e_3, e_4\}$.

Thus the non-bondage set of a fuzzy graph G need not be an edge dominating set of G .

Note 5.6. b_n - set is a maximum non-bondage set.

Theorem 5.7. For any fuzzy graph G , $b_n(G) = |E| - |V| + \gamma(G)$.

Proof. Let D be a minimum dominating set of G and so $|D| = \gamma(G)$. For every node $v \in V - D$, choose exactly one strong arc which is incident to v and to a node in D . Let S_1 be the set of all such strong arcs. Then clearly $S - S_1$ is a b_n -set of G if G has no non-strong arcs. Suppose G has non-strong arcs, then each non-strong arc will become a strong arc by deleting corresponding strong arcs in G . Therefore

$$\begin{aligned} b_n(G) &= |S| - [|V| - \gamma(G)] + |S'| \\ &= |S| + |E - S| + \gamma(G) - |V| \\ &= |E| - |V| + \gamma(G). \end{aligned}$$

Corollary 5.8. For any fuzzy graph G , $b_n(G) \leq |E| - \Delta_S(G)$.

Proof. From above theorem, $b_n(G) = |E| - |V| + \gamma(G)$. And we know that $\gamma(G) \leq |V| - \Delta_S(G)$ Therefore

$$\begin{aligned} b_n(G) &\leq |E| - |V| + |V| - \Delta_S(G), \\ b_n(G) &\leq |E| - \Delta_S(G). \end{aligned}$$

Theorem 5.9. If a fuzzy graph G does not have a bondage set then $b_n(G) = |S|$.

Proof. Let G be a fuzzy graph and it does not have a bondage set. i.e., there does not exist any set $X \subseteq S$ such that $\gamma(G - X) > \gamma(G)$. Thus deletion of all

strong arcs from G does not increase the domination number of G . Now delete the set of all strong arcs, S , and the domination number will be $\gamma(G-S) = \gamma(G)$. Therefore $b_n(G) = |S|$.

Theorem 5.10. *If a fuzzy graph G has a bondage set then $b(G) \leq b_n(G) + 1$.*

Proof. Let G be a fuzzy graph which has a bondage set.

A b_n -set is a maximum non-bondage set i.e., deletion of all arcs in a b_n -set results in $\gamma(G) = \gamma(G - b_n)$. So deletion of any strong arc $e \notin b_n$ with the edges in the set b_n results in $\gamma(G - \{b_n \cup e\}) > \gamma(G)$ which implies $\{b_n \cup e\}$ is a bondage set.

Thus

$$b(G) \leq |b_n \cup e| = b_n(G) + 1 \implies b(G) \leq b_n(G) + 1.$$

Theorem 5.11. *If a non-bondage set of G is an edge dominating set of G then $b_n(G) \geq \gamma(G)/2$.*

Proof. Let G be a fuzzy graph. Let D be a non-bondage set of G and it is also an edge dominating set of G . Clearly $|D| \geq \gamma'(G)$ and $|D| \leq b_n(G)$,

$$\gamma'(G) \leq |D| \leq b_n(G)$$

$$\gamma'(G) \leq b_n(G).$$

We know that

$$\gamma(G) \leq 2\gamma'(G)$$

$$\gamma(G) \leq 2\gamma'(G) \leq 2b_n(G).$$

Therefore $\gamma(G) \leq 2b_n(G)$, and so $\gamma(G)/2 \leq b_n(G)$.

Theorem 5.12. *If G is a complete fuzzy graph with n vertices or nodes then $b_n(G) = (n-1)(n-2)/2$.*

Proof. Let G be a complete fuzzy graph with n vertices. In G , all arcs are strong arcs. Thus the total number of (strong) arcs in G are $n(n-1)/2$.

We know that $\gamma(G) = 1$. Each node will dominate all other nodes. Therefore we need minimum $n-1$ arcs to keep $\gamma(G) = 1$. Thus we can almost delete $|S| - (n-1)$ arcs.

Therefore

$$\begin{aligned}
 b_n(G) &= |S| - (n - 1) \\
 &= n(n - 1)/2 - (n - 1) \\
 &= (n - 1)(n/2 - 1) \\
 &= (n - 1)((n - 2)/2), \\
 b_n(G) &= (n - 1)(n - 2)/2.
 \end{aligned}$$

Theorem 5.13. *If G is a fuzzy graph and G^* is a star then $b_n(G) = 0$.*

Proof. Let G be a fuzzy graph and G^* be a star. Then the domination number of G is 1 i.e., $\gamma(G) = 1$. Thus the node in the centre of G dominates all other nodes in G .

So deletion of any one arc of G will result $\gamma(G) = 2$ since all arcs of G are strong arcs in G . Thus we don't have a non-bondage set for G . Therefore $b_n(G) = 0$.

6. Conclusion

We discussed about the bondage number of fuzzy graphs and its upper bound. We also given bondage number of a complete fuzzy graph. The non-bondage number of a fuzzy graph is also defined. The exact value of the non-bondage number of the fuzzy graphs and the relation between bondage number and non-bondage number of the fuzzy graphs are given. Using these concepts some future work are to find the bondage number and the non-bondage number for fuzzy trees and also to find the lower bound of both bondage and non-bondage number.

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