**ON THE NON-EXISTENCE OF LIMIT CYCLES
FOR A CUBIC KOLMOGOROV SYSTEMS**

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Abstract: In this paper we characterize the integrability and the non-existence of limit cycles of cubic Kolmogorov systems of the form

$$\begin{cases} x' = x (\lambda + ax^2 + bxy + cy^2) \\ y' = y (\mu + ux^2 + vxy + wy^2) \end{cases},$$

where $\lambda, \mu, a, b, c, u, v, w \in \mathbb{R}$.

AMS Subject Classification: 34C07, 37C27, 37K10

Key Words: cubic Kolmogorov system, first integral, periodic orbits, limit cycle

1. Introduction and Statement of the Main Results

The autonomous polynomial differential system on the plane given by

Received: April 7, 2015

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$$\begin{cases} x' = \frac{dx}{dt} = xP(x, y), \\ y' = \frac{dy}{dt} = yQ(x, y). \end{cases} \quad (1)$$

known as Kolmogorov system, where P and Q are two polynomials of $\mathbb{R}[x, y]$ ($\mathbb{R}[x, y]$ denotes the ring of polynomials in the variables x and y with real coefficients), the derivatives are performed with respect to the time variable. By definition, the degree of the system 1 is $n = 1 + \max(\deg(P), \deg(Q))$. In what follows a planar polynomial differential systems 1 of degree 3 will be call a Cubic Kolmogorov System see [3, 4, 5, 9, 12, 13, 14].

There are many natural phenomena which can be modeled the Kolmogorov systems such as mathematical ecology and population dynamics see [6, 8, 11, 15, 17] chemical reactions, plasma physics see [16], hydronamics see [7], economics, etc.

In the qualitative theory of planar dynamical systems see [18], one of the most important topics is related to the second part of the unsolved Hilbert 16th problem, There is a huge literature about limit cycles, most of them deal essentially with their detection, their number and their stability and rare are papers concerned by giving them explicitly see [1, 2, 10].

We recall that in the phase plane, a limit cycle of system 1 is an isolated periodic orbit in the set of all periodic orbits of system 1 .

System 1 is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non constant C^1 function $H : \Omega \rightarrow \mathbb{R}$, called a first integral of the system on Ω , which is constant on the trajectories of the polynomial system 1 contained in Ω , i.e. if

$$\frac{dH(x, y)}{dt} = \frac{\partial H(x, y)}{\partial x} xP(x, y) + \frac{\partial H(x, y)}{\partial y} yQ(x, y) \equiv 0 \quad \text{in the points of } \Omega.$$

Moreover, $H = h$ is the general solution of this equation, where h is an arbitrary constant. It is well known that for differential systems defined on the plane $\mathbb{R}^2$ the existence of a first integral determines their phase portrait see [6, 10].

In this paper we are intersted in studying the integrability and the periodic orbits of the 2-dimensional cubic kolmogorov systems of the for

$$\begin{cases} x' = x(\lambda + ax^2 + bxy + cy^2) \\ y' = y(\mu + ux^2 + vxy + wy^2) \end{cases} \quad (2)$$

where $\lambda, \mu, a, b, c, u, v, w \in \mathbb{R}$.

We define the trigonometric polynomials

$$A(\theta) = a(\cos \theta)^2 + b(\cos \theta)(\sin \theta) + c(\sin \theta)^2$$

$$B(\theta) = u(\cos \theta)^2 + v(\cos \theta)(\sin \theta) + w(\sin \theta)^2$$

Our main result on the integrability and the periodic orbits of the cubic kolmogorov system 2 is the following.

Theorem 1. *Consider a polynomial Komogorov system 2, then the following statements hold.*

(a) *if $\mu = \lambda$ and $A(\theta) \neq B(\theta)$, then system 2 has the first integral*

$$\begin{aligned} H(x, y) = & (x^2 + y^2) \exp \left(-2 \int^{\arctan(\frac{y}{x})} \left(\frac{A(s) \cos^2 s + B(s) \sin^2 s}{(B(s) - A(s)) \cos s \sin s} \right) ds \right) - \\ & \int^{\arctan(\frac{y}{x})} \exp \left(- \int^{\omega} \left(2 \frac{A(s) \cos^2 s + B(s) \sin^2 s}{(B(s) - A(s)) \cos s \sin s} \right) ds \right) \\ & \times \left(\frac{2\lambda}{(B(\omega) - A(\omega)) \cos \omega \sin \omega} \right) d\omega \end{aligned}$$

Moreover, the system 2 has no limit cycle.

(b) *if $\mu \neq \lambda$ and $A(\theta) = B(\theta)$ for all $\theta \in \mathbb{R}$, then system 2 has the first integral*

$$\begin{aligned} H(x, y) = & \frac{1}{x^2 + y^2} \exp \left(2 \int^{\arctan(\frac{y}{x})} \frac{(\lambda \cos^2 s + \mu \sin^2 s)}{(\mu - \lambda) \cos s \sin s} ds \right) + \\ & 2 \int^{\arctan(\frac{y}{x})} \exp \left(2 \int^{\omega} \left(\frac{(\lambda \cos^2 s + \mu \sin^2 s)}{(\mu - \lambda) \cos s \sin s} \right) ds \right) \left(\frac{A(\omega)}{(\mu - \lambda) \cos \omega \sin \omega} \right) d\omega \end{aligned}$$

Moreover, the system 2 has no limit cycle.

(c) *if $\mu = \lambda$ and $B(\theta) = A(\theta)$ for all $\theta \in \mathbb{R}$, then system 2 has the first integral $H = \frac{y}{x}$, Moreover, the system 2 has no limit cycle.*

Proof. In order to prove our results we write the polynomial differential system 2 in polar coordinates (r, θ) , defined by $x = r \cos \theta$, and $y = r \sin \theta$, then system becomes

$$\begin{cases} r' = (\lambda \cos^2 \theta + \mu \sin^2 \theta) r + (A(\theta) \cos^2 \theta + B(\theta) \sin^2 \theta) r^3 \\ \theta' = (\mu - \lambda) \cos \theta \sin \theta + ((B(\theta) - A(\theta)) \cos \theta \sin \theta) r^2 \end{cases} \quad (3)$$

where $r' = \frac{dr}{dt}$ and $\theta' = \frac{d\theta}{dt}$.

If $\mu = \lambda$ and $B(\theta) \neq A(\theta)$, then the differential system 3 becomes

$$\begin{cases} r' = \lambda r + (A(\theta) \cos^2 \theta + B(\theta) \sin^2 \theta) r^3 \\ \theta' = ((B(\theta) - A(\theta)) \cos \theta \sin \theta) r^2 \end{cases} \quad (4)$$

Taking as new independent variable the coordinate θ , this differential system 4 writes

$$\frac{dr}{d\theta} = \frac{A(\theta) \cos^2 \theta + B(\theta) \sin^2 \theta}{(B(\theta) - A(\theta)) \cos \theta \sin \theta} r + \frac{\lambda}{(B(\theta) - A(\theta)) \cos \theta \sin \theta} \frac{1}{r} \quad (5)$$

which is a Bernoulli equation. By introducing the standard change of variables $\rho = r^2$ we obtain the linear equation

$$\frac{d\rho}{d\theta} = 2 \frac{A(\theta) \cos^2 \theta + B(\theta) \sin^2 \theta}{(B(\theta) - A(\theta)) \cos \theta \sin \theta} \rho + \frac{2\lambda}{(B(\theta) - A(\theta)) \cos \theta \sin \theta} \quad (6)$$

The general solution of linear equation 6 is

$$\begin{aligned} \rho(\theta) = & \exp \left(\int^\theta \left(2 \frac{A(s) \cos^2 s + B(s) \sin^2 s}{(B(s) - A(s)) \cos s \sin s} \right) ds \right) \\ & \left(\alpha + \int^\theta \exp \left(\int^\omega \left(2 \frac{A(s) \cos^2 s + B(s) \sin^2 s}{(B(s) - A(s)) \cos s \sin s} \right) ds \right) \right. \\ & \left. \times \left(\frac{2\lambda}{(B(\omega) - A(\omega)) \cos \omega \sin \omega} \right) d\omega \right), \end{aligned}$$

where $\alpha \in \mathbb{R}$ which has the first integral

$$\begin{aligned} H(x, y) = & (x^2 + y^2) \exp \left(- \int^{\arctan(\frac{y}{x})} \left(2 \frac{A(s) \cos^2 s + B(s) \sin^2 s}{(B(s) - A(s)) \cos s \sin s} \right) ds \right) - \\ & \int^{\arctan(\frac{y}{x})} \exp \left(- 2 \int^\omega \left(\frac{A(s) \cos^2 s + B(s) \sin^2 s}{(B(s) - A(s)) \cos s \sin s} \right) ds \right) \\ & \times \left(\frac{2\lambda}{(B(\omega) - A(\omega)) \cos \omega \sin \omega} \right) d\omega. \end{aligned}$$

The equilibrium points of the Kolmogorov system 2 are located at the origin, or on the x or y axes, or in some of the open four quadrants obtained from $\mathbb{R}^2$ removing the x and y axes. Since the axes x and y are formed by trajectories of the system 2, surrounding the equilibria located on these axes cannot be periodic orbits, and consequently no limit cycle. Let γ be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let $h_\gamma = H(\gamma)$.

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system 2, can be written as

$$r(\theta) = \sqrt{h \exp f(\theta) + \exp f(\theta) \int^\theta \exp(-f(\omega)) g(\omega) d\omega}$$

$$\text{where } f(\theta) = 2 \int^\theta \left(\frac{A(s) \cos^2 s + B(s) \sin^2 s}{(B(s) - A(s)) \cos s \sin s} \right) ds, \text{ and } g(\theta) = \frac{2\lambda}{(B(\theta) - A(\theta)) \cos \theta \sin \theta}$$

Therefore the periodic orbit γ is contained in the curve

$$r(\theta) = \sqrt{h_\gamma \exp f(\theta) + \exp f(\theta) \int^\theta \exp(-f(\omega)) g(\omega) d\omega}$$

But this curve cannot contain the periodic orbit γ , and consequently no limit cycle contained in one of open quadrants because this curve at most have a unique point on every ray $\theta = \theta^*$ for all $\theta^* \in [0, 2\pi)$. Hence statement (a) of theorem 1 is proved.

Suppose now that $\mu \neq \lambda$ and $B(\theta) = A(\theta)$ for all $\theta \in \mathbb{R}$, Then the system 3 becomes

$$\begin{cases} r' = (\lambda \cos^2 \theta + \mu \sin^2 \theta) r + A(\theta) r^3 \\ \theta' = (\mu - \lambda) \cos \theta \sin \theta \end{cases}$$

Taking as new independent variable the coordinare θ , this differential system writes

$$\frac{dr}{d\theta} = \frac{\lambda \cos^2 \theta + \mu \sin^2 \theta}{(\mu - \lambda) \cos \theta \sin \theta} r + \frac{A(\theta)}{(\mu - \lambda) \cos \theta \sin \theta} r^3$$

which is a Bernoulli equation. By introducing the standard change of variables $\rho = \frac{1}{r^2}$ we obtain the linear equation

$$\frac{d\rho}{d\theta} = -2 \frac{(\lambda \cos^2 \theta + \mu \sin^2 \theta)}{(\mu - \lambda) \cos \theta \sin \theta} \rho - 2 \frac{A(\theta)}{(\mu - \lambda) \cos \theta \sin \theta}$$

The general solution of linear equation,

$$\begin{aligned} \rho(\theta) = & \exp \left(-2 \int^\theta \left(\frac{(\lambda \cos^2 s + \mu \sin^2 s)}{(\mu - \lambda) \cos s \sin s} \right) ds \right) \\ & \left(\alpha - 2 \int^\theta \exp \left(\int^\omega \left(2 \frac{(\lambda \cos^2 s + \mu \sin^2 s)}{(\mu - \lambda) \cos s \sin s} \right) ds \right) \right. \\ & \left. \left(\frac{A(\omega)}{(\mu - \lambda) \cos \omega \sin \omega} \right) d\omega \right), \end{aligned}$$

where $\alpha \in \mathbb{R}$, which has the first integral

$$\begin{aligned} H(x, y) = & \frac{1}{x^2 + y^2} \exp \left(2 \int^{\arctan(\frac{y}{x})} \frac{(\lambda \cos^2 s + \mu \sin^2 s)}{(\mu - \lambda) \cos s \sin s} ds \right) + \\ & 2 \int^{\arctan(\frac{y}{x})} \exp \left(2 \int^\omega \left(\frac{(\lambda \cos^2 s + \mu \sin^2 s)}{(\mu - \lambda) \cos s \sin s} \right) ds \right) \\ & \times \left(\frac{A(\omega)}{(\mu - \lambda) \cos \omega \sin \omega} \right) d\omega \end{aligned}$$

Let γ be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let $h_\gamma = H(\gamma)$. The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system 2, can be written as

$$r(\theta) = \sqrt{\frac{\exp\left(2 \int^\theta \frac{(\lambda \cos^2 s + \mu \sin^2 s)}{(\mu - \lambda) \cos s \sin s} ds\right)}{h - 2 \int^\theta \exp\left(2 \int^\omega \left(\frac{(\lambda \cos^2 s + \mu \sin^2 s)}{(\mu - \lambda) \cos s \sin s}\right) ds\right) \left(\frac{A(\omega)}{(\mu - \lambda) \cos \omega \sin \omega}\right) d\omega}}$$

Therefore the periodic orbit γ is contained in the curve

$$r(\theta) = \sqrt{\frac{\exp\left(2 \int^\theta \frac{(\lambda \cos^2 s + \mu \sin^2 s)}{(\mu - \lambda) \cos s \sin s} ds\right)}{h_\gamma - 2 \int^\theta \exp\left(2 \int^\omega \left(\frac{(\lambda \cos^2 s + \mu \sin^2 s)}{(\mu - \lambda) \cos s \sin s}\right) ds\right) \left(\frac{A(\omega)}{(\mu - \lambda) \cos \omega \sin \omega}\right) d\omega}}$$

But this curve cannot contain the periodic orbit γ , and consequently no limit cycle contained in one of open quadrants because this curve at most has a unique point on every ray $\theta = \theta^*$ for all $\theta^* \in [0, 2\pi)$. Hence statement (b) of theorem 1 is proved.

Assume now that $\mu = \lambda$ and $B(\theta) = A(\theta)$ then, from (3) it follows that $\theta' = 0$. So the straight lines through the origin of coordinates of the differential system 2 are invariant by the flow of this system. Hence, $\frac{y}{x}$ is a first integral of the system, then since all straight lines through the origin are formed by trajectories, clearly the system has no periodic orbits, and consequently no limit cycle. This completes the proof of statement (c) of Theorem 1. $\square$

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