CONTINUED FRACTION FOR TRIBONACCI RATIO

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Abstract: We study generalized continued fractions for the expression of tribonacci ratios $\frac{T_{n+k}}{T_n}$ when $n$ is large, and find $n$th convergents of the continued fraction explicitly for each $n$.

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Key Words: tribonacci sequence, generalized continued fraction

1. Introduction

The tribonacci sequence $\{T_n\}$ is a generalized fibonacci sequence defined by $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ with $T_0 = 0$ and $T_1 = T_2 = 1$, so $\{T_n\} = \{0, 1, 1, 3, 4, 7, 13, \cdots \}$. It is well known that the fibonacci ratio $\frac{T_{n+1}}{T_n}$ with large $n$ satisfies the fibonacci polynomial $x^2 = x + 1$. Due to continued fraction (CF for short) expansion for quadratic polynomial, the ratio is exhibited by $\langle 1; 1, 1, \cdots \rangle$. Similar to this, the tribonacci ratio shows

$$\frac{T_{n+1}}{T_n} = \frac{T_n + T_{n-1} + T_{n-2}}{T_n} = 1 + \frac{1}{\frac{T_n}{T_{n-1}} + \frac{T_{n-1}}{T_{n-2}}},$$

so when $n$ is large, $\frac{T_{n+1}}{T_n}$ holds the tribonacci polynomial $x^3 = x^2 + x + 1$.

The CF has been used to solve many problems not only in mathematics but
in engineering, physics and computer field, since it provides good approximations of roots of quadratic polynomials. However real roots of cubic polynomials may not be naturally explained by means of classical CF. A question of generalization of CF to higher degree polynomial has been raised by Hermite in 1850 [7]. However, as is mentioned in [6] the CF expression even for cubic polynomial has not been studied extensively. Among few researches, [4] and [5] made use of bifurcating CF for cubic polynomial. It was called as a ternary CF in [3].

This work is devoted to investigating tribonacci ratios with respect to CF expressions. We will study generalized CFs for the expression of $\frac{T_{n+k}}{T_n}$ when $n$ is large, and also find $n$th convergent of CF explicitly for each $n$.

2. Continued Fraction for Cubic Polynomial

We shall denote a finite CF

$$a_0 + \frac{b_1}{a_1 + \frac{b_2}{\ldots + \frac{b_n}{a_n}}} = a_0 + \frac{b_1}{a_1 + \frac{b_2}{\ldots + \frac{b_n}{a_n}}}$$

by a double CF $\langle a_0; b_1, b_2, \ldots, b_n \rangle$. If all $b_i = 1$ then $\langle a_0; \frac{1}{a_1}, \ldots, \frac{1}{a_n} \rangle$ represent a simple CF $\langle a_0; a_1, \ldots, a_n \rangle$. Moreover an infinite double CF is defined by

$$\langle a_0; b_1, b_2, \ldots \rangle = \lim_{n \to \infty} \langle a_0; b_1, a_2, \ldots, b_n \rangle$$

Let $\alpha$ be a real root of $x^3 = x^2 + x + 1$. Then

$$\alpha - 1 = \frac{\alpha^2(\alpha - 1)}{\alpha^2} = \frac{\alpha + 1}{\alpha^2} = \frac{1}{(\alpha - 1) + \frac{1}{\alpha + 1}}$$

shows a simple CF $\alpha = \langle 1; \alpha - 1, \alpha + 1 \rangle$. And we also have

$$\alpha = 1 + \frac{1}{\alpha + 1} = 1 + \frac{1}{\alpha(\alpha + 1) - \alpha} = 1 + \frac{1}{\alpha - 1 + \frac{1}{\alpha}} = \langle 1; \alpha, -1, -\alpha \rangle.$$

On the other hand since $\alpha + 1 = 2 + \frac{1}{\alpha - 1 + \frac{1}{\alpha}}$, it follows that

$$\alpha = 1 + \frac{1}{(\alpha - 1) + \frac{1}{\alpha + 1}} = 1 + \frac{1}{\frac{1}{\alpha - 1 + \frac{1}{\alpha}} + \frac{1}{2 + \frac{1}{\alpha - 1 + \frac{1}{\alpha}}}}.$$
So if we let \( \frac{1}{\alpha - \frac{1}{1 + \frac{1}{\alpha}}} = \beta \) then \( \alpha = 1 + \frac{1}{\beta + \frac{1}{\beta + 2}} = \langle 1; \beta, \beta + 2 \rangle \) where \( \beta = \frac{1}{\alpha - \frac{1}{1 + \frac{1}{\alpha}}} = \langle 0; \alpha, -1, -\alpha \rangle \).

Besides these, there could be various simple CF expressions for the root of \( x^3 = x^2 + x + 1 \). But they do not seem like satisfactory forms to find real root explicitly. We now shall deal with double CF expansion.

**Theorem 1.** For a real root \( \alpha \) of \( x^3 = x^2 + x + 1 \), a double CF expansion is \( \alpha = \langle 1; \beta_1, \beta_1, \beta_1, \ldots \rangle \) with \( \beta = \langle 1; 1, 1, 1, \ldots \rangle \).

**Proof.** The polynomial \( x^3 = x^2 + x + 1 \) yields

\[
\alpha = 1 + \frac{\alpha + 1}{\alpha^2} = 1 + \frac{\alpha + 1}{\alpha} \frac{1}{\alpha}.
\]

Let \( \beta = \frac{\alpha + 1}{\alpha} \). Then

\[
\alpha = 1 + \frac{\beta}{\alpha} = 1 + \frac{\beta}{1 + \frac{\beta}{\alpha}} = 1 + \frac{\beta}{1 + \frac{\beta}{\alpha}} = \cdots = \langle 1; \beta, \beta, \beta \ldots \rangle
\]

while \( \beta = 1 + \frac{1}{\alpha} = 1 + \frac{1}{1 + \frac{1}{\alpha}} = \langle 1; 1, 1, 1, \ldots \rangle \).

Let \( a_i, b_i (i \geq 0) \) be integers and \( \alpha_i, \beta_i (i \geq 1) \) be nonzero integers. Suppose that a real number \( \alpha = \alpha_0 \) bifurcates into \( a_0 \) and \( \frac{\beta_1}{\alpha_1} \), i.e., \( \alpha_0 = a_0 + \frac{\beta_1}{\alpha_1} \). And assume that \( \alpha_i = a_i + \frac{\beta_{i+1}}{\alpha_{i+1}} \) and \( \beta_i = b_i + \frac{1}{\alpha_{i+1}} \) for all \( i \geq 1 \). Then the first few level of bifurcations of \( \alpha \) and \( \beta \) are as follows.

\[
\alpha = a_0 + \frac{\beta_1}{\alpha_1} = a_0 + \frac{b_1 + \frac{1}{a_2}}{a_1 + \frac{\beta_2}{a_2}} = a_0 + \frac{b_1 + \frac{1}{a_2 + \frac{1}{a_3}}}{a_1 + \frac{b_2 + \frac{1}{a_3}}{a_2 + \frac{1}{a_3}}} = \cdots
\]

\[
\beta = \beta_0 = b_0 + \frac{1}{\alpha_1} = b_0 + \frac{1}{a_1 + \frac{\beta_2}{a_2}} = b_0 + \frac{1}{a_1 + \frac{b_2 + \frac{1}{a_3}}{a_2 + \frac{1}{a_3}}} = \cdots
\]

The pair of objects representing \( \alpha \) and \( \beta \) is called a bifurcating CF and denoted by \( (\alpha, \beta) = \{(a_0, a_1, \ldots), \{b_0, b_1, \ldots\}\} \) in [1]. Hence the real root \( \alpha \) of \( x^3 = x^2 + x + 1 \) in Theorem 1 is expressed by \( (\alpha, \beta) = \{(1, 1, \cdots), (1, 1, \cdots)\} \).
3. Continued Fraction for Tribonacci Ratio

Let \( \alpha_i = a_i + \frac{\beta_{i+1}}{\alpha_{i+1}} \) and \( \beta_i = b_i + \frac{1}{\alpha_{i+1}} \) for every \( i > 0 \). Then due to (\( \dagger \)), the \( i \)th bifurcation \( \alpha^{[i]} \) of \( \alpha = \alpha_0 \) can be written by

\[
\alpha = \alpha^{[i]} = a_0 + \frac{N_i(i)}{D_i(i)}
\]

where \( \alpha^{[1]} = a_0 + \frac{N_1(1)}{D_1(1)} \) with \( N_1(1) = \beta_1, D_1(1) = \alpha_1 \),

\( \alpha^{[2]} = a_0 + \frac{N_2(2)}{D_2(2)} \) with \( N_2(2) = b_1 + \frac{1}{\alpha_2}, D_2(2) = a_1 + \frac{\beta_2}{\alpha_2} \),

\( \alpha^{[3]} = a_0 + \frac{N_3(3)}{D_3(3)} \) with \( N_3(3) = b_1 + \frac{1}{a_2 + \frac{\beta_3}{\alpha_3}} \), \( D_3(3) = a_1 + \frac{b_2 + \frac{1}{a_2 + \frac{\beta_3}{\alpha_3}}}{\frac{\beta_3}{\alpha_3}} \),

and so on.

Now for \( i \leq j \), let \( N_i(j) \) denote the same formula of \( N_i(i) \) where all values \( a_s, b_s, \alpha_s \) and \( \beta_s \) in \( N_i(i) \) are replaced by \( a_{s+(j-i)}, b_{s+(j-i)}, \alpha_{s+(j-i)} \) and \( \beta_{s+(j-i)} \) in \( N_i(j) \). We also define \( D_i(j) \) analogously from \( D_i(i) \). For instance \( N_3(6) = b_4 + \frac{1}{a_5 + \frac{\beta_6}{\alpha_6}} \) and \( D_3(5) = a_3 + \frac{b_4 + \frac{1}{a_5 + \frac{\beta_6}{\alpha_6}}}{\frac{\beta_6}{\alpha_6}} \).

**Theorem 2.** With the same context above, the \( i \)th bifurcation of \( \alpha = \alpha_0 \) forms \( \alpha^{[i]} = a_0 + \frac{N_i(i)}{D_i(i)} \), where \( N_i(i), D_i(i) \) are written by double CF that

\[
N_1(1) = N_2(2) = \langle b_1; D_1(2) \rangle, \quad D_1(1) = D_2(2) = \langle a_1; N_1(1), D_1(2) \rangle
\]

and for all \( i > 2 \),

\[
N_i(i) = \langle b_1; a_2, \frac{1}{a_3}, \ldots, \frac{N_i-2(i)}{a_{i-1}}, D_i(i) \rangle = \langle b_1; D_{i-1}(i) \rangle,
\]

\[
D_i(i) = \langle a_1; a_2, \frac{1}{a_3}, \ldots, \frac{N_i-1(i)}{a_{i-1}}, D_{i-1}(i) \rangle = \langle a_1; D_{i-1}(i) \rangle.
\]

**Proof.** The \( N_i(i), D_i(i) \) in \( \alpha^{[i]} \) follow inductively from (\( \dagger \)):

(i) \( N_1(1) = \beta_1 \) and \( D_1(1) = \alpha_1 \).

(ii) \( N_2(2) = b_1 + \frac{1}{\alpha_2} = b_1 + \frac{1}{D_1(2)} = \langle b_1; D_1(2) \rangle \),

\[
D_2(2) = a_1 + \frac{\beta_2}{\alpha_2} = \langle a_1; N_1(2), D_1(2) \rangle.
\]

(iii) \( N_3(3) = b_1 + \frac{1}{a_2 + \frac{\beta_3}{\alpha_3}} = b_1 + \frac{1}{a_2 + \frac{N_1(3)}{D_1(3)}} = \langle b_1; a_2, D_1(3) \rangle \)

\[
= b_1 + \frac{1}{D_2(3)} = \langle b_1; D_2(3) \rangle
\]

\[
D_3(3) = a_1 + \frac{b_2 + \frac{1}{a_2 + \frac{\beta_3}{\alpha_3}}}{\frac{\beta_3}{\alpha_3}} = a_1 + \frac{N_2(3) N_1(3)}{a_2 + D_1(3)} = \langle a_1; D_2(3) \rangle
\]

\[
= a_1 + \frac{N_2(3)}{D_2(3)} = \langle a_1; N_2(3), D_2(3) \rangle.
\]
(iv) \( N_4(4) = b_1 + \frac{1}{a_2 + \frac{b_3 + \frac{1}{a_4 + \frac{\beta_4}{a_4}}}{a_3 + \frac{\beta_4}{a_4}}} = b_1 + \frac{1}{a_2 + \frac{N_2(4) \cdot N_1(4)}{D_1(4)}} \)

\[
= \langle b_1; a_2, a_3, D_1(4) \rangle = b_1 + \frac{1}{D_3(4)} = \langle b_1; D_3(4) \rangle
\]

\[
D_4(4) = a_1 + \frac{b_2 + \frac{1}{a_2 + \frac{b_3 + \frac{1}{a_3 + \frac{\beta_4}{a_4}}}{a_3 + \frac{\beta_4}{a_4}}}}{a_2 + \frac{\beta_4}{a_4}} = a_1 + \frac{N_3(4) \cdot N_2(4) \cdot N_1(4)}{D_3(4)}
\]

Continuing this process, \( N_i(i) \) and \( D_i(i) \) follow that

\[
b_1 + \frac{1}{a_2 + \frac{N_{i-2}(i)}{a_3 + \frac{N_{i-1}(i)}{a_4 + \frac{D_{i-1}(i)}}}} = \langle b_1; a_2, a_3, a_i \rangle
\]

respectively.

Moreover we define \( \hat{N}_i(i) \) and \( \hat{D}_i(i) \) by the same \( N_i(i) \) and \( D_i(i) \) respectively, in which the last fractions \( \frac{\beta_i}{a_i} \) and \( \frac{1}{a_i} \) are omitted.

**Theorem 3.** \( \hat{N}_i(i) \) and \( \hat{D}_i(i) \) satisfy the followings.

1. For \( i = 1, 2 \), \( \hat{N}_i(i) = b_1 \) and \( \hat{D}_i(i) = a_1 \).

2. For \( i > 2 \), \( \hat{N}_i(i) = \langle b_1; \hat{D}_{i-1}(i) \rangle \) and \( \hat{D}_i(i) = \langle a_1; \hat{D}_{i-1}(i) \rangle \). In particular

\[
\hat{N}_i(i) = \langle 1; \hat{D}_{i-1}(i-1) \rangle \quad \text{and} \quad \hat{D}_i(i) = \langle 1; \hat{D}_{i-1}(i-1) \rangle, \text{ if } a_i = b_i = 1 \text{ for all i}.
\]

3. \( \lim_{i \to \infty} \frac{\hat{N}_i(i)}{\hat{D}_i(i)} = \lim_{i \to \infty} \frac{N_i(i)}{D_i(i)} \).

**Proof.** It is clear that \( \hat{N}_1(1) = \hat{N}_2(2) = b_1 \) and \( \hat{D}_1(1) = \hat{D}_2(2) = a_1 \). And \( \hat{N}_3(3) = b_1 + \frac{1}{a_2} = \langle b_1; \hat{D}_2(3) \rangle \) and \( \hat{D}_3(3) = a_1 + \frac{b_2}{a_2} = \langle a_1; \hat{N}_2(3) \rangle \). Similarly

\[
\hat{N}_4(4) = b_1 + \frac{1}{a_2 + \frac{\beta_4}{a_3}} = \langle b_1; \hat{D}_3(4) \rangle \quad \text{and} \quad \hat{D}_4(4) = a_1 + \frac{b_2 + \frac{\beta_4}{a_3}}{a_2 + \frac{\beta_4}{a_3}} = \langle a_1; \hat{N}_3(4) \rangle.
\]

So we have \( \hat{N}_i(i) = \langle b_1; \hat{D}_{i-1}(i) \rangle \) and \( \hat{D}_i(i) = \langle a_1; \hat{D}_{i-1}(i) \rangle \) for all \( i > 2 \).

Obviously \( \hat{N}_2(2) \) and \( \hat{D}_2(2) \) are the 1st convergents of \( N_2(2) \) and \( D_2(2) \) respectively. Moreover \( \hat{N}_3(3) = \langle b_1; \frac{1}{a_2} \rangle \) and \( \hat{D}_3(3) = \langle a_1; \frac{b_2}{a_2} \rangle = \langle a_1; \hat{N}_2(3) \rangle \) are the 2nd convergents of \( N_3(3) = \langle b_1; \frac{1}{a_2}, \frac{\beta_3}{a_3} \rangle \) and \( D_3(3) = \langle a_1; \frac{N_2(3) \cdot N_1(3)}{a_2}, D_1(3) \rangle \) respectively. Continuing this consideration we may say \( \hat{N}_i(i) \) and \( \hat{D}_i(i) \) are...
the \((i - 1)\)th convergents of \(N_i(i)\) and \(D_i(i)\) respectively. Hence it follows that
\[
\lim_{i \to \infty} \frac{\hat{N}_i(i)}{\hat{D}_i(i)} = \lim_{i \to \infty} \frac{N_i(i)}{D_i(i)}.
\]
In particular if \(a_i = b_i = 1\) then the bifurcation
\[
\alpha = \alpha_0 = 1 + \frac{1 + \frac{1}{\alpha_1}}{1 + \frac{1}{\alpha_2}} = 1 + \frac{1 + \frac{1}{\alpha_3}}{1 + \frac{1}{\alpha_4}} = \cdots
\]
implies that
\[
\hat{N}_2(2) = \hat{D}_2(2) = 1 \text{ and } \hat{N}_3(3) = \hat{D}_3(3) = 1 + \frac{1}{\alpha} = 2,
\]
\[
\hat{N}_4(4) = 1 + \frac{1}{1 + \frac{1}{\alpha}} = 1 + \frac{1}{D_3(3)}, \quad \hat{D}_4(4) = 1 + \frac{1 + \frac{1}{\alpha}}{1 + \frac{1}{\alpha}} = 1 + \frac{\hat{N}_3(3)}{D_3(3)},
\]
and so on. Therefore we have
\[
\hat{N}_i(i) = 1 + \frac{1}{D_{i-1}(i-1)} = \langle 1; \hat{D}_{i-1}(i-1) \rangle \quad \hat{D}_i(i) = 1 + \frac{\hat{N}_{i-1}(i-1)}{D_{i-1}(i-1)} = \langle 1; \hat{N}_{i-1}(i-1) \rangle.
\]

**Theorem 4.** Let \(\alpha = \lim_{n \to \infty} T_{n+1}^3 T_n\). Then we have the followings.

1. The double CF \(\alpha\) is \(\alpha = \langle 1; \beta, \beta, \cdots \rangle\) with \(\beta = \langle 1; 1, 1, 1, \cdots \rangle\).

2. The \(i\)th bifurcation is \(\alpha^{[i]} = 1 + \frac{N_i(i)}{D_i(i)}\), where \(N_1(1) = \beta, D_1(1) = \alpha\) and
\[
N_i(i) = 1 + \frac{1}{D_{i-1}(i-1)}, \quad D_i(i) = 1 + \frac{N_{i-1}(i-1)}{D_{i-1}(i-1)} \text{ for all } i > 1.
\]

3. \(\lim_{i \to \infty} \frac{N_i(i)}{D_i(i)} = \lim_{i \to \infty} \frac{\hat{N}_i(i)}{\hat{D}_i(i)}\), and \(\lim_{n \to \infty} \frac{T_{n+1}^3 T_n}{T_n^3} = 1.8392 \cdots\).

**Proof.** When \(n\) is large, \(\alpha = \frac{T_{n+1}^3 T_n}{T_n^3}\) satisfies \(x^3 = x^2 + x + 1\), so (1) is due to Theorem 1. Moreover by letting \(\beta = \frac{\alpha + 1}{\alpha}\),
\[
\alpha = 1 + \frac{\beta}{\alpha} = 1 + \frac{\beta}{1 + \frac{\beta}{\alpha}} = 1 + \frac{1 + \beta}{1 + \frac{\beta}{\alpha}} = \cdots
\]
implies \(\alpha\) bifurcates into 1 and \(\frac{\beta}{\alpha}\), while \(\beta\) into 1 and \(\frac{1}{\alpha}\). Thus
\[
\alpha = 1 + \frac{\beta}{\alpha} = 1 + \frac{1 + \frac{1}{\alpha}}{1 + \frac{1}{\alpha}} = 1 + \frac{1 + \frac{1}{\alpha}}{1 + \frac{1}{\alpha}} = \cdots
\]
Now owing to Theorem 2 and 3, we have \(\alpha^{[i]} = 1 + \frac{N_i(i)}{D_i(i)}\) where \(N_1(1) = \beta, D_1(1) = \alpha\)
$N_2(2) = 1 + \frac{1}{\alpha} = 1 + \frac{1}{D_1(1)}, \quad D_2(2) = 1 + \frac{\beta}{\alpha} = 1 + \frac{N_1(1)}{D_1(1)}$

$N_3(3) = 1 + \frac{1}{1 + \frac{\gamma}{\beta}} = 1 + \frac{1}{D_2(2)}, \quad D_3(3) = 1 + \frac{1 + \frac{\gamma}{\beta}}{1 + \frac{\gamma}{\beta}} = 1 + \frac{N_2(2)}{D_2(2)}$

$N_4(4) = 1 + \frac{1}{1 + \frac{\delta}{\gamma}} = 1 + \frac{1}{D_3(3)}, \quad D_4(4) = 1 + \frac{N_3(3)}{D_3(3)}$, ...

thus $N_i(i) = 1 + \frac{1}{D_{i-1}(i-1)}$, $D_i(i) = 1 + \frac{N_{i-1}(i-1)}{D_{i-1}(i-1)}$ for all $i > 1$. Moreover for $\hat{N}_i(i)$ and $\hat{D}_i(i)$, we have $\hat{N}_2(2) = 1$, $\hat{D}_2(2) = 1$, and $\hat{N}_3(3) = 1 + \frac{1}{\hat{T}_1} = 1 + \frac{1}{D_2(2)}$, $\hat{D}_3(3) = 1 + \frac{1}{\hat{T}_1} = 1 + \frac{\hat{N}_2(2)}{\hat{D}_2(2)}$, $\hat{N}_4(4) = 1 + \frac{1}{\hat{T}_1} = 1 + \frac{1}{D_3(3)}$, $\hat{D}_4(4) = 1 + \frac{1}{\hat{T}_1} = 1 + \frac{\hat{N}_3(3)}{\hat{D}_3(3)}$, so $\hat{N}_i(i) = 1 + \frac{1}{D_{i-1}(i-1)}$ and $\hat{D}_i(i) = 1 + \frac{\hat{N}_{i-1}(i-1)}{D_{i-1}(i-1)}$ for all $i > 2$. So by computing the first some $\hat{N}_i(i)$ and $\hat{D}_i(i)$ explicitly, we have

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\hat{N}_i(i)$</th>
<th>$\hat{D}_i(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{7}{2}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{11}{7}$</td>
<td>$\frac{24}{13}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{37}{24}$</td>
<td>$\frac{84}{50}$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{163}{92}$</td>
<td>$\frac{380}{227}$</td>
</tr>
<tr>
<td>7</td>
<td>$\frac{684}{404}$</td>
<td>$\frac{1597}{927}$</td>
</tr>
<tr>
<td>8</td>
<td>$\frac{353}{205}$</td>
<td>$\frac{8333}{4832}$</td>
</tr>
<tr>
<td>9</td>
<td>$\frac{8392}{4833}$</td>
<td>$\frac{18392}{10667}$</td>
</tr>
<tr>
<td>10</td>
<td>$\frac{18392}{10667}$</td>
<td>$\frac{40667}{23392}$</td>
</tr>
<tr>
<td>11</td>
<td>$\frac{40667}{23392}$</td>
<td>$\frac{91392}{52392}$</td>
</tr>
</tbody>
</table>

So the table shows $\frac{\hat{N}_i(i)}{\hat{D}_i(i)}$ ($2 \leq i \leq 13$) approximates to 0.8392 $\cdots$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\hat{N}_i(i)/\hat{D}_i(i)$</th>
<th>$i$</th>
<th>$\hat{N}_i(i)/\hat{D}_i(i)$</th>
<th>$i$</th>
<th>$\hat{N}_i(i)/\hat{D}_i(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\frac{3}{2}$ = 1</td>
<td>3</td>
<td>$\frac{7}{4}$ = 1.75</td>
<td>4</td>
<td>$\frac{3}{2}$ = 1.5</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{6}{5} = 1.2$</td>
<td>6</td>
<td>$\frac{7}{5} = 1.4$</td>
<td>7</td>
<td>$\frac{4}{3} = 1.33$</td>
</tr>
<tr>
<td>8</td>
<td>$\frac{37}{30} = 1.2333$</td>
<td>9</td>
<td>$\frac{84}{68} = 1.2558$</td>
<td>10</td>
<td>$\frac{125}{102} = 1.2255$</td>
</tr>
<tr>
<td>11</td>
<td>$\frac{83941}{48333} = 1.73$</td>
<td>12</td>
<td>$\frac{353}{205} = 1.73$</td>
<td>13</td>
<td>$\frac{83926}{48328} = 1.73$</td>
</tr>
</tbody>
</table>

Hence we have $\lim_{n \to \infty} \frac{T_{n+1}}{T_n} = \alpha = \lim_{i \to \infty} \alpha^{[i]} = 1 + \lim_{i \to \infty} \frac{\hat{N}_i(i)}{\hat{D}_i(i)} = 1.8392 \cdots$ by Theorem 3.

Compare it to the real root $\frac{1}{3} (1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}})$ of $x^3 = x^2 + x + 1$ which approximates to 1.8392 $\cdots$.

**Corollary 5.** Let $\frac{N_n(n)}{D_n(n)} = \frac{p_n}{q_n}$ with $p_n, q_n \in \mathbb{N}$. Then $\{q_n\}$ is the (original) tribonacci sequence having initials 1, 2, 4, while $\{p_n\}$ is a tribonacci type sequence having initials 1, 2, 3. In particular, $1 + \frac{N_n(n)}{D_n(n)} = \frac{T_{n+1}}{T_n}$ for every $n$.

**Proof.** It is easy to see that $\{p_n + q_n\}$ is a tribonacci sequence with initials 2, 4, 7. Thus $\{p_n + q_n\} = \{T_{n+1}\}$ and $1 + \frac{N_n(n)}{D_n(n)} = 1 + \frac{p_n}{q_n} = \frac{T_{n+1}}{T_n}$. $\square$
For instance 1 + \( \frac{\hat{N}_{16}(16)}{D_{16}(16)} = 1 + \frac{4841}{5768} = \frac{10609}{5768} = \frac{T_{17}}{T_{16}}. \)

### 4. Continued Fraction for \( k \) Step Tribonacci Ratio

According to [8], ratios that appear in phyllotaxis are fractions of fibonacci numbers spaced 2 steps such as 8/3, 13/5, 21/8, \( \cdots \) rather than consecutive terms. The 2 step fibonacci ratio is expressed by CF that \( \frac{F_{n+2}}{F_n} = 1 + \frac{F_{n+1}}{F_n} = 1 + \langle 1; 1, \cdots, 1 \rangle = \langle 2; 1, \cdots, 1 \rangle. \) However when \( n \) is large, any \( k \) step fibonacci ratio \( \frac{F_{n+k}}{F_n} = \frac{F_{n+k}}{F_{n+k-1}} \cdots \frac{F_{n+1}}{F_n} = \langle 1; 1, \cdots \rangle^k \) may not be simply represented by CF since arithmetic algorithms of CFs are not trivial. In this section we shall study \( k \) step tribonacci ratios \( \frac{T_{n+k}}{T_n} \) with CF expansions. Let us begin with \( k = 4. \)

**Theorem 6.** For 4 step tribonacci ratio \( \frac{T_{n+4}}{T_n}, \lim_{n \to \infty} \frac{T_{n+4}}{T_n} = \langle 1; \beta, \beta, \cdots \rangle \) with \( \beta = \langle 5; 11, 11, \cdots \rangle. \)

**Proof.** \( T_n \) satisfies the 4 step recurrence rule \( T_{n+4} = 11T_n + 5T_{n-4} + T_{n-8} \) (see [2]), hence we have

\[
\frac{T_{n+4}}{T_n} = 11 + \frac{5}{T_{n-4}} + \frac{1}{T_{n-4}T_{n-8}}.
\]

Then \( \alpha = \lim_{n \to \infty} \frac{T_{n+4}}{T_n} \) is a root of \( x^3 = 11x^2 + 5x + 1. \) So we have \( \alpha = 11 + \frac{5\alpha + 1}{\alpha} = 11 + \frac{\beta}{\alpha} \) by letting \( \beta = \frac{5\alpha + 1}{\alpha}. \) Hence \( \alpha \) bifurcates into 11 and \( \frac{\beta}{\alpha} \) that

\[
\alpha = 11 + \frac{\beta}{11 + \frac{\beta}{\alpha}} = 11 + \frac{\beta}{11 + \frac{\beta}{11 + \frac{\beta}{\alpha}}} = \cdots = \langle 11; \beta, \beta, \cdots \rangle
\]

while \( \beta \) bifurcates into 5 and \( \frac{1}{\alpha} \) that

\[
\beta = 5 + \frac{1}{11 + \frac{\beta}{\alpha}} = 5 + \frac{1}{11 + \frac{\beta}{11 + \frac{\beta}{\alpha}}} = \cdots = \langle 5; 11, 11, \cdots \rangle.
\]

We now shall compute \( \frac{T_{n+4}}{T_n} \) explicitly.
Theorem 7. Let \( n = 4t + r \) \((t, r \in \mathbb{N}, 1 \leq r \leq 4)\) and \( \alpha_t = \frac{T_{n+r}}{T_n} \) be the \( 4 \) step tribonacci ratio. Then we have the followings.

(1) The double CF expansion of \( \alpha_t \) is \( \alpha_t = \langle 11; \beta_{t-1}^{11}, \ldots, \beta_2 \beta_1^{11} \rangle \) for \( t \geq 2 \), where \( \beta_i = \langle 5; \frac{1}{11}, \frac{1}{11}, \ldots, \frac{1}{11}, \alpha_1^1 \rangle \) for \( 3 \leq i < t \). In particular \( \alpha_0 = \frac{T_{4+r}}{T_r} \), \( \alpha_1 = \frac{T_{4+2r}}{T_{4+r}} \) and \( \beta_1 = \langle 5; \alpha_{i-1} \rangle \) if \( i = 1, 2 \).

(2) The \( i \)th bifurcation of \( \alpha_t \) is \( \alpha_t[i] = 11 + \frac{N_i(i)}{D_i(i)} \), where double CFs of \( N_i(i) \) and \( D_i(i) \) are \( N_1(1) = \beta_{t-1}^{11} \) and \( D_1(1) = \alpha_{t-1} \) and

\[
\begin{align*}
N_2(2) &= \langle 5; D_1(2) \rangle \\
D_1(2) &= \langle 11; N_2(2) \rangle \\
N_3(i) &= \langle 5; D_1(2) \rangle \\
D_1(i) &= \langle 11; N_2(i) \rangle
\end{align*}
\]

for \( 3 \leq i < t \). Here \( N_i(j) \) [resp. \( D_i(j) \)] for \( i \leq j \) is defined from \( N_i(i) \) [resp. \( D_i(i) \)] where \( \alpha_{t-i} \) and \( \beta_{t-i} \) in \( N_i(i) \) are replaced by \( \alpha_{t-j} \) and \( \beta_{t-j} \) in \( N_i(j) \).

(3) Moreover \( \lim_{n \to \infty} \frac{T_{n+r}}{T_n} = 11.4452 \cdot \ldots \).

Proof. The \( T_{4(t+1)+r} = 11T_{4t+r} + 5T_{4(t-1)+r} + T_{4(t-2)+r} \) shows

\[
\alpha_t = \frac{T_{4(t+1)+r}}{T_{4t+r}} = 11 + \frac{5}{T_{4(t-1)+r}} + \frac{1}{T_{4(t-2)+r}}
\]

\[
= 11 + \frac{5}{\alpha_{t-1}} + \frac{1}{\alpha_{t-1} \alpha_{t-2}} = 11 + \left(5 + \frac{1}{\alpha_{t-2}}\right)\frac{1}{\alpha_{t-1}},
\]

so we have \( \alpha_t = 11 + \frac{\beta_{t-1}}{\alpha_{t-1}} \) with \( \beta_{t-1} = 5 + \frac{1}{\alpha_{t-2}} \). Thus

\[
\alpha_t = 11 + \frac{\beta_{t-1}}{11 + \frac{\beta_{t-2}}{11 + \frac{\beta_{t-3}}{\alpha_{t-3}}}} = \cdots = \langle 11; \beta_{t-1}^{11}, \beta_{t-2}, \ldots, \beta_2 \beta_1 \rangle
\]

for \( t \geq 2 \), while for \( 3 \leq i \leq t - 1 \),

\[
\beta_i = 5 + \frac{1}{11 + \frac{\beta_{i-2}}{11 + \frac{\beta_{i-3}}{\alpha_{i-3}}}} = \cdots = \langle 5; \frac{1}{11}, \frac{1}{11}, \ldots, \frac{1}{11}, \alpha_1^1 \rangle.
\]

In particular \( \alpha_0 = \frac{T_{4+r}}{T_r} \), \( \alpha_1 = \frac{T_{4+2r}}{T_{4+r}} \) and \( \beta_2 = 5 + \frac{1}{\alpha_1} = \langle 5; \alpha_1 \rangle \), \( \beta_1 = 5 + \frac{1}{\alpha_0} = \langle 5; \alpha_0 \rangle.\)
\[ \alpha_t - 11 = \frac{\beta_{t-1}}{\alpha_{t-1}} = \frac{5 + \frac{1}{\alpha_{t-2}}}{11 + \frac{\beta_{t-3}}{\alpha_{t-2}}} = \frac{5 + \frac{1}{11 + \frac{\beta_{t-3}}{\alpha_{t-2}}}}{11 + \frac{5 + \frac{1}{\alpha_{t-3}}}{11 + \frac{\beta_{t-3}}{\alpha_{t-3}}}} = \cdots \]

shows the \( i \)th level bifurcation \( \alpha_t[i] = 11 + \frac{N_t(i)}{D_t(i)} \) as follows.

\[
N_1(1) = \beta_{t-1}, \quad D_1(1) = \alpha_{t-1}
\]

\[
N_2(2) = \langle 5; \alpha_{t-2} \rangle = \langle 5; D_1(2) \rangle, \quad D_2(2) = \langle 11; \beta_{t-2} \rangle = \langle 11; D_1(2) \rangle
\]

\[
N_3(3) = \langle 5; \frac{1}{11'} \beta_{t-3} \rangle = \langle 5; \frac{1}{11'} N_1(3) \rangle, \quad D_3(3) = \langle 11; \frac{N_3(3)}{11}, D_1(3) \rangle
\]

\[
N_4(4) = \langle 5; \frac{1}{11'} N_2(4) N_1(4) \rangle, \quad D_4(4) = \langle 11; \frac{N_4(4)}{11}, D_1(4) \rangle
\]

and so on, where for \( i \leq j \), the \( N_i(j) \) and \( D_i(j) \) come from \( N_i(i) \) and \( D_i(i) \) in which \( \alpha_{t-i}, \beta_{t-i} \) are substituted by \( \alpha_{t-j}, \beta_{t-j} \). Thus for all \( 3 \leq i < t \), we have

\[
N_i(i) = \langle 5; \frac{1}{11'}, \frac{N_{i-2}(i)}{11}, \cdots, \frac{N_2(i)}{11}, \frac{N_1(i)}{11}, D_1(i) \rangle,
\]

\[
D_i(i) = \langle 11; \frac{N_{i-1}(i)}{11'}, \cdots, \frac{N_2(i)}{11}, \frac{N_1(i)}{11}, D_1(i) \rangle.
\]

Furthermore by considering \( \hat{N}_i(i) \) and \( \hat{D}_i(i) \) that are the \((i-1)th\) convergent of \( N_i(i) \) and \( D_i(i) \) respectively, we have

\[
\hat{N}_2(2) = 5, \quad \hat{D}_2(2) = 11, \quad \frac{\hat{N}_2(2)}{\hat{D}_2(2)} = \frac{5}{11} \approx 0.454545,
\]

\[
\hat{N}_3(3) = 5 + \frac{1}{11'}, \quad \hat{D}_3(3) = 11 + \frac{5}{11'}, \quad \frac{\hat{N}_3(3)}{\hat{D}_3(3)} = \frac{56}{126} \approx 0.444444,
\]

\[
\hat{N}_4(4) = \frac{641}{126}, \quad \hat{D}_4(4) = \frac{1442}{126}, \quad \frac{\hat{N}_4(4)}{\hat{D}_4(4)} = \frac{641}{1442} \approx 0.444521,
\]

and so on. Therefore it follows that

\[
\lim_{n \to \infty} \frac{T_{n+4}}{T_n} = \lim_{t \to \infty} \alpha_t = 11 + \lim_{t \to \infty} \frac{N_{t-1}(t-1)}{D_{t-1}(t-1)} = 11.44452 \cdots.
\]

Recall the real root \( \frac{1}{3}(11 + 2\sqrt{199 + 3\sqrt{33}} + 2\sqrt[4]{199 - 3\sqrt{33}}) \) of \( x^3 = 11x^2 + 5x + 1 \) that approximates to 11.44452 \cdots. And it equals \((1.8392 \cdots)^4\).

**Example.** For \( \frac{T_{n+4}}{T_n}, \hat{N}_1(1) = \hat{N}_2(2) = 5 \) and \( \hat{D}_1(1) = \hat{D}_2(2) = 11 \). And \( \hat{N}_i(i) = 5 + \frac{1}{\hat{D}_{i-1}(i-1)}, \hat{D}_i(i) = 11 + \frac{\hat{N}_{i-1}(i-1)}{\hat{D}_{i-1}(i-1)} \) yield the table:
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<table>
<thead>
<tr>
<th>(i)</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N_t(i))</td>
<td>5</td>
<td>11</td>
<td>641</td>
<td>7336</td>
<td>83957</td>
<td>960848</td>
</tr>
<tr>
<td>(D_t(i))</td>
<td>11</td>
<td>1442</td>
<td>1442</td>
<td>16503</td>
<td>188869</td>
<td>2161516</td>
</tr>
<tr>
<td>(\hat{N}_t(i))</td>
<td>56</td>
<td>126</td>
<td>641</td>
<td>7336</td>
<td>83957</td>
<td>960848</td>
</tr>
<tr>
<td>(\hat{D}_t(i))</td>
<td>11</td>
<td>126</td>
<td>1442</td>
<td>16503</td>
<td>188869</td>
<td>2161516</td>
</tr>
</tbody>
</table>

Hence, for instance, the ratio \(\frac{T_{38}}{T_{34}} = \frac{T_{4(9)+2}}{T_{4(8)+2}} = \alpha_8\) is obtained by the 7th bifurcation \(\alpha_8^{[7]}\) that \(\alpha_8 = \alpha_8^{[7]} = 11 + \frac{\hat{N}_t(7)}{\hat{D}_t(7)} \approx 11.44452 \cdots\).

Moreover all \(4t\) subscripted tribonacci numbers can be retrieved in this way.

**Corollary 8.** For any \(t > 1\), \(11 + \frac{\hat{N}_t(t)}{\hat{D}_t(t)}\) retrieves \(\frac{T_{4(t+1)}}{T_{4t}}\).

**Proof.** Write \(11 + \frac{\hat{N}_t(t)}{\hat{D}_t(t)} = \frac{p_t}{q_t}\). Then the above table shows

<table>
<thead>
<tr>
<th>(t)</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{p_t}{q_t})</td>
<td>126/11</td>
<td>1442/126</td>
<td>16503/1442</td>
<td>188869/16503</td>
<td>2161516/188869</td>
<td>2161516/2161516</td>
</tr>
</tbody>
</table>

So we have the set of fractions \(\frac{p_t}{q_t}\) for \(2 \leq t \leq 7\) that

\[
\{ \frac{504}{44}, \frac{5768}{504}, \frac{66012}{3768}, \frac{755476}{66012}, \frac{8646064}{755476}, \frac{98950064}{8646064} \}
\]

which equals \(\left\{ \frac{T_{4(t+1)}}{T_{4t}} \right\}\). Moreover since \(p_t = q_{t+1}\) and the set \(\{q_t\} = \{11, 126, 1442, 16503, 188869, 2161516\}\) satisfy the recurrence \(q_{t+1} = 11q_t + 5q_{t-1} + q_{t-2}\), it follows that \(\frac{4p_t}{4q_t} = \frac{T_{4(t+1)}}{T_{4t}}\) for all \(t > 2\). \(\square\)

**Corollary 9.** If \(n\) is divisible by 4 then so is \(T_n\).

**Proof.** \(T_4 = 4\), \(T_8 = 44\) and \(T_{12} = 504\) are all zeros by mod 4. So if \(T_{4i} \equiv 0(mod 4)\) for \(1 \leq i \leq t\) then \(T_{4(t+1)} = 11T_{4t} + 5T_{4(t-1)} + T_{4(t-2)} \equiv 0(mod 4)\). \(\square\)

It is now obvious by Theorem 4 and 7 that any \(k\) step tribonacci ratio \(\frac{T_{n+k}}{T_n}\) depends on the \(k\) step recurrence rule for \(T_n\).

**Lemma 1.** [2] Any \(k\) step recurrence satisfies \(T_{n+k} = a_kT_n + b_kT_{n-k} + T_{n-2k}\) with \(a_k = 3T_k - T_{k-6}\) and \(b_k = -a_{k-1}\). The \(\{a_k\}\) and \(\{b_k\}\) satisfy \(a_{k+3} = a_{k+2} + a_{k+1} + a_k\) and \(b_k = b_{k+1} + b_{k+2} + b_{k+3}\) with initial 1, 3, 7 and 1, 1, -5, respectively.

In particular the pairs \((a_k, b_k)\) for \(1 \leq k \leq 10\) are \((1, 1), (3, 1), (7, -5), (11, 5), (21, 1), (39, -11), (71, 15), (131, -3), (241, -23)\) and \((443, 41)\). So for example, \(T_{n+3} = 7T_n - 5T_{n-3} + T_{n-6}\) and \(T_{n+10} = 443T_n + 41T_{n-10} + T_{n-20}\). And the next theorem follows immediately.
Theorem 10. Let \( a_k, b_k \) be integers satisfying \( T_{n+k} = a_k T_n + b_k T_{n-k} + T_{n-2k} \). When \( n \) is large, we have the followings.

1. The \( k \) step tribonacci ratio \( \frac{T_{n+k}}{T_n} \) holds \( x^3 = a_k x^2 + b_k x + 1 \).

2. If \( n = kt + r \) (\( 1 \leq r \leq k \)) then \( \frac{T_{n+k}}{T_n} = \langle a_k; \beta_{t-1}, \ldots, \beta_2, \beta_1 \rangle \) for \( t \geq 2 \), \( a_k \) with \( \beta_i = \langle b_k; a_k, \alpha_{i-1} \rangle \) if \( i = 1, 2 \). In particular \( \alpha_0 = \frac{T_{k+r}}{T_r} \), \( \alpha_k = T_{k+r} \), and \( \beta_i = \langle b_k; \alpha_{i-1} \rangle \) if \( i = 1, 2 \).

3. \[ \lim_{n \to \infty} \frac{T_{n+k}}{T_n} = \lim_{t \to \infty} \alpha_t[i] = a_k + \lim_{t \to \infty} \frac{N_{t-1}(t-1)}{D_{t-1}(t-1)}. \]

For instance, \( \frac{T_{80}}{T_{70}} = \frac{T_{10(8)}}{T_{10(7)}} = \alpha_7 = \alpha_7^{[6]} = 443 + \frac{\hat{N}_6(6)}{\hat{D}_6(6)} \), where \( \hat{N}_2(2) = 41 \), \( \hat{D}_2(2) = 443 \), \( \hat{N}_3(3) = 41 + \frac{1}{443} = \frac{18164}{443}, \hat{D}_3(3) = 443 + \frac{41}{443} = \frac{196290}{443}, \hat{N}_4(4) = 41 + \frac{443}{196290} = \frac{804833}{196290} ; \hat{D}_4(4) = 443 + \frac{18164}{196290} = \frac{86974634}{196290}, \hat{N}_5(5) = \frac{38537811195}{38537811195}, \hat{D}_5(5) = \frac{38537811195}{38537811195} \), \( \hat{N}_6(6) = \frac{38537811195}{38537811195} \), and \( \hat{D}_6(6) = \frac{38537811195}{38537811195} \). These fractions converge to 0.09253, so \( \frac{T_{80}}{T_{70}} \approx 443.09253 \). It can be compared to the root \( \frac{1}{3}(443 + \sqrt[3]{87022054 + 1914\sqrt[3]{33} + \sqrt[3]{87022054}} - 1914\sqrt[3]{33}) \approx 443.09253 \) of \( x^3 = 443x^2 + 41x + 1 \) as well as to \( 1.839286755^{10} \approx 443.09253 \).

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References


