MULTIVARIATE HOMOGENEOUS MATRIX PADE-TYPE APPROXIMANTS AND PADE APPROXIMANTS

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Abstract: Matrix Padé-type approximants and Padé approximants has been studied by several authors in the scalar case. Some definitions in multivariate case have been introduced. In the current work, we define a multivariate homogeneous matrix-type approximants of which generating polynomials have shift. We consider higher-order approximants and introduce multivariate homogeneous matrix Padé approximants. We derive some recurrence relations for generating polynomials and associated polynomials. Finally, numerical example is given to illustrate our results.

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1. Introduction

The matrix Padé approximants has been studied for a long time by several authors [8],[9],[13],[12],[14],[15] to cite just a few. These approximants are used extensively in many fields : dynamic system control, multiport network synthesis, design of multi-input and multi-output digital filters, reduction of a high degree multivariable system.

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Bose and Basu [6] studied the existence, non uniqueness, and recursive computation of 2-D matrix Padé approximants with the inverse matrix. Gu [10] considered a bivariate matrix-value rational interpolant in Thiele-type continued fraction form with scalar denominator and matrix-value numerator.

Let F be a rectangular matrix of size $r \times s$, defined by :

$$F(x, y) = \begin{pmatrix} f^{(1,1)}(x, y) & \dots & f^{(1,s)}(x, y) \\ \vdots & & \vdots \\ f^{(r,1)}(x, y) & \dots & f^{(r,s)}(x, y) \end{pmatrix} \quad (1.1)$$

It is assumed that the $f^{(k,l)}$ has a formal power series expansion. Therefore

$$F(x, y) = \sum_{i,j=0}^{+} C_{i,j} x^i y^j, \quad C_{i,j} \in \mathcal{M}_{r,s}, \quad (1.2)$$

where $\mathcal{M}_{r,s}$ is the space of rectangular matrices with real or complex coefficients of size $r \times s$.

C. D. Zheng et al. [14],[15] have defined the multivariate matrix Padé-type approximants and Padé approximants. They have generalized the concept of Brezinski in scalar cases [5]. Their definition is based on the bilinear functional c acting on $\mathcal{P}$ (the vector space of all bivariate polynomials whose coefficients are real or complex) defined from its moments by $c(u^i v^j) = C_{i,j}$. In the paper [1] we introduced a new family of multivariate matrix Padé approximants.

The paper is organized as follows. In Section 2, we recall the definition of multivariate homogeneous Padé approximants introduced by A. Cuyt in [7].

In Section 3, we give a definition of the multivariate homogeneous matrix Padé-type approximants. It generalizes directly the definition in [7].

In Section 4 we study the connexion between the homogeneous multivariate matrix Padé-type approximants and homogeneous multivariate matrix Padé approximants.

The recurrence relations are also established and proved for the generator polynomials and the associated polynomials in Section 5. In the last Section, we give a numerical example to illustrate our results.

We restrict our description to the bivariate case, only for notational convenience, although we may use the term multivariate in the text.

2. Multivariate Homogeneous Padé Approximants

The scalar homogeneous multivariate Padé-type approximants and Padé approximants are defined in [2],[4],[7],[11]. In this section, we recall the definition

of approximants introduced by A. Cuyt in [7] of which the polynomials numerator and denominator have shift.

Let $f(x, y)$ be a formal power series:

$$f(x, y) = \sum_{i,j=0}^{+} d_{ij} x^i y^j$$

For completeness we repeat that the multivariate homogeneous Padé approximant $[m - 1/m]_f$ is defined as an irreducible form $\frac{p}{q}$ with

$$p(x, y) = \sum_{i+j=(m-1)m}^{(m-1)m+m-1} a_{ij} x^i y^j \quad (2.1)$$

$$q(x, y) = \sum_{i+j=(m-1)m}^{(m-1)m+m} b_{ij} x^i y^j \quad (2.2)$$

$$(fq - p)(x, y) = \sum_{i+j=(m-1)m+2m} e_{ij} x^i y^j \quad (2.3)$$

One of the great advantage of this homogeneous definition is that it results in a unique irreducible form, whatever solution of (2.1), (2.2) and (2.3) is considered.

3. Multivariate Homogeneous Matrix Padé-type Approximants

Let F be a rectangular matrix of size $r \times s$ with two variables, defined by:

$$F(x, y) = \sum_{i,j=0}^{+} C_{i,j} x^i y^j, \quad C_{i,j} \in \mathcal{M}_{r,s} \quad (3.1)$$

This series can be written

$$F(x, y) = \left(f^{(k,l)}(x, y) \right)_{\substack{1 \leq k \leq r \\ 1 \leq l \leq s}} \quad (3.2)$$

It is assumed that the $f^{(k,l)}$ has a formal power series expansion.

$$f^{(k,l)}(x, y) = \sum_{i,j=0}^{+} c_{i,j}^{(k,l)} x^i y^j$$

We introduce the notations bellow

$$(x, y) = (\lambda_1 z, \lambda_2 z), \quad x, y, z \in \mathbb{C}, \quad \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2, \quad \|\lambda\| = 1$$

$$c_i^{(k,l)}(\lambda) = \sum_{j=0}^i c_{i-j,j}^{(k,l)} \lambda_1^{i-j} \lambda_2^j$$

$$C_i(\lambda) = \left(c_i^{(k,l)}(\lambda) \right)_{\substack{1 \leq k \leq r \\ 1 \leq l \leq s}}$$

Now the series can also be formally viewed as

$$F(x, y) = F_\lambda(z) = \sum_{i=0}^+ C_i(\lambda) z^i = \left(\sum_{i=0}^+ c_i^{(k,l)}(\lambda) z^i \right)_{\substack{1 \leq k \leq r \\ 1 \leq l \leq s}} \quad (3.3)$$

For $n, m \in \mathbb{N}$, we consider an arbitrary function $V_{n,m}(x, y)$ of the form

$$V_{n,m}(x, y) = \mathcal{V}_{n,m}(z) = \sum_{i=0}^m b_{nm+m-i}(\lambda) z^i$$

$$= \sum_{i=0}^m \left(\sum_{j=0}^{nm+m-i} b_{nm+m-i-j,j} \lambda_1^{nm+m-i-j} \lambda_2^j \right) z^i$$

$\mathcal{V}_{n,m}(z)$ is a polynomial of degree m in z and its coefficients are homogeneous polynomials. Let $W_{n,m}$ be defined by

$$W_{n,m}(x, y) = \mathcal{W}_{n,m}(z) = \Gamma_{n,m} \left(\frac{\mathcal{V}_{n,m}(u) - \mathcal{V}_{n,m}(z)}{u - z} \right)$$

where the linear functional $\Gamma_{n,m}$ acting on the space of bivariate polynomials in variable u , as

$$\Gamma_{n,m}(u^i) = \left(\Gamma_{n,m}^{(k,l)}(u^i) \right)_{\substack{1 \leq k \leq r \\ 1 \leq l \leq s}} = \left(c_{n-m+1+i}^{(k,l)}(\lambda) \right)_{\substack{1 \leq k \leq r \\ 1 \leq l \leq s}} \quad (3.4)$$

with the convention that $c_i^{(k,l)}(\lambda) = 0$ if $i < 0$.

Proposition. *The polynomial $\mathcal{W}_{n,m}$ is of the form*

$$\mathcal{W}_{n,m}(z) = \sum_{i=0}^{m-1} A_{nm+n-i}(\lambda) z^i = \left(\sum_{i=0}^{m-1} a_{nm+n-i}^{(k,l)}(\lambda) z^i \right)_{\substack{1 \leq k \leq r \\ 1 \leq l \leq s}}.$$

Proof. $\mathcal{W}_{n,m}$ is the matrix polynomial of degree $m - 1$ in z .

$$\begin{aligned} \frac{\mathcal{V}_{n,m}(u) - \mathcal{V}_{n,m}(z)}{u - z} &= \sum_{i=1}^m b_{nm+m-i}(\lambda) \frac{u^i - z^i}{u - z} \\ &= \sum_{i=1}^m b_{nm+m-i}(\lambda) \sum_{\mu=0}^{i-1} u^{i-1-\mu} z^{\mu} \end{aligned}$$

$$\begin{aligned} \mathcal{W}_{n,m}(z) &= \Gamma_{n,m} \left(\frac{\mathcal{V}_{n,m}(u) - \mathcal{V}_{n,m}(z)}{u - z} \right) \\ &= \sum_{i=1}^m b_{nm+m-i}(\lambda) \sum_{\mu=0}^{i-1} \Gamma_{n,m}(u^{i-1-\mu}) z^{\mu} \\ &= \sum_{i=1}^m b_{nm+m-i}(\lambda) \sum_{\mu=0}^{i-1} C_{n-m+i-\mu}(\lambda) z^{\mu} \end{aligned}$$

If these sums were developed, is obtained by identifying the coefficients matrix A_{nm+n-i} , $0 \leq i \leq m - 1$, of the polynomial $\mathcal{W}_{n,m}(z)$. More precisely

$$A_{nm+n-i}^{(k,l)} = \sum_{j=0}^{m-1-i} b_{nm+m-1-i-j}(\lambda) C_{n-m+1+j}(\lambda).$$

□

Let

$$\widetilde{V}_{n,m}(x, y) = \widetilde{\mathcal{V}}_{n,m}(z) = z^{nm+m} \mathcal{V}_{n,m}(z^{-1}) = \sum_{i=0}^m b_{nm+i}(\lambda) z^{nm+i}$$

$$\widetilde{W}_{n,m}(x, y) = \widetilde{\mathcal{W}}_{n,m}(z) = z^{nm+m-1} \mathcal{W}_{n,m}(z^{-1}) = \sum_{i=0}^{m-1} A_{nm+n-i}(\lambda) z^{nm+m-1-i}$$

Define the multivariate homogeneous matrix Padé-type approximant $(n/m)_F^H$ by:

$$(n/m)_F^H(x, y) = \sum_{i=0}^{n-m} C_i(\lambda) z^i + z^{n-m+1} \frac{\widetilde{\mathcal{W}}_{n,m}(z)}{\widetilde{\mathcal{V}}_{n,m}(z)} \quad (3.5)$$

The numerator and denominator of this fraction are polynomials in x and y containing only terms of total degree respectively between nm and $nm+n$, and between nm and $nm+m$.

It is assumed that the sum is null if $n < m$.

Let $\mathcal{V}_m(z) = \sum_{i=0}^m b_{m^2-i}^{(m)}(\lambda) z^i$ be the generator polynomial of the $(m-1/m)_F^H$.

We have:

Theorem 1.

$$\begin{aligned} F(x, y) - (m-1/m)_F^H(x, y) &= \frac{z^{m^2}}{\widetilde{\mathcal{V}}_m(z)} \Gamma\left(\frac{\mathcal{V}_m(u)}{1-uz}\right) \\ &= \left(\frac{z^{m^2}}{\widetilde{\mathcal{V}}_m(z)} \Gamma^{(k,l)}\left(\frac{\mathcal{V}_m(u)}{1-uz}\right) \right)_{\substack{1 \leq k \leq r \\ 1 \leq l \leq s}} \end{aligned} \quad (3.6)$$

Proof. We note $\Gamma_{m-1,m} = \Gamma$.

Firstly

$$F(x, y) = \sum_{i=0}^+ C_i(\lambda) z^i = \Gamma\left(\frac{1}{1-uz}\right) = \frac{z^{m^2}}{\widetilde{\mathcal{V}}_m(z)} \Gamma\left(\frac{\mathcal{V}_m(z^{-1})}{1-uz}\right)$$

Secondly

$$\begin{aligned} (m-1/m)_F^H(x, y) &= \frac{\widetilde{\mathcal{W}}_{m-1}(z)}{\widetilde{\mathcal{V}}_m(z)} \\ &= \frac{z^{m^2-1}}{\widetilde{\mathcal{V}}_m(z)} \Gamma\left(\frac{\mathcal{V}_m(u) - \mathcal{V}_m(z^{-1})}{u - z^{-1}}\right) \\ &= \frac{z^{m^2}}{\widetilde{\mathcal{V}}_m(z)} \Gamma\left(\frac{\mathcal{V}_m(u) - \mathcal{V}_m(z^{-1})}{uz - 1}\right) \end{aligned}$$

Then

$$\begin{aligned} F(x, y) - (m-1/m)_F^H(x, y) &= \frac{z^{m^2}}{\widetilde{\mathcal{V}}_m(z)} \Gamma\left(\frac{\mathcal{V}_m(u)}{1-uz}\right) \\ &= \left(\frac{z^{m^2}}{\widetilde{\mathcal{V}}_m(z)} \Gamma^{(k,l)}\left(\frac{\mathcal{V}_m(u)}{1-uz}\right) \right)_{\substack{1 \leq k \leq r \\ 1 \leq l \leq s}} \end{aligned}$$

□

4. Multivariate Homogeneous Matrix Padé Approximants

In this part, we will assume that orthogonality conditions are met, in order to reach high order estimates. These conditions will be used to calculate the denominator $\tilde{\mathcal{V}}_m(z)$ of the matrix Padé approximant $[m-1/m]_F^H$.

The level of the approximation is determined by the theorem bellow:

Theorem 2. Let $F_\lambda(z) = \sum_{i=0}^+ C_i(\lambda)z^i$ given in (3.3), we have

$$\tilde{\mathcal{V}}_m(z)F_\lambda(z) - \tilde{\mathcal{W}}_{m-1}(z) = O(z^{m^2})$$

Proof. By using the relation (3.6), we have

$$\begin{aligned} \tilde{\mathcal{V}}_m(z)F_\lambda(z) - \tilde{\mathcal{W}}_{m-1}(z) &= z^{m^2}\Gamma\left(\frac{\mathcal{V}_m(u)}{1-uz}\right) \\ &= z^{m^2}\Gamma\left(\mathcal{V}_m(u)\sum_{i=0}^+ u^i z^i\right) \\ &= z^{m^2}\Gamma\left(\left(\sum_{j=0}^m b_{m^2-j}^{(m)}(\lambda)u^j\right)\sum_{i=0}^+ u^i z^i\right) \\ &= z^{m^2}\sum_{i=0}^+ \Gamma\left(\sum_{j=0}^m b_{m^2-j}^{(m)}(\lambda)u^{i+j}\right) z^i \\ &= z^{m^2}\sum_{i=0}^+ \left(\sum_{j=0}^m b_{m^2-j}^{(m)}(\lambda)C_{i+j}(\lambda)\right) z^i \\ &= O(z^{m^2}) \end{aligned}$$

□

So, we obtain:

$$\begin{aligned} F(x,y)-(m-1/m)_F^H(x,y) &= \frac{z^{m^2}}{\tilde{\mathcal{V}}_m(z)}\Gamma\left(\frac{\mathcal{V}_m(u)}{1-uz}\right) \\ &= \frac{z^{m^2}}{\tilde{\mathcal{V}}_m(z)}\left(\Gamma^{(k,l)}\left(\frac{\mathcal{V}_m(u)}{1-uz}\right)\right) \begin{matrix} 1 \leq k \leq r \\ 1 \leq l \leq s \end{matrix} \end{aligned}$$

$$= \frac{z^{m^2}}{\tilde{\mathcal{V}}_m(z)} \left(\Gamma^{(k,l)}(\mathcal{V}_m(u)) + \Gamma^{(k,l)}(u\mathcal{V}_m(u))z + \dots \right. \\ \left. + \Gamma^{(k,l)}(u^{m-1}\mathcal{V}_m(u))z^{m-1} + z^m \Gamma^{(k,l)}\left(\frac{u^m \mathcal{V}_m(u)}{1-uz}\right) \right) \quad \begin{matrix} 1 \leq k \leq r \\ 1 \leq l \leq s \end{matrix}$$

We write

$$\begin{cases} m = p_m \times rs + n_m \\ n_m = q_m \times s + t_m \end{cases} \quad (4.1)$$

where p_m and n_m are respectively the quotient and rest of the Euclidean division of m by rs . Also q_m and t_m are respectively the quotient and rest of the Euclidean division of n_m by s .

Let α_m be the integer:

$$\alpha_m = \frac{m(m-1) + (p_m+1)(n_m+m)}{2},$$

and $\mathcal{V}_m(z) = \sum_{i=0}^m b_{\alpha_m-i}^{(m)}(\lambda) z^i$ be the polynomial generator of $(m-1/m)_F^H$

where $b_{\alpha_m-i}^{(m)}(\lambda) = \sum_{j=0}^{\alpha_m-i} b_{\alpha_m-i-j,j}^{(m)} \lambda_1^{\alpha_m-i-j} \lambda_2^j$.

The approximation error is given as follows:

$$\begin{aligned} F(x, y) - (m-1/m)_F^H(x, y) &= \frac{z^{\alpha_m}}{\tilde{\mathcal{V}}_m(z)} \Gamma\left(\frac{\mathcal{V}_m(u)}{1-uz}\right) \\ &= \frac{z^{\alpha_m}}{\tilde{\mathcal{V}}_m(z)} \left(\Gamma^{(k,l)}\left(\frac{\mathcal{V}_m(u)}{1-uz}\right) \right) \quad \begin{matrix} 1 \leq k \leq r \\ 1 \leq l \leq s \end{matrix} \\ &= \frac{z^{\alpha_m}}{\tilde{\mathcal{V}}_m(z)} \left\{ \Gamma^{(k,l)}(\mathcal{V}_m(u)) + \Gamma^{(k,l)}(u\mathcal{V}_m(u))z + \dots + \Gamma^{(k,l)}(u^{p_m-1}\mathcal{V}_m(u))z^{p_m-1} \right. \\ &\quad \left. + \Gamma^{(k,l)}(u^{p_m}\mathcal{V}_m(u))z^{p_m} + \Gamma^{(k,l)}\left(\frac{u^{p_m+1}\mathcal{V}_m(u)}{1-uz}\right)z^{p_m+1} \right\} \quad \begin{matrix} 1 \leq k \leq r \\ 1 \leq l \leq s \end{matrix} \end{aligned}$$

If we choose the polynomial $\mathcal{V}_m$ in the way that $(\Gamma^{(k,l)}(u^i \mathcal{V}_m(u))) \quad \begin{matrix} 1 \leq k \leq r \\ 1 \leq l \leq s \end{matrix} =$

$0, i = 0, \dots, j-1 \leq p_m-1$, we will obtain an approximation which is a $O(z^{\alpha_m+j})$

as the j first terms of the previous expression will vanish.

The $\mathcal{V}_m$ polynomial has $(m+1)\alpha_m - \frac{m(m-1)}{2} + 1$ arbitrary coefficients. It is almost entirely determined by the conditions:

$$\Gamma^{(k,l)}(u^j \mathcal{V}_m(u)) = 0, \quad 1 \leq k \leq r, \quad 1 \leq l \leq s, \quad 0 \leq j \leq p_m - 1 \quad (4.2)$$

$$\Gamma^{(k,l)}(u^{p_m} \mathcal{V}_m(u)) = 0, \quad 1 \leq k \leq q_m, \quad 1 \leq l \leq s \quad (4.3)$$

$$\Gamma^{(q_m+1,l)}(u^{p_m} \mathcal{V}_m(u)) = 0, \quad 1 \leq l \leq t_m \quad (4.4)$$

Then, we will have:

$$\begin{aligned} F(x, y) - (m-1/m)F_F^H(x, y) &= \frac{z^{\alpha_m+p_m}}{\tilde{\mathcal{V}}_m(z)} \left(\Gamma^{(k,l)}(u^{p_m} \mathcal{V}_m(u)) \right)_{\substack{1 \leq k \leq r \\ 1 \leq l \leq s}} \\ &+ \frac{z^{\alpha_m+p_m+1}}{\tilde{\mathcal{V}}_m(z)} \left(\Gamma^{(k,l)} \left(\frac{u^{p_m+1} \mathcal{V}_m(u)}{1-uz} \right) \right)_{\substack{1 \leq k \leq r \\ 1 \leq l \leq s}} \end{aligned} \quad (4.5)$$

where the q_m first lines and the t_m first components of the line $q_m + 1$ of the matrix $(\Gamma^{(k,l)}(u^{p_m} \mathcal{V}_m(u)))_{\substack{1 \leq k \leq r \\ 1 \leq l \leq s}}$ are null.

The conditions (4.2), (4.3) and (4.4) are represented in the systems follows:

$$\sum_{i=0}^m b_{\alpha_m-i}^{(m)}(\lambda) c_{i+j}^{(k,l)}(\lambda) = 0, \quad 1 \leq k \leq r, \quad 1 \leq l \leq s, \quad 0 \leq j \leq p_m - 1 \quad (4.6)$$

$$\sum_{i=0}^m b_{\alpha_m-i}^{(m)}(\lambda) c_{i+p_m}^{(k,l)}(\lambda) = 0, \quad 1 \leq k \leq q_m, \quad 1 \leq l \leq s \quad (4.7)$$

$$\sum_{i=0}^m b_{\alpha_m-i}^{(m)}(\lambda) c_{i+p_m}^{(q_m+1,l)}(\lambda) = 0, \quad 1 \leq l \leq t_m \quad (4.8)$$

We have a total $(m+1)\alpha_m - \frac{(m-1)m}{2}$ equations and $(m+1)\alpha_m - \frac{(m-1)m}{2} + 1$ unknown variables.

With the $C_i(\lambda)$, we can define the polynomial Hankel determinants generalizing the determinants as in [1],

$$H_0^{(j)}(\lambda) = 1$$

and for $m \geq 1$

$$H_m^{(j)}(\lambda) = \begin{vmatrix} c_{p_j}^{(q_j+1, t_j+1)}(\lambda) & c_{p_{j+1}}^{(q_j+1, t_j+1)}(\lambda) & \dots & c_{p_{j+m-1}}^{(q_j+1, t_j+1)}(\lambda) \\ c_{p_{j+1}}^{(q_{j+1}+1, t_{j+1}+1)}(\lambda) & c_{p_{j+1}+1}^{(q_{j+1}+1, t_{j+1}+1)}(\lambda) & \dots & c_{p_{j+1}+m-1}^{(q_{j+1}+1, t_{j+1}+1)}(\lambda) \\ \vdots & \vdots & \dots & \vdots \\ c_{p_{j+m-1}}^{(q_{j+m-1}+1, t_{j+m-1}+1)}(\lambda) & c_{p_{j+m-1}+1}^{(q_{j+m-1}+1, t_{j+m-1}+1)}(\lambda) & \dots & c_{p_{j+m-1}+m-1}^{(q_{j+m-1}+1, t_{j+m-1}+1)}(\lambda) \end{vmatrix}$$

If for every $m \geq 0$, $H_m^{(j)}(\lambda) \neq 0$, we say that the linear functional $\Gamma = (\Gamma^{(k,l)})_{\substack{1 \leq k \leq r \\ 1 \leq l \leq s}}$ is well defined.

Remark. If we suppose that the linear functional Γ is well defined, the linear systems (4.6), (4.7) and (4.8) that gives $b_{\alpha_m-i}^{(m)}(\lambda)$ coefficients has a solution giving by:

$$\mathcal{V}_0(z) = 1$$

$$\mathcal{V}_m(z) = \begin{vmatrix} c_{p_0}^{(q_0+1, t_0+1)}(\lambda) & c_{p_0+1}^{(q_0+1, t_0+1)}(\lambda) & \dots & c_{p_0+m}^{(q_0+1, t_0+1)}(\lambda) \\ c_{p_1}^{(q_1+1, t_1+1)}(\lambda) & c_{p_1+1}^{(q_1+1, t_1+1)}(\lambda) & \dots & c_{p_1+m}^{(q_1+1, t_1+1)}(\lambda) \\ \vdots & \vdots & \dots & \vdots \\ c_{p_{m-1}}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda) & c_{p_{m-1}+1}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda) & \dots & c_{p_{m-1}+m}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda) \\ 1 & z & \dots & z^m \end{vmatrix}$$

So we have

$$\mathcal{W}_{m-1}(z) = \Gamma \left(\frac{\mathcal{V}_m(u) - \mathcal{V}_m(z)}{u - z} \right)$$

$$\begin{aligned}
&= \left| \begin{array}{cccc} c_{p_0}^{(q_0+1, t_0+1)}(\lambda) & c_{p_0+1}^{(q_0+1, t_0+1)}(\lambda) & \dots & c_{p_0+m}^{(q_0+1, t_0+1)}(\lambda) \\ c_{p_1}^{(q_1+1, t_1+1)}(\lambda) & c_{p_1+1}^{(q_1+1, t_1+1)}(\lambda) & \dots & c_{p_1+m}^{(q_1+1, t_1+1)}(\lambda) \\ \vdots & \vdots & \dots & \vdots \\ c_{p_{m-1}}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda) & c_{p_{m-1}+1}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda) & \dots & c_{p_{m-1}+m}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda) \\ 0 & C_0(\lambda) & \dots & \sum_{i=0}^{m-1} C_{m-1-i}(\lambda) z^i \end{array} \right| \\
&= \left(\left| \begin{array}{cccc} c_{p_0}^{(q_0+1, t_0+1)}(\lambda) & c_{p_0+1}^{(q_0+1, t_0+1)}(\lambda) & \dots & c_{p_0+m}^{(q_0+1, t_0+1)}(\lambda) \\ c_{p_1}^{(q_1+1, t_1+1)}(\lambda) & c_{p_1+1}^{(q_1+1, t_1+1)}(\lambda) & \dots & c_{p_1+m}^{(q_1+1, t_1+1)}(\lambda) \\ \vdots & \vdots & \dots & \vdots \\ c_{p_{m-1}}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda) & c_{p_{m-1}+1}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda) & \dots & c_{p_{m-1}+m}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda) \\ 0 & c_0^{(k, l)}(\lambda) & \dots & \sum_{i=0}^{m-1} c_{m-1-i}^{(k, l)}(\lambda) z^i \end{array} \right| \right) \begin{array}{l} 1 \leq k \leq r \\ 1 \leq l \leq s \end{array}
\end{aligned}$$

$$\tilde{\mathcal{V}}_m(z) = z^m \mathcal{V}_m(z^{-1})$$

$$= \left| \begin{array}{cccc} c_{p_0}^{(q_0+1, t_0+1)}(\lambda) & c_{p_0+1}^{(q_0+1, t_0+1)}(\lambda) & \dots & c_{p_0+m}^{(q_0+1, t_0+1)}(\lambda) \\ c_{p_1}^{(q_1+1, t_1+1)}(\lambda) & c_{p_1+1}^{(q_1+1, t_1+1)}(\lambda) & \dots & c_{p_1+m}^{(q_1+1, t_1+1)}(\lambda) \\ \vdots & \vdots & \dots & \vdots \\ c_{p_{m-1}}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda) & c_{p_{m-1}+1}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda) & \dots & c_{p_{m-1}+m}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda) \\ z^m & z^{m-1} & \dots & 1 \end{array} \right|$$

Likewise

$$\tilde{\mathcal{W}}_{m-1}(z) = z^{m-1} \mathcal{W}_{m-1}(z^{-1})$$

$$= \begin{vmatrix} c_{p_0}^{(q_0+1, t_0+1)}(\lambda) & c_{p_0+1}^{(q_0+1, t_0+1)}(\lambda) & \dots & c_{p_0+m}^{(q_0+1, t_0+1)}(\lambda) \\ c_{p_1}^{(q_1+1, t_1+1)}(\lambda) & c_{p_1+1}^{(q_1+1, t_1+1)}(\lambda) & \dots & c_{p_1+m}^{(q_1+1, t_1+1)}(\lambda) \\ \vdots & \vdots & \dots & \vdots \\ c_{p_{m-1}}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda) & c_{p_{m-1}+1}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda) & \dots & c_{p_{m-1}+m}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda) \\ z^m N_{-1}(z) & z^{m-1} N_0(z) & \dots & N_{m-1}(z) \end{vmatrix},$$

where $N_i(z) = \sum_{k=0}^i C_k(\lambda) z^k$ and $N_{-1}(z) \equiv 0$.

We notice that $\tilde{\mathcal{W}}_{m-1}(z)$ is a polynomial in z of degree $m-1$. We divide the j th column ($j = 1, \dots, m+1$) of the two previous determinants by z^{m-j+1} and we multiply the $i = 1, \dots, m$ line by z^m .

The quotient of the two new determinants is the homogeneous matrix Padé approximant $[m-1/m]_F^H$

$$[m-1/m]_F^H(\lambda_1 z, \lambda_2 z) = \frac{1}{\tilde{\mathcal{V}}_m(z)} \cdot \tilde{\mathcal{W}}_{m-1}(z)$$

$$\begin{array}{c}
\begin{array}{cccc}
c_{p_0}^{(q_0+1, t_0+1)}(\lambda) & c_{p_0+1}^{(q_0+1, t_0+1)}(\lambda)z & \dots & c_{p_0+m}^{(q_0+1, t_0+1)}(\lambda)z^m \\
c_{p_1}^{(q_1+1, t_1+1)}(\lambda) & c_{p_1+1}^{(q_1+1, t_1+1)}(\lambda)z & \dots & c_{p_1+m}^{(q_1+1, t_1+1)}(\lambda)z^m \\
\vdots & \vdots & \dots & \vdots \\
c_{p_{m-1}}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda) & c_{p_{m-1}+1}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda)z & \dots & c_{p_{m-1}+m}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda)z^m
\end{array} \\
= \frac{\begin{array}{cccc}
N_{-1}(z) & N_0(z) & \dots & N_{m-1}(z)
\end{array}}{\begin{array}{cccc}
c_{p_0}^{(q_0+1, t_0+1)}(\lambda) & c_{p_0+1}^{(q_0+1, t_0+1)}(\lambda)z & \dots & c_{p_0+m}^{(q_0+1, t_0+1)}(\lambda)z^m \\
c_{p_1}^{(q_1+1, t_1+1)}(\lambda) & c_{p_1+1}^{(q_1+1, t_1+1)}(\lambda)z & \dots & c_{p_1+m}^{(q_1+1, t_1+1)}(\lambda)z^m \\
\vdots & \vdots & \dots & \vdots \\
c_{p_{m-1}}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda) & c_{p_{m-1}+1}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda)z & \dots & c_{p_{m-1}+m}^{(q_{m-1}+1, t_{m-1}+1)}(\lambda)z^m \\
1 & 1 & \dots & 1
\end{array}}
\end{array}$$

5. Recurrence Relations

In the sequel, we shall assume that the linear functional Γ is well defined.

Lemma. *The polynomials $\{\mathcal{V}_m\}$ are linearly independent in $\mathbb{C}(\lambda)[z]$, the space of rational functions in the variable z with coefficients being complex-valued bivariate polynomials in λ .*

Proof. Let's suppose the existence of a combinaison of factors $\eta_0(\lambda), \eta_1(\lambda), \dots \in \mathbb{C}(\lambda)$ as

$$\sum_{i=0}^{+} \eta_i(\lambda) \mathcal{V}_i(z) = 0 \quad \forall z \in \mathbb{C}$$

Also for $j \geq 0$, we obtain

$$\sum_{i=0}^{+} \eta_i(\lambda) \Gamma^{(q_j+1, t_j+1)}(z^{p_j} \mathcal{V}_i(z)) \equiv 0$$

Taking (4.2) into account, we obtain for $j \geq 0$

$$\sum_{i=0}^j \eta_i(\lambda) \Gamma^{(q_j+1, t_j+1)}(z^{p_j} \mathcal{V}_i(z)) \equiv 0$$

For $j = 0$, the combinaison is reduced to

$$\eta_0(\lambda) \Gamma^{(1,1)}(\mathcal{V}_0(z)) \equiv 0$$

This result is that $\eta_0(\lambda) = 0$ because $\Gamma^{(1,1)}(\mathcal{V}_0(z)) = H_1^{(0)}(\lambda) \neq 0$.

For $j \geq 1$ the proof that $\eta_j(\lambda) = 0$ is by induction. $\square$

Theorem 3. *The polynomials $\mathcal{V}_m(z)$ and $\mathcal{W}_m(z)$ satisfy the recurrence relations:*

$$z\mathcal{V}_m(z) = \sum_{i=\max(0, m-rs)}^{m+1} \varphi_i^{(m+1)}(\lambda) \mathcal{V}_i(z), \quad \mathcal{V}_0(z) = 1 \quad (5.1)$$

$$z\mathcal{W}_{m-1}(z) + \Gamma(\mathcal{V}_m(z)) = \sum_{i=\max(0, m-rs)}^{m+1} \varphi_i^{(m+1)}(\lambda) \mathcal{W}_{i-1}(z), \quad \mathcal{W}_{-1}(z) = 0 \quad (5.2)$$

The $\varphi_i^{(m+1)}(\lambda)$, $0 \leq i \leq m$, are computed using the triangular system:

$$\Gamma^{(q_j+1, t_j+1)}(z^{p_j+1} \mathcal{V}_m(z)) = \sum_{i=0}^j \varphi_i^{(m+1)}(\lambda) \Gamma^{(q_j+1, t_j+1)}(z^{p_j} \mathcal{V}_i(z)), \quad 0 \leq j \leq m$$

Proof. The polynomials $z\mathcal{V}_m(z)$ can be written as linear combinaison:

$$z\mathcal{V}_m(z) = \sum_{i=0}^{m+1} \varphi_i^{(m+1)}(\lambda) \mathcal{V}_i(z)$$

where $\varphi_i^{(m+1)}(\lambda)$ are the rational functions of the variable λ .

We multiply left and right hand sides with z^{p_j} and apply the linear functional Γ to obtain:

$$\varphi_i^{(m+1)}(\lambda) = 0, \quad i = 0, \dots, m - rs - 1 \quad (\text{if } m \geq rs + 1)$$

$$\Gamma^{(q_j+1, t_j+1)}(z^{p_j+1} \mathcal{V}_m(z)) = \sum_{i=0}^j \varphi_i^{(m+1)}(\lambda) \Gamma^{(q_j+1, t_j+1)}(z^{p_j} \mathcal{V}_i(z)), \quad 0 \leq j \leq m$$

On the other hand, we have

$$\mathcal{V}_m(z) = H_m^{(0)}(\lambda)z^m + \dots,$$

So that consequently

$$H_m^{(0)}(\lambda) = \varphi_{m+1}^{(m+1)}(\lambda)H_{m+1}^{(0)}(\lambda).$$

For the associated polynomials $\{\mathcal{W}_m(z)\}$ we have, using

$$\frac{u\mathcal{V}_m(u) - z\mathcal{V}_m(z)}{u - z} = z \frac{\mathcal{V}_m(u) - \mathcal{V}_m(z)}{u - z} + \mathcal{V}_m(u),$$

$$\Gamma \left(z \frac{\mathcal{V}_m(u) - \mathcal{V}_m(z)}{u - z} + \mathcal{V}_m(u) \right) = \sum_{i=\max(0, m-rs)}^{m+1} \varphi_i^{(m+1)}(\lambda) \Gamma \left(\frac{\mathcal{V}_i(u) - \mathcal{V}_i(z)}{u - z} \right)$$

which gives the expected results. $\square$

To compute the polynomial Hankel determinants, we need the following theorem proved in [3].

Theorem 4. *Let A be a matrix, and let $A_{\mu k}$ denote the matrix with row μ and column k deleted. Also let $A_{\mu\nu, kl}$ denote the matrix A with rows μ and ν and columns k and l deleted. Provided $\mu < \nu$ and $k < l$,*

$$\det A \det A_{\mu\nu, kl} = \det A_{\mu k} \det A_{\nu l} - \det A_{\mu l} \det A_{\nu k}.$$

Theorem 5. *The polynomial Hankel determinants $\{H_m^{(j)}\}$ satisfies the recurrence relation*

$$H_{m+1}^{(j)}(\lambda)H_{m-1}^{(rs+j+1)}(\lambda) = H_m^{(j)}(\lambda)H_m^{(rs+j+1)}(\lambda) - H_m^{(j+1)}(\lambda)H_m^{(rs+j)}(\lambda) \quad (5.3)$$

for $m, j = 1, 2, \dots$

$$H_{-1}^{(j)}(\lambda) = 0, \quad H_0^{(j)}(\lambda) = 1.$$

Proof. Let A denote the matrix

$$A = \begin{pmatrix} c_{p_j}^{(q_j+1, t_j+1)}(\lambda) & c_{p_{j+1}}^{(q_j+1, t_j+1)}(\lambda) & \dots & c_{p_{j+m}}^{(q_j+1, t_j+1)}(\lambda) \\ c_{p_{j+1}}^{(q_{j+1}+1, t_{j+1}+1)}(\lambda) & c_{p_{j+1}+1}^{(q_{j+1}+1, t_{j+1}+1)}(\lambda) & \dots & c_{p_{j+1}+m}^{(q_{j+1}+1, t_{j+1}+1)}(\lambda) \\ \vdots & \vdots & \dots & \vdots \\ c_{p_{j+m}}^{(q_{j+m}+1, t_{j+m}+1)}(\lambda) & c_{p_{j+m}+1}^{(q_{j+m}+1, t_{j+m}+1)}(\lambda) & \dots & c_{p_{j+m}+m}^{(q_{j+m}+1, t_{j+m}+1)}(\lambda) \end{pmatrix}$$

We take $\mu = 1$, $\nu = m + 1$, $k = 1$ and $l = m + 1$. The theorem is a consequence of the previous theorem of fact that

$$\det A = H_{m+1}^{(j)}(\lambda), \quad \det A_{\mu\nu,kl} = H_{m-1}^{(rs+j+1)}(\lambda), \quad \det A_{\mu k} = H_m^{(rs+j+1)}(\lambda),$$

$$\det A_{\nu l} = H_m^{(j)}(\lambda), \quad \det A_{\mu l} = H_m^{(j+1)}(\lambda), \quad \det A_{\nu k} = H_m^{(rs+j)}(\lambda)$$

and the follows relations satisfies by p_j, q_j, n_j and t_j introduced in (4.1),

$$\begin{aligned} p_{j+k.rs} &= p_j + k, & n_{j+k.rs} &= n_j \\ q_{j+k.rs} &= q_j + k, & t_{j+k.rs} &= t_j \end{aligned} \quad j, k = 0, 1, \dots \quad (5.4)$$

By the previous theorem, we deduce the recursive relation:

$$H_{m+1}^{(j)}(\lambda)H_{m-1}^{(rs+j+1)}(\lambda) = H_m^{(j)}(\lambda)H_m^{(rs+j+1)}(\lambda) - H_m^{(j+1)}(\lambda)H_m^{(rs+j)}(\lambda)$$

for $m, j = 1, 2, \dots$

□

6. Numerical Example

Based on the analysis above, we consider present the following example for computing the multivariate homogeneous matrix Padé approximants to the matrix function.

$$F(x, y) = \begin{pmatrix} -(xy + 1)e^{-x^2 - y^2} & -\frac{3y}{1 + x^2 + y^2} \\ \sqrt{4 - x^2 - y^2} & -x^2 - y^2 + \ln(1 + x + y) \end{pmatrix}$$

The exact value of F at the point $(x_0, y_0) = (\frac{\sqrt{3}}{200}, \frac{1}{200})$ is given by

$$\begin{aligned} F(x_0, y_0) &= \begin{pmatrix} -(1 + \frac{\sqrt{3}}{40000})e^{10000} & -\frac{150}{10001} \\ \frac{\sqrt{39999}}{100} & -\frac{1}{10000} + \ln(\frac{201}{200} + \frac{\sqrt{3}}{200}) \end{pmatrix} \\ &\approx \begin{pmatrix} -0.99994330194011203964 & -0.014998500149985001500 \\ 1.9999749998437480468 & 0.013467793835957699252 \end{pmatrix} \end{aligned}$$

m	$[m - 1/m]_F(x_0, y_0)$	$\ (F - [m - 1/m]_F)(x_0, y_0)\ _2$
1	$\begin{pmatrix} -1 & 0 \\ 2 & 0 \end{pmatrix}$	0.0201579
2	$\begin{pmatrix} -0.999943304484753 & -0.014999149567271296 \\ 1.999886608969506 & 0.013659479562703306 \end{pmatrix}$	0.000211085
3	$\begin{pmatrix} -0.9999433069394953 & -0.014998500149985002 \\ 1.99997500249975 & 0.013465606207034462 \end{pmatrix}$	2.18763×10^{-6}
4	$\begin{pmatrix} -0.9999433019669471 & -0.01499850002173361 \\ 1.9999749998580398 & 0.013467793874129514 \end{pmatrix}$	1.35594×10^{-10}
5	$\begin{pmatrix} -0.9999433019401114 & -0.014998500147536242 \\ 1.9999749998438265 & 0.013467793836360091 \end{pmatrix}$	2.48165×10^{-12}
6	$\begin{pmatrix} -0.9999433019401119 & -0.014998500149985 \\ 1.9999749998437488 & 0.013467793835961195 \end{pmatrix}$	3.6221×10^{-15}
7	$\begin{pmatrix} -0.9999433019401117 & -0.014998500149985 \\ 1.9999749998437482 & 0.01346779383595773 \end{pmatrix}$	3.33072×10^{-16}
8	$\begin{pmatrix} -0.9999433019401123 & -0.014998500149985002 \\ 1.999974999843748 & 0.013467793835957701 \end{pmatrix}$	3.17599×10^{-17}

The exact value of F at the point $(x_0, y_0) = (\frac{\sqrt{2}}{200}, \frac{\sqrt{2}}{200})$ is given by

$$\begin{aligned}
 F(x_0, y_0) &= \begin{pmatrix} -\frac{20001}{20000}e^{10000} & -\frac{150}{10001}\sqrt{2} \\ \frac{39999}{100} & -\frac{1}{10000} + \ln(1 + \frac{1}{50}\sqrt{2}) \end{pmatrix} \\
 &\approx \begin{pmatrix} -0.99995000000008332917 & -0.021211082327363689363 \\ 1.9999749998437480468 & 0.013943068544592249131 \end{pmatrix}
 \end{aligned}$$

m	$[m - 1/m]_F(x_0, y_0)$	$\ (F - [m - 1/m]_F)(x_0, y_0)\ _2$
1	$\begin{pmatrix} -1 & 0 \\ 2 & 0 \end{pmatrix}$	0.0253835
2	$\begin{pmatrix} -0.999950002499875 & -0.021212142828455002 \\ 1.99990000499975 & 0.014141428552303336 \end{pmatrix}$	0.000212066
3	$\begin{pmatrix} -0.9999500049994999 & -0.021211082327363687 \\ 1.9999750024997491 & 0.01394074154957599 \end{pmatrix}$	2.327×10^{-6}
4	$\begin{pmatrix} -0.999950000022005 & -0.021211082118386446 \\ 1.9999749998554406 & 0.013943068560980125 \end{pmatrix}$	2.10666×10^{-10}
5	$\begin{pmatrix} -0.9999500000000815 & -0.02121108232250809 \\ 1.9999749998438086 & 0.013943068544664601 \end{pmatrix}$	4.85614×10^{-12}
6	$\begin{pmatrix} -0.9999500000000834 & -0.021211082327363698 \\ 1.999974999843749 & 0.013943068544594393 \end{pmatrix}$	2.26543×10^{-15}
7	$\begin{pmatrix} -0.9999500000000828 & -0.021211082327363687 \\ 1.9999749998437477 & 0.01394306854459228 \end{pmatrix}$	7.11118×10^{-16}
8	$\begin{pmatrix} -0.9999500000000833 & -0.021211082327363694 \\ 1.9999749998437482 & 0.01394306854459225 \end{pmatrix}$	1.11328×10^{-16}

For (x, y) varying over the domains represented by the figure 1-4, the absolute values of all entries of the error matrix $(F - [3/4]_F^H)(x, y)$ are less than or equal to 10^{-10} .

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