A RADICAL PROPERTY OF KRASNER TERNARY HYPERRINGS

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Abstract: In 2010, Davvaz and Mirvakili [6] defined the concept of Krasner \((m, n, )\)-hyperring, defined a fundamental relation on the structure and proved the isomorphism theorems where the hyper ideals considered in the construction of its quotient class is not necessarily normal. Recently, Castillo and Vilela [5] considered a particular case where \(m = 2\) and \(n = 3\) called a Krasner ternary hyperring and proved the isomorphism theorems without the normality condition. In this article, regularity is defined in Krasner ternary hyperrings and proved that this property is radical in these classes of algebraic hyperstructure.

AMS Subject Classification: 03E20, 05C25
Key Words: Krasner ternary hyperring, Krasner \((m, n, )\)-hyperring

1. Introduction

Hyperstructure theory was introduced in 1934 by the French Mathematician F. Marty [9] at the 8th Congress of Scandinavian Mathematicians where he defined...
the notion of hyperoperation on groups. In a classical algebraic structure, the binary operation of two elements of a set is again an element of the same set, while in an algebraic hyperstructure, the hyperoperation of two elements, is a subset of the same set. If this hyperoperation sends two elements to a singleton, then the hyperoperation coincides with the classical binary operation.

In literature, a number of different hyperstructure theories are widely studied since these represent a suitable and natural generalization of classical algebraic structures such as groups, rings and modules and for their applications to many areas of pure and applied mathematics and computer science.

In [3] Assokumar and Velrajnan, investigates the properties of regular multiplicative hyperrings. In this article, we extend the concept of regularity in the context of Krasner ternary hyperrings and prove that regularity is a radical property in these classes of algebraic hyperstructures.

2. Preliminaries

This section presents some definitions in hyperstructure theory and the preliminary concepts of this study.

**Definition 2.1.** [7] Let $H$ be a nonempty set and let $\wp^*(H)$ be the set of all nonempty subsets of $H$. A map $*: H \times H \to \wp^*(H)$ is called a hyperoperation on the set $H$. The couple $(H, *)$ is called a hypergroupoid. If $A$ and $B$ are nonempty subsets of $H$, we define

i. $A * B = \bigcup_{a \in A, b \in B} a * b$;

ii. $x * A = \bigcup_{a \in A} x * a$;

iii. $B * x = \bigcup_{b \in B} b * x$.

**Example 2.2.** Let $\mathbb{N}$ be the set of positive integers. Define a hyperoperation “*” on $\mathbb{N}$ by $x * y = \{x, y\}$ for all $x, y \in \mathbb{N}$. Then $(\mathbb{N}, *)$ is a hypergroupoid.

**Definition 2.3.** [7] A hypergroupoid $(H, *)$ is commutative if $a * b = b * a$ for all $a, b \in H$.

**Definition 2.4.** [7] A hypergroupoid $(H, *)$ is called a semihypergroup if for all $x, y, z \in H$, $(x * y) * z = x * (y * z)$.
**Definition 2.5.** [7] A nonempty subset $B$ of a semihypergroup $H$ is called a *subsemihypergroup* of $H$ if $(B, *)$ is itself a semihypergroup.

**Definition 2.6.** [7] Let $(H, *)$ be a semihypergroup. Then $H$ is called a *hypergroup* if it satisfies the *reproduction axiom*, that is, $x * H = H * x = H$ for all $x \in H$.

**Definition 2.7.** [7] A nonempty set $H$ with a hyperoperation “+” is said to be a *canonical hypergroup* if the following are satisfied:

i. For every $x, y \in H$, $x + y = y + x$;

ii. For every $x, y, z \in H$, $x + (y + z) = (x + y) + z$;

iii. $\exists 0_H \in H$ (called the *neutral element of $H$*) such that $x + 0_H = \{x\}$ and $0_H + x = \{x\}$ for all $x \in H$;

iv. for each $x \in H$ there exists a unique element denoted by $-x \in H$ such that $0_H \in [x + (-x)] \cap [(-x) + x]$;

v. for every $x, y, z \in H$, $z \in x + y$ implies $y \in -x + z$ and $x \in -y + z$.

The pair $(H, +)$ is called a *canonical hypergroup*.

**Lemma 2.8.** [7] If $(H, +)$ is a canonical hypergroup then the following hold:

i. $-(-a) = a$;

ii. $0$ is the unique element such that for every $a \in H$, there is an element $-a \in H$ with the property $0 \in a + (-a)$;

iii. $-0 = 0$;

iv. $-(a + b) = -a - b$;

for all $a, b \in H$.

**Example 2.9.** Let $H = [0, 1]$. We define a hyperaddition “+” on $H$ as follows:

$$x + y = \begin{cases} [0, x], & \text{if } x = y, \\ \max \{x, y\}, & \text{if } x \neq y. \end{cases}$$

**Definition 2.10.** A *Krasner ternary hyperring* is an algebraic structure $(R, +, \cdot)$ consisting of a nonempty set $R$, a hyperoperation “+” and a ternary multiplication “." satisfying the following:
i. \((R, +)\) is a canonical hypergroup;

ii. \((a \cdot b \cdot c) \cdot d \cdot e = a \cdot (b \cdot c \cdot d) \cdot e = a \cdot b \cdot (c \cdot d \cdot e)\);

iii. \((a + b) \cdot c \cdot d = a \cdot c \cdot d + b \cdot c \cdot d\);

iv. \(a \cdot (b + c) \cdot d = a \cdot b \cdot d + a \cdot c \cdot d\);

v. \(a \cdot b \cdot (c + d) = a \cdot b \cdot c + a \cdot b \cdot d\),

vi. \(a \cdot 0 \cdot 0 = 0 \cdot a \cdot 0 = 0 \cdot 0 \cdot a = 0\).

for every \(a, b, c, d, e \in R\). If without ambiguity, \(a \cdot b \cdot c\) can be written as \(abc\) unless otherwise specified.

**Example 2.11.** Consider the canonical hypergroup \(([0, 1], +)\) in Example 2.9. If we define "+" to be the ordinary multiplication in \(\mathbb{R}\), then \(([0, 1], +, \cdot)\) is a ternary hyperring.

**Definition 2.12.** Let \((R, +, \cdot)\) be a Krasner ternary hyperring. A subset \(S\) of \(R\) is called a **Krasner subternary hyperring** if \((S, +, \cdot)\) is itself a Krasner ternary hyperring.

**Remark 2.13.** \(\{0\}\) is a subternary hyperring of \(R\) called the **trivial subternary hyperring** of \(R\).

**Definition 2.14.** A Krasner subternary hyperring \(I\) of a Krasner ternary hyperring \(R\) is called a

i. **right hyper ideal** of \(R\) if \(abi \in I\) for all \(a, b \in R\) and \(i \in I\);

ii. **left hyper ideal** of \(R\) if \(iab \in I\) for all \(a, b \in R\) and \(i \in I\);

iii. **lateral hyper ideal** of \(R\) if \(aib \in I\) for all \(a, b \in R\) and \(i \in I\);

iv. **two sided hyper ideal**, if \(I\) is both a right and left hyper ideal;

v. **hyper ideal** if \(I\) is left, right and lateral hyper ideal.

**Theorem 2.15.** (Hyper ideal Criterion) [5] Let \(R\) be a ternary hyperring. A nonempty subset \(I\) of \(R\) is a right [resp. left and lateral] hyper ideal of \(R\) if and only if for all \(i, j \in I\) and \(a, b \in R\)

i. \(i - j \subseteq I\),

ii. \(abi \in I\) [resp. \(iab \in R\) and \(aib \in R\)].
Example 2.16. Let \([0, 1], +, \cdot\) be the ternary hyperring in Example 2.11 and \(I = [0, \frac{1}{2}]\). It can be verified by using Theorem 2.15 that \(I = [0, \frac{1}{2}]\) is a hyper ideal of \([0, 1]\).

Definition 2.17. [5] Let \((R_1, +, \cdot)\) and \((R_2, \oplus, *)\) be two ternary hyper-rings. A mapping \(\phi : R_1 \rightarrow R_2\) is called a homomorphism if the following are satisfied:

i. \(\phi(a + b) = \phi(a) \oplus \phi(b)\) for all \(a, b \in H\);

ii. \(\phi(a_1 \cdot a_2 \cdot a_3) = \phi(a_1) * \phi(a_2) * \phi(a_3)\) for all \(a_1, a_2, a_3 \in R_1\);

iii. \(\phi(0_{R_1}) = 0_{R_2}\).

Lemma 2.18. [5] Let \((H, +)\) be a canonical hypergroup and \(N \subseteq H\). Then \(N\) is a subcanonical hypergroup of \(H\) if and only if \(x - y \subseteq N\) for all \(x, y \in N\).

Theorem 2.19. [5] Let \(I\) and \(J\) be hyper ideals of a ternary hyperring \(R\). Then

i. \(I\) is a hyper ideal of \(I + J\),

ii. \(I \cap J\) is a hyper ideal of \(I\).

Theorem 2.20. [5] Let \(R\) be a ternary hyperring, \(I\) a hyper ideal of \(R\) and \(x, y \in R\) then, \(x + I = y + I\) if and only if \(x \in y + I\).

Theorem 2.21. [5] Let \(I\) and \(J\) be right [resp. left and lateral] hyper ideals of a Krasner ternary hyperring \(R\) such that \(I \subseteq J\). Then \(J/I\) is a hyper ideal of \(R/I\).

Theorem 2.22. (Second Isomorphism Theorem) [5] Let \(R\) be a ternary hyperring and \(I\) and \(J\) be hyper ideals of \(R\). Then \(I/(I \cap J) \cong (I + J)/J\).

3. Main Results

This section presents the main results generated in this study.

Let us recall that an element \(x\) of a ring \(R\) is regular if \(x = xrx\) for some \(r \in R\). A ring \(R\) is called a regular ring if all of its element is regular. We define regularity in Krasner ternary hyperring as follows:
Definition 3.1. An element \(a\) in a Krasner ternary hyperring \(R\) is called \textit{regular} if there exists an element \(x\) in \(R\) such that \(a = axa\). A Krasner ternary hyperring is called \textit{regular} if all of its elements are regular.

In ring theory, a classic way of obtaining reasonable structure results is to determine all rings up to isomorphism and to single out some “bad” properties of these rings and study only those rings that do not have these properties. We follow this procedure in Krasner ternary hyperring. In order to do this, some additional assumptions must be made.

Let \(P\) be a property of a Krasner ternary hyperring \(R\), the \(P\)-radical of \(R\) is the largest hyper ideal of \(R\) that possesses property \(P\). We denote by \(P(R)\) the \(P\)-hyperradical of \(R\).

Definition 3.2. A \textit{radical property} \(P\) of a Krasner ternary hyperring is one that satisfies the following conditions:

i. the homomorphic image of a \(P\)-hyper ideal is a \(P\)-hyper ideal;

ii. every Krasner ternary hyperring \(R\) contains a \(P\)-hyperradical;

iii. the \(P\)-hyperradical of the quotient Krasner ternary hyperring \(R/P(R)\) is zero;

iv. the \(P\)-hyperradical of the Krasner ternary hyperring \(P(R)\) is \(P(R)\).

Remark 3.3. Let \(R\) be a Krasner ternary hyperring and \(a \in R\). Then \(aRa = \{ara \mid r \in R\}\) is a canonical hypergroup.

To see this, let \(x, y \in aRa\). Then \(x = ar_ia\) and \(y = ar_ia\) for some \(r_i, r_j \in R\). Thus, \(x + (-y) = ar_ia + (-ar_2a) = a(r_1 + (-r_2))a \subseteq aRa\). By Lemma 2.18, \(aRa\) is a canonical hypergroup.

Lemma 3.4. Let \(R\) be a Krasner ternary hyperring,
\[
S = \{r \in R \mid r \text{ is regular}\}
\]
and \(a \in R\). If \(S \cap (a + (-axa)) \neq \emptyset\) for some \(x \in R\), then \(a\) is a regular element of \(R\).

Proof. Let \(a \in R\) and \(S \cap (a + (-axa)) \neq \emptyset\) for some \(x \in R\). Then there exists \(s \in a + (-axa)\) such that \(s = sts\) for some \(t \in R\). By Definition 2.7(v), \(a \in s + axa\). Hence, by Remark 3.3
\[
a \in s + axa
= sts + axa
\]
∈ (a + (−axa))t(a + (−axa)) + axa
= ata + (−axata) + (−ataxa) + axatax + axa
= a(t + (−xat) + (−tax) + xatx + x)a
⊆ aRa

Hence, a ∈ aRa. Therefore, a = ara for some r ∈ R and a is regular. □

**Theorem 3.5.** Let R be a Krasner ternary hyperring and \{It\}_{t ∈ N} the family of right [resp. left and lateral] hyper ideals of R. Then

\[ M = \bigcup \left\{ \sum_{t=1}^{n} a_t \mid a_t \in I_t, t, n \in \mathbb{N} \right\} \]

is a right [resp. left and lateral] hyper ideal of R.

**Proof.** Let a, b ∈ M. Then a ∈ \( \sum_{t=1}^{m} a_t \) and b ∈ \( \sum_{s=1}^{n} b_s \) for some \( m, n \in \mathbb{N} \), \( a_t \in I_t, a_s \in I_s \), and \( s, t \in \mathbb{N} \). Thus,

\[ a + (-b) \subseteq \sum_{t=1}^{m} a_t + \left[ -\sum_{s=1}^{n} b_s \right] = \sum_{t=1}^{m} a_t + \sum_{s=1}^{n} (-b_s) = \sum_{j=1}^{m+n} d_j \subseteq M, \]

where

\[ d_j = \begin{cases} a_s, & s = 1, 2, \ldots, m, \text{ if } j \leq m \\ -b_t, & t = m + 1, m + 2, \ldots, n, \text{ if } j > m \end{cases} \]

Also, for all \( x, y \in R \), and \( a \in M \), \( xy a \in xy \sum_{t=1}^{m} a_t = \sum_{t=1}^{n} xya_t \subseteq M \). Therefore, by Theorem 2.15, M is a right hyper ideal of R. Similarly, M is a left and lateral hyper ideal of R. □

**Theorem 3.6.** Let R be a Krasner ternary hyperring and I be a hyper ideal of R. Then R is regular if and only if both R and R/I are regular.

**Proof.** Suppose that both R/I and I are regular. Let \( x \in R \). Then \( x + I \in R/I \) and since R/I is regular, there exists \( y + I \in R/I \) such that \( (x + I)(y + I)(x + I) = x + I \). So, \( xyx + I = x + I \) and by Remark 2.20, \( xyx \in x + I \). Thus, \( xyx \in x + i \) for some \( i \in I \). By Definition 2.7(v), \( i \in x + (−xyx) \). Since i is regular, \( x + (−xyx) \cap S \neq \emptyset \), where S is the
set containing all regular elements of $R$. Therefore, by Lemma 3.4, $x$ is regular. Since $x$ is arbitrary, $R$ is regular. Conversely, if $R$ is regular, then $I$ is also regular. Let $x + I \in R/I$. Then $x \in R$ and there exists $y \in R$ such that $x = yx$. Thus, $x + I = xy + I = (x + I)(y + I)(x + I)$. Therefore, $R/I$ is regular. □

**Theorem 3.7.** The homomorphic image of a regular Krasner ternary hypering is regular.

**Proof.** Let $R$ be a Krasner ternary hyperring, $\phi$ a homomorphism on $R$ and $y \in \text{Im}\phi$. Then there exists $x \in R$ such that $\phi(x) = y$. Since $R$ is regular, there exists $a \in R$ such that $xax = x$. Thus,

$$y = \phi(x) = \phi(xax) = \phi(x)\phi(a)\phi(x) = y\phi(a)y.$$

Therefore, $y$ is regular and $\text{Im}\phi$ is regular. □

**Theorem 3.8.** Let $R$ be a Krasner ternary hyperring. If $I$ and $J$ are regular hyper ideals of $R$, then $I + J$ is regular.

**Proof.** Let $\phi : J \to (I + J)/I$ be defined by $\phi(j) = j + I$. Then from the proof of Theorem 2.22, $\phi$ is a homomorphism and $\text{Im}\phi = (I + J)/I$. Thus, $(I + J)/I$ is the homomorphic image of a regular hyper ideal $J$. Hence, by Theorem 3.7, $(I + J)/I$ is regular. Since $I$ is a regular hyper ideal of $I + J$, by Theorem 3.6, $I + J$ is regular. □

By induction on $n$, the next remark follows directly from Theorem 3.8.

**Remark 3.9.** Let $R$ be a Krasner ternary hyperring and $n \in \mathbb{N}$. If $I_1, \ldots, I_n$ are regular hyper ideals of $R$, then $I_1 + I_2 + \ldots + I_n$ is regular.

**Theorem 3.10.** Any Krasner ternary hyperring $R$ has a regular hyper radical.

**Proof.** Let $R$ be a Krasner ternary hyperring and $\{I_\omega\}_{\omega \in \Omega}$ the family of regular hyper ideals of $R$. By Theorem 3.5,

$$M = \bigcup \left\{ \sum_{t=1}^{n} a_t \mid a_t \in I_t, t \in \Omega, n \in \mathbb{N} \right\}$$

is a hyper ideal of $R$. If $x \in M$, then $x \in a_1 + a_2 + \ldots + a_n$ for some $n \in \mathbb{N}$ and $a_i \in I_i$. By Remark 3.9, $I_1 + I_2 + \ldots + I_n$ is regular. Therefore, $x$ is regular. Consequently, $M$ is regular. Since $M$ contains all regular hyper ideals of $R$, $M$ is the regular hyperradical of $R$. □
Theorem 3.11. Let $R$ be a Krasner ternary hyperring and $P(R)$ the regular hyperradical of $R$. Then every regular hyper ideal of $R/P(R)$ is zero. That is, $P(R/P(R)) = 0_{R/P(R)}$.

Proof. Let $J$ be a regular hyper ideal of $R/P(R)$. Then by Theorem 2.21, $J = K/P(R)$ is a hyper ideal of $R/P(R)$. Thus, both $K/P(R)$ and $P(R)$ are regular. By Theorem 3.6, $K$ is regular. Hence, $K \subseteq P(R)$. Since $K$ contains $P(R)$, $K = P(R)$. Therefore, $J = P(R)/P(R) = 0_{R/P(R)}$. Hence, $P(R/P(R)) = 0_{R/P(R)}$.

The next remark follows from the regularity and maximality of $P(R)$.

Remark 3.12. Let $R$ be a Krasner ternary hyperring. Then

$$P(P(R)) = P(R).$$

The next corollary follows from Theorems 3.7, 3.10, 3.11, and Remark 3.12.

Corollary 3.13. Regularity is a radical property on Krasner ternary hyperrings.

References


