ON ONE-SIDED BASES OF A TERNARY SEMIGROUP

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Abstract: In this paper, the notions of left bases and right bases of a ternary semigroup are introduced and the structure of a ternary semigroup containing right bases will be studied. For the structure of a ternary semigroup containing left bases can be considered similarly.

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1. Preliminaries

The notion of a ternary semigroup generalized the notion of a ternary group was defined as follows: a *ternary semigroup* (see, [7]) is a non-empty set T together with a ternary operation, written as $(a, b, c) \rightarrow [abc]$, satisfying the associative law, that is, for all $a, b, c, u, v \in T$,

$$[[abc]uv] = [a[bcu]v] = [ab[cuv]].$$

This notion has been widely studied (see [1], [2], [3], [4], [6], [7], [8], [9], [10], [11], [12]). If A_1, A_2, A_3 are non-empty subsets of a ternary semigroup T , the set product $[A_1 A_2 A_3]$ of A_1, A_2, A_3 is defined by:

$$[A_1 A_2 A_3] = \{[a_1 a_2 a_3] \mid a_1 \in A_1, a_2 \in A_2, a_3 \in A_3\}.$$

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If $A_1 = \{a\}$, we write $[A_1 A_2 A_3]$ as $[a A_2 A_3]$, and similarly for $A_2 = \{a\}$ or $A_3 = \{a\}$. A non-empty subset A of T is called a *ternary subsemigroup* of T if $[AAA] \subseteq A$. That is, if $[a_1 a_2 a_3] \in A$ for all $a_1, a_2, a_3 \in A$.

The ideal theory of ternary semigroups was defined (see, [11], [7]) as follows: a non-empty subset A of a ternary semigroup T is called a *left ideal* (resp. *right ideal*) of T if $[TTA] \subseteq A$ (resp. $[ATT] \subseteq A$). A left ideal A of T is said to be *proper* if $A \subset T$. The symbol $\subset$ stands for proper inclusion for sets.

It is known (see, [9], [11]) that the following hold for a non-empty subset A of a ternary semigroup T :

- $A_l = A \cup [TTA]$ is the left ideal generated by A of T ;
- $A_r = A \cup [ATT]$ is the right ideal generated by A of T .

If $A = \{a\}$, we write A_l (resp. A_r) as $(a)_l$ (resp. $(a)_r$).

We introduce the *quasi-ordering* on a ternary semigroup T by: for any a, b in T ,

$$a \leq_l b \text{ if and only if } (a)_l \subseteq (b)_l.$$

Tamura [13] introduced the notions of left bases and right bases of a semigroup. Fabrici [5] studied the structure of a semigroup containing one-sided bases. In this paper, we introduce the notions of one-sided bases, left bases and right bases, of a ternary semigroup and study the structure of a ternary semigroup containing right bases. For the structure of a ternary semigroup containing left bases can be considered dually.

2. One-Sided Bases

We define left and right bases of a ternary semigroup as follows.

Definition 1. A subset A of a ternary semigroup T is called a *right base* (resp. *left base*) of T if it satisfies the following conditions:

- (i) $A_l = T$ (resp. $A_r = T$);
- (ii) there exists no a proper subset B of A such that $B_l = T$ (resp. $B_r = T$).

We now provide some examples.

Example 2. Let $T = \{-i, 0, i\}$. Then T is a ternary semigroup under the multiplication over complex numbers (see, [3]). We have $\{i\}$ and $\{-i\}$ are both the left and right bases of T .

Example 3. Let $T = \{0, a, b, c, 1\}$. Define the ternary operation on T by, for all $a, b, c \in T$, $[abc] = a * (b * c)$ where $*$ is the binary operation on T defined by

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	b	b	b
c	0	0	b	c	c
1	0	a	b	c	1

Then T is a ternary semigroup [2]. We have $\{1\}$ is a right and a left base of T .

Example 4. Let T be a non-empty set such that $|T| \geq 2$. Then T is a ternary semigroup under the ternary operation which is defined by $[xyz] = x$ for all $x, y, z \in T$. This is called the *left zero ternary semigroup* [10]. Then, for all $a \in T$, $\{a\}$ is a right base of T .

Example 5. Let T be a non-empty set such that $|T| > 3$. Choose an element $0 \in T$ and define the ternary operation on T by, for any $x, y, z \in T$,

$$[xyz] = \begin{cases} x & \text{if } x = y = z; \\ 0 & \text{otherwise.} \end{cases}$$

Then T is a ternary semigroup [10]. We have $T \setminus \{0\}$ is both the right and the left base of T .

Example 6. Let $T = \{0, 1, 2, 3, 4, 5\}$. Define the ternary operation on T by, for all $a, b, c \in T$, $[abc] = (a * b) * c$ where $*$ is the binary operation on T defined by

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	3	1	1
3	0	1	1	1	2	3
4	0	1	4	5	1	1
5	0	1	1	1	4	5

Then T is a ternary semigroup [10]. We have $\{2, 3\}$, $\{2, 5\}$, $\{3, 4\}$, $\{4, 5\}$ are the right bases of T . And, $\{2, 4\}$, $\{2, 5\}$, $\{3, 4\}$, $\{3, 5\}$ are the left bases of T .

We now prove some elementary results.

Lemma 7. *Let A be a right base of a ternary semigroup T . If $a, b \in A$ and $a \in [TTb]$, then $a = b$.*

Proof. Let $a, b \in A$ be such that $a \in [TTb]$. Suppose that $a \neq b$. Let $B = A \setminus \{a\}$. Then $B \subset A$. Let $x \in A$. If $x \notin B$, then

$$x = a \in [TTb] \subset [TTB] \cup B.$$

Thus

$$A \subseteq [TTB] \cup B \subseteq B_1.$$

Hence $T = A_1 \subseteq B_1$. This is a contradiction. Thus $a = b$. $\square$

Lemma 8. *A non-empty subset A of a ternary semigroup T is a right base of T if and only if A satisfies the following conditions:*

- (1) *for any $x \in T$ there exists $a \in A$ such that $x \leq_1 a$;*
- (2) *for any $a, b \in A$, if $a \neq b$, then neither $a \leq_1 b$ nor $b \leq_1 a$.*

Proof. Assume that A is a right base of T . Let $x \in T$. Then $x \in A_1 = [TTA] \cup A$, that is, $x \in A$ or $x \in [TTA]$. If $x \in A$, then there exists $a = x \in A$ such that $x \leq_1 a$. If $x \in [TTA]$, then $x = [sta] \in (a)_1$ for some $s, t \in T, a \in A$. Thus $(x)_1 \subseteq (a)_1$. This shows that (1) holds. Let $a, b \in A$ be such that $a \neq b$. Suppose that $a \leq_1 b$. Then $(a)_1 \subseteq (b)_1$. Since $a \neq b$, we have $a \in (b)_1 \setminus \{b\}$. By Lemma 7, $a = b$. This is a contradiction. Similarly, $b \leq_1 a$ implies $a = b$. This is a contradiction. Therefore, (2) holds.

Conversely, assume that the conditions (1) and (2) hold. It follows from (1) that $T = A_1$. Suppose that $T = B_1$ for some a proper subset B of A . Let $a \in A \setminus B$. We have $a \in B_1$. By (1), there exists $b \in B$ such that $a \leq_1 b$. This contradicts to (2). Hence A is a right base of T . $\square$

3. Main Results

A ternary semigroup T is said to be *right singular* if, for any $x, y, z \in T$, $[xyz] = z$. An element a of T is called an *selfpotent* if $[aaa] = a$.

In general, a right base of a ternary semigroup need not be a ternary subsemigroup. The following theorem characterizes when a right base of a ternary semigroup is a ternary subsemigroup.

Theorem 9. *Let A be a right base of a ternary semigroup T . Then A is a ternary subsemigroup of T if and only if A is right singular.*

Proof. Assume that A is a ternary subsemigroup of T . Let $a, b, c \in A$. Thus $[abc] \in A$, and so $[abc] = d$ for some $d \in A$. Then $d \in [TTC]$. By Lemma 7, $d = c$. Therefore, A is right singular.

Conversely, if A is right singular, then, for $a, b, c \in A$, $[abc] = c \in A$. Hence A is a ternary subsemigroup of T . $\square$

By Theorem 9, we have the following.

Corollary 10. *If a right base A of a ternary semigroup T is a ternary subsemigroup of T , then T contains at least one selfpotent.*

The following theorem shows that for any two right bases of a ternary semigroup have the same cardinality.

Theorem 11. *Any two right bases of a ternary semigroup T have the same cardinality.*

Proof. Let A and B be right bases of a ternary semigroup T . Let $a \in A$. Since B is a right base of T , we have $a \leq_1 b$ for some $b \in B$. For $a \in A$, we choose and fix $b \in B$ such that $a \leq_1 b$ and define a mapping

$$f : A \rightarrow B \text{ by } f(a) = b \text{ for all } a \in A.$$

If $a_1, a_2 \in A$ such that $f(a_1) = f(a_2) = b$, then $a_1 \leq_1 b$ and $a_2 \leq_1 b$. Since A is a right base of T , we have $b \leq_1 a$ for some a in A . Thus $a_1 \leq_1 a$, $a_2 \leq_1 a$, and $a_1, a_2, a \in A$. Thus $a_1 = a = a_2$ by Lemma 8. Hence f is one to one.

Now, let $b \in B$. Then there exists $a \in A$ such that $b \leq_1 a$. Similarly, there exists $b \in B$ such that $a \leq_1 b$. Then $b \leq_1 b$. By Lemma 8, we have $b = b$. Thus $a \leq_1 b = b$. Let $f(a) = c$. Then $a \leq_1 c$ and $a \leq_1 b$. Since $c, b \in T$ and A is a right base of T , there exist $a, a \in A$ such that $c \leq_1 a$ and $b \leq_1 a$. Then $a \leq_1 a$ and $a \leq_1 a$. By Lemma 8, $a = a = a$. Then $b \leq_1 a = a \leq_1 c$, and so $b = c$ by Lemma 8. Hence f is onto. $\square$

Lemma 12. *Let A be a right base of a ternary semigroup T . Let $a \in A$. If $(a)_1 = (b)_1$ for some $b \in T$ and $b \neq a$, then b is an element of a right base of T which is distinct from A .*

Proof. Let $B = (A \setminus \{a\}) \cup \{b\}$. It is clear that $B \subset A$. To show that B is a right base of T , it suffices to show that B satisfies the conditions (1) and (2) of Lemma 8. Let $x \in T$. Since A is a right base of T , there exists $c \in A$ such that $x \leq_1 c$. If $c \neq a$, then $c \in B$. If $c = a$, then $(c)_1 = (a)_1$. Thus $(c)_1 = (b)_1$, and so

$$(x)_1 \subseteq (c)_1 = (b)_1.$$

It follows that $x \leq_1 b$ and $b \in B$. It means that B satisfies the condition (1) of Lemma 8. Now, let $b_1, b_2 \in B$ be such that $b_1 \neq b_2$. If b_1, b_2 are distinct from b , then $b_1, b_2 \in A$. Since A is a right base of T , so neither $b_1 \leq_1 b_2$ nor $b_2 \leq_1 b_1$. Let $b_1 = b$. If $b_1 \leq_1 b_2$, then $a \leq_1 b_2$ where $a, b_2 \in A$. This is impossible. If $b_2 \leq_1 b_1$, then $(b_2)_1 \subseteq (b_1)_1 = (a)_1$. Thus $b_2 \leq_1 a$ where $a, b_2 \in A$. This is impossible. It means that B satisfies the condition (2) of Lemma 8. Therefore, B is a right base of T . $\square$

This is a consequence of the lemma above.

Corollary 13. *Let A be a right base of a ternary semigroup T . Let $a \in A$. If $(a)_1 = (b)_1$ for some $b \in T$ and $b \neq a$, then T contains at least two right bases.*

Theorem 14. *Let $\mathcal{A}$ be the union of all right bases of a ternary semigroup T . If $L = T \setminus \mathcal{A}$ is non-empty, then L is a left ideal of T .*

Proof. Let $x, y \in T$ and $a \in L = T \setminus \mathcal{A}$. Suppose that $[xya] \notin L$. Then there exists $b \in \mathcal{A}$ such that $b = [xya] \subseteq [TTa]$. That is, $(b)_1 \subseteq (a)_1$. Then $b \leq_1 a$. Since $b \in \mathcal{A}$, so $b \in B$ for some a right base B of T . Since B is a right base, there exists $b_1 \in B$ such that $a \leq_1 b_1$. Then $b \leq_1 a \leq_1 b_1$, and thus $b \leq_1 b_1$. This contradicts to the condition (2) of Lemma 8. Hence $[xya] \in L$. $\square$

A proper left ideal M of a ternary semigroup T is said to be *maximal* if there is no a proper left ideal A of T such that $M \subset A$.

Theorem 15. *Let $\mathcal{A}$ be the union of all right bases of a ternary semigroup T such that $\mathcal{A} \neq \emptyset$. Then $T \setminus \mathcal{A}$ is a maximal proper left ideal of T if and only if $\mathcal{A} \neq T$ and $\mathcal{A} \subseteq (a)_1$ for all $a \in \mathcal{A}$.*

Proof. Let $L = T \setminus \mathcal{A}$ be a maximal proper left ideal of a ternary semigroup T . Then $\mathcal{A} \neq T$. Let $a \in \mathcal{A}$. Suppose that $\mathcal{A} \not\subseteq (a)_1$. Since $\mathcal{A} \not\subseteq (a)_1$, there exists $x \in \mathcal{A}$ such that $x \notin (a)_1$. Thus $x \notin T \setminus \mathcal{A}$ and $x \notin (a)_1$. This implies $(T \setminus \mathcal{A}) \cup (a)_1 \neq T$. Then $(T \setminus \mathcal{A}) \cup (a)_1 \neq T$ is a proper left ideal of T such that $(T \setminus \mathcal{A}) \subset (T \setminus \mathcal{A}) \cup (a)_1$. This contradicts to the maximality of $T \setminus \mathcal{A}$. Hence $\mathcal{A} \subseteq (a)_1$.

Conversely, let $\mathcal{A} \subseteq (a)_1$ for all $a \in \mathcal{A}$, and $\mathcal{A} \neq T$. We have to prove that $T \setminus \mathcal{A}$ is a maximal proper left ideal of T . By Theorem 14, $T \setminus \mathcal{A}$ is a proper left ideal of T . Let L be a left ideal of T such that $T \setminus \mathcal{A} \subset L$. Then $L \cap \mathcal{A} \neq \emptyset$. Let $a \in L \cap \mathcal{A}$. Then $a \in L$, and it follows that $(a)_1 \subseteq L$. Since $\mathcal{A} \subseteq (a)_1$, $\mathcal{A} \subseteq L$. Consequently, $\mathcal{A} \subseteq L, T \setminus \mathcal{A} \subset L$. Therefore $T = L$. $\square$

Theorem 16. *Let $\mathcal{A}$ be the union of all right base of a ternary semigroup T such that $\emptyset \neq \mathcal{A} \subset T$. Let M be a proper left ideal of T containing every proper left ideal of T . The following statements are equivalent:*

- (1) $T \setminus \mathcal{A}$ is a maximal left ideal of T ;
- (2) $\mathcal{A} \subseteq (a)_l$ for every $a \in \mathcal{A}$;
- (3) $T \setminus \mathcal{A} = M$;
- (4) every right base of T is a singleton set.

Proof. (1) $\Leftrightarrow$ (2). This follows from Theorem 15.

(3) $\Leftrightarrow$ (4). Assume that $T \setminus \mathcal{A} = M$. Then $T \setminus \mathcal{A}$ is a maximal left ideal of T . Let $a \in \mathcal{A}$. By Theorem 15, we have $\mathcal{A} \subseteq (a)_l$. Then $T = \mathcal{A}_l \subseteq (a)_l$. This implies $T = (a)_l$. Hence $\{a\}$ is a right base of T . Let B be a right base of T , and let $a, b \in B$. Then $B \subseteq \mathcal{A}$, that is, $a, b \in \mathcal{A}$. Hence $b \in T = (a)_l$. By Lemma 7, $a = b$ (i.e., B is a singleton set).

Conversely, assume that every right base of T is a singleton set. Then $T = (a)_l$ for all $a \in \mathcal{A}$. Let M be a left ideal of T such that M is not contained in $T \setminus \mathcal{A}$. Then there exists $x \in \mathcal{A} \cap M$. Since $x \in M$, we have $T = (x)_l \subseteq M$, and so $T = M$.

(1) $\Leftrightarrow$ (3). Assume that $T \setminus \mathcal{A}$ is a maximal left ideal of T . Let M be a left ideal of T such that M is not contained in $T \setminus \mathcal{A}$. Hence, there exists $x \in M \cap \mathcal{A}$. Then $\mathcal{A} \subseteq (x)_l \subseteq M$. Thus $M = \mathcal{A} \cup X$ for some $X \subseteq T \setminus \mathcal{A}$. Let $y \in T$. Then there exists $c \in \mathcal{A}$ such that $y \leq c$; hence $y \in (y)_l \subseteq (c)_l \subseteq M$. Thus $M = T$. Therefore, $T \setminus \mathcal{A} = M$.

The converse statement is obvious. □

Theorem 17. *Let $\mathcal{A}$ be the union of all right bases of a ternary semigroup T . If $\emptyset \neq \mathcal{A} \neq T$ and $T \setminus \mathcal{A}$ is a maximal left ideal of T , then every right base A of T from neither $[TTA] = T$ (i.e., $A_l = [TTA]$) or there is unique a right base A of T such that $A \subseteq T \setminus [TTA]$.*

Proof. Assume that $T \setminus \mathcal{A}$ is a maximal left ideal of T and A is a right base of T . By Theorem 16, $A = \{a\}$ for some $a \in T$ and $\mathcal{A} \subseteq (x)_l$ for all $x \in \mathcal{A}$. If there exist $x, y \in \mathcal{A}$ such that $x \in [TTx]$ and $y \notin [TTY]$, then

$$T \setminus \mathcal{A} \subseteq [TTY] \subset T$$

where $T \setminus \mathcal{A} \neq [TTY]$, because $x \in [TTY]$. Hence $T \setminus \mathcal{A} \neq [TTY]$. This is contradicts to the maximality of $T \setminus \mathcal{A}$. Hence, there are two cases to consider:

Case 1: $x \in [TTx]$ for all $x \in \mathcal{A}$. We have $[TTA] = T$. That is, $A_l = [TTA]$.

Case 2: $x \notin [TTx]$ for all $x \in \mathcal{A}$. We have $A \subseteq T \setminus [TTA]$. Suppose that T contains at least two right bases, $A_1 = \{a_1\}, A_2 = \{a_2\}$, such that $a_1 \notin [TTa_1], a_2 \notin [TTa_2]$ and $a_1, a_2 \in \mathcal{A}$. Thus

$$T \setminus \mathcal{A} \subseteq T \setminus \{a_1\} = [TTa_1].$$

Since $a_2 \in [TTa_1], T \setminus \mathcal{A} \neq [TTa_1]$. This contradicts to the maximality of $T \setminus \mathcal{A}$. Hence, there is unique a right base A of T such that $A \subseteq T \setminus [TTA]$. $\square$

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