ON ONE-SIDED BASES OF A TERNARY SEMIGROUP

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Abstract: In this paper, the notions of left bases and right bases of a ternary semigroup are introduced and the structure of a ternary semigroup containing right bases will be studied. For the structure of a ternary semigroup containing left bases can be considered similarly.

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1. Preliminaries

The notion of a ternary semigroup generalized the notion of a ternary group was defined as follows: a ternary semigroup (see, [7]) is a non-empty set T together with a ternary operation, written as \((a, b, c) \rightarrow [abc]\), satisfying the associative law, that is, for all \(a, b, c, u, v \in T\),

\[
[[abc]uv] = [ab[cuv]] = [a[bcu]]v = [abcu].
\]

This notion has been widely studied (see [1], [2], [3], [4], [6], [7], [8], [9], [10], [11], [12]). If \(A_1, A_2, A_3\) are non-empty subsets of a ternary semigroup \(T\), the set product \([A_1A_2A_3]\) of \(A_1, A_2, A_3\) is defined by:

\[
[A_1A_2A_3] = \{[a_1a_2a_3] \mid a_1 \in A_1, a_2 \in A_2, a_3 \in A_3\}.
\]
If $A_1 = \{a\}$, we write $[A_1 A_2 A_3]$ as $[a A_2 A_3]$, and similarly for $A_2 = \{a\}$ or $A_3 = \{a\}$. A non-empty subset $A$ of $T$ is called a ternary subsemigroup of $T$ if $[AAA] \subseteq A$. That is, if $[a_1 a_2 a_3] \in A$ for all $a_1, a_2, a_3 \in A$.

The ideal theory of ternary semigroups was defined (see, [11], [7]) as follows: a non-empty subset $A$ of a ternary semigroup $T$ is called a left ideal (resp. right ideal) of $T$ if $[T TA] \subseteq A$ (resp. $[ATT] \subseteq A$). A left ideal $A$ if $T$ is said to be proper if $A \subset T$. The symbol $\subseteq$ stands for proper inclusion for sets.

It is known (see, [9], [11]) that the following hold for a non-empty subset $A$ of a ternary semigroup $T$:

- $A_l = A \cup [TTA]$ is the left ideal generated by $A$ of $T$;
- $A_r = A \cup [ATT]$ is the right ideal generated by $A$ of $T$.

If $A = \{a\}$, we write $A_l$ (resp. $A_r$) as $(a)_l$ (resp. $(a)_r$).

We introduce the quasi-ordering on a ternary semigroup $T$ by: for any $a, b$ in $T$,

$$a \leq_l b \text{ if and only if } (a)_l \subseteq (b)_l.$$ 

Tamura [13] introduced the notions of left bases and right bases of a semigroup. Fabrici [5] studied the structure of a semigroup containing one-sided bases. In this paper, we introduce the notions of one-sided bases, left bases and right bases, of a ternary semigroup and study the structure of a ternary semigroup containing right bases. For the structure of a ternary semigroup containing left bases can be considered dually.

2. One-Sided Bases

We define left and right bases of a ternary semigroup as follows.

**Definition 1.** A subset $A$ of a ternary semigroup $T$ is called a right base (resp. left base) of $T$ if it satisfies the following conditions:

(i) $A_l = T$ (resp. $A_r = T$);

(ii) there exists no a proper subset $B$ of $A$ such that $B_l = T$ (resp. $B_r = T$).

We now provide some examples.

**Example 2.** Let $T = \{-i, 0, i\}$. Then $T$ is a ternary semigroup under the multiplication over complex numbers (see, [3]). We have $\{i\}$ and $\{-i\}$ are both the left and right bases of $T$.  

Example 3. Let $T = \{0, a, b, c, 1\}$. Define the ternary operation on $T$ by, for all $a, b, c \in T$, $[abc] = a \ast (b \ast c)$ where $\ast$ is the binary operation on $T$ defined by

\[
\begin{array}{c|ccccc}
\ast & 0 & a & b & c & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 & a & a \\
b & 0 & 0 & b & b & b \\
c & 0 & 0 & b & c & c \\
1 & 0 & a & b & c & 1
\end{array}
\]

Then $T$ is a ternary semigroup [2]. We have $\{1\}$ is a right and a left base of $T$.

Example 4. Let $T$ be a non-empty set such that $|T| \geq 2$. Then $T$ is a ternary semigroup under the ternary operation which is defined by $[xyz] = x$ for all $x, y, z \in T$. This is called the left zero ternary semigroup [10]. Then, for all $a \in T$, $\{a\}$ is a right base of $T$.

Example 5. Let $T$ be a non-empty set such that $|T| > 3$. Choose an element $0 \in T$ and define the ternary operation on $T$ by, for any $x, y, z \in T$,

\[
[xyz] = \begin{cases} 
 x & \text{if } x = y = z; \\
 0 & \text{otherwise.}
\end{cases}
\]

Then $T$ is a ternary semigroup [10]. We have $T \setminus \{0\}$ is both the right and the left base of $T$.

Example 6. Let $T = \{0, 1, 2, 3, 4, 5\}$. Define the ternary operation on $T$ by, for all $a, b, c \in T$, $[abc] = (a \ast b) \ast c$ where $\ast$ is the binary operation on $T$ defined by

\[
\begin{array}{c|ccccc}
\ast & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 \\
2 & 0 & 1 & 2 & 3 & 1 & 1 \\
3 & 0 & 1 & 1 & 1 & 2 & 3 \\
4 & 0 & 1 & 4 & 5 & 1 & 1 \\
5 & 0 & 1 & 1 & 1 & 4 & 5
\end{array}
\]

Then $T$ is a ternary semigroup [10]. We have $\{2, 3\}$, $\{2, 5\}$, $\{3, 4\}$, $\{4, 5\}$ are the right bases of $T$. And, $\{2, 4\}$, $\{2, 5\}$, $\{3, 4\}$, $\{3, 5\}$ are the left bases of $T$.

We now prove some elementary results.
Lemma 7. Let $A$ be a right base of a ternary semigroup $T$. If $a, b \in A$ and $a \in [TTb]$, then $a = b$.

Proof. Let $a, b \in A$ be such that $a \in [TTb]$. Suppose that $a \neq b$. Let $B = A \setminus \{a\}$. Then $B \subset A$. Let $x \in A$. If $x \notin B$, then

$$x = a \in [TTb] \subset [TTB] \cup B.$$  

Thus

$$A \subseteq [TTB] \cup B \subseteq B_1.$$  

Hence $T = A_1 \subseteq B_1$. This is a contradiction. Thus $a = b$. \hfill $\square$

Lemma 8. A non-empty subset $A$ of a ternary semigroup $T$ is a right base of $T$ if and only if $A$ satisfies the following conditions:

(1) for any $x \in T$ there exists $a \in A$ such that $x \leq_1 a$;

(2) for any $a, b \in A$, if $a \neq b$, then neither $a \leq_1 b$ nor $b \leq_1 a$.

Proof. Assume that $A$ is a right base of $T$. Let $x \in T$. Then $x \in A_1 = [TTA] \cup A$, that is, $x \in A$ or $x \in [TTA]$. If $x \in A$, then there exists $a = x \in A$ such that $x \leq_1 a$. If $x \in [TTA]$, then $x = [sta] \in (a)_1$ for some $s, t \in T, a \in A$. Thus $(x)_1 \subseteq (a)_1$. This shows that (1) holds. Let $a, b \in A$ be such that $a \neq b$. Suppose that $a \leq_1 b$. Then $(a)_1 \subseteq (b)_1$. Since $a \neq b$, we have $a \in (b)_1 \setminus \{b\}$. By Lemma 7, $a = b$. This is a contradiction. Similarly, $b \leq_1 a$ implies $a = b$. This is a contradiction. Therefore, (2) holds.

Conversely, assume that the conditions (1) and (2) hold. It follows from (1) that $T = A_1$. Suppose that $T = B_1$ for some a proper subset $B$ of $A$. Let $a \in A \setminus B$. We have $a \in B_1$. By (1), there exists $b \in B$ such that $a \leq_1 b$. This contradicts to (2). Hence $A$ is a right base of $T$. \hfill $\square$

3. Main Results

A ternary semigroup $T$ is said to be right singular if, for any $x, y, z \in T$, $[xyz] = z$. An element $a$ of $T$ is called an selfpotent if $[aaa] = a$.

In general, a right base of a ternary semigroup need not be a ternary subsemigroup. The following theorem characterizes when a right base of a ternary semigroup is a ternary subsemigroup.

Theorem 9. Let $A$ be a right base of a ternary semigroup $T$. Then $A$ is a ternary subsemigroup of $T$ if and only if $A$ is right singular.
Proof. Assume that $A$ is a ternary subsemigroup of $T$. Let $a, b, c \in A$. Thus $[abc] \in A$, and so $[abc] = d$ for some $d \in A$. Then $d \in [TTc]$. By Lemma 7, $d = c$. Therefore, $A$ is right singular.

Conversely, if $A$ is right singular, then, for $a, b, c \in A$, $[abc] = c \in A$. Hence $A$ is a ternary subsemigroup of $T$.

By Theorem 9, we have the following.

Corollary 10. If a right base $A$ of a ternary semigroup $T$ is a ternary subsemigroup of $T$, then $T$ contains at least one selfpotent.

The following theorem shows that for any two right bases of a ternary semigroup have the same cardinality.

Theorem 11. Any two right bases of a ternary semigroup $T$ have the same cardinality.

Proof. Let $A$ and $B$ be right bases of a ternary semigroup $T$. Let $a \in A$. Since $B$ is a right base of $T$, we have $a \leq_1 b$ for some $b \in B$. For $a \in A$, we choose and fix $b \in B$ such that $a \leq_1 b$ and define a mapping

$$f : A \to B \text{ by } f(a) = b \text{ for all } a \in A.$$ 

If $a_1, a_2 \in A$ such that $f(a_1) = f(a_2) = b$, then $a_1 \leq_1 b$ and $a_2 \leq_1 b$. Since $A$ is a right base of $T$, we have $b \leq_1 a$ for some $a \in A$. Thus $a_1 \leq_1 a$, $a_2 \leq_1 a$, and $a_1, a_2, a \in A$. Thus $a_1 = a = a_2$ by Lemma 8. Hence $f$ is one to one.

Now, let $b \in B$. Then there exists $a \in A$ such that $b \leq_1 a$. Similarly, there exists $b' \in B$ such that $a \leq_1 b'$. Then $b \leq_1 b'$. By Lemma 8, we have $b = b'$. Thus $a \leq_1 b' = b$. Let $f(a) = c$. Then $a \leq_1 c$ and $a \leq_1 b$. Since $c, b \in T$ and $A$ is a right base of $T$, there exist $a, a' \in A$ such that $c \leq_1 a'$ and $b \leq_1 a'$. Then $a \leq_1 a'$ and $a \leq_1 a''$. By Lemma 8, $a = a' = a''$. Then $b \leq_1 a'' = a \leq_1 c$, and so $b = c$ by Lemma 8. Hence $f$ is onto.

Lemma 12. Let $A$ be a right base of a ternary semigroup $T$. Let $a \in A$. If $(a)_{|T} = (b)_{|T}$ for some $b \in T$ and $b \neq a$, then $b$ is an element of a right base of $T$ which is distinct from $A$.

Proof. Let $B = (A \setminus \{a\}) \cup \{b\}$. It is clear that $B \subseteq A$. To show that $B$ is a right base of $T$, it suffices to show that $B$ satisfies the conditions (1) and (2) of Lemma 8. Let $x \in T$. Since $A$ is a right base of $T$, there exists $c \in A$ such that $x \leq_1 c$. If $c \neq a$, then $c \in B$. If $c = a$, then $(c)_{|T} = (a)_{|T}$. Thus $(c)_{|T} = (b)_{|T}$, and so $(x)_{|T} \subseteq (c)_{|T} = (b)_{|T}$. 


It follows that \( x \leq b \) and \( b \in B \). It means that \( B \) satisfies the condition (1) of Lemma 8. Now, let \( b_1, b_2 \in B \) be such that \( b_1 \neq b_2 \). If \( b_1, b_2 \) are distinct from \( b \), then \( b_1, b_2 \in A \). Since \( A \) is a right base of \( T \), so neither \( b_1 \leq b_2 \) nor \( b_2 \leq b_1 \). Let \( b_1 = b \). If \( b_1 \leq b_2 \), then \( a \leq b_2 \) where \( a, b_2 \in A \). This is impossible. If \( b_2 \leq b_1 \), then \( (b_2)_l \subseteq (b_1)_l = (a)_l \). Thus \( b_2 \leq a \) where \( a, b_2 \in A \). This is impossible. It means that \( B \) satisfies the condition (2) of Lemma 8. Therefore, \( B \) is a right base of \( T \).

This is a consequence of the lemma above.

**Corollary 13.** Let \( A \) be a right base of a ternary semigroup \( T \). Let \( a \in A \). If \( (a)_l = (b)_l \) for some \( b \in T \) and \( b \neq a \), then \( T \) contains at least two right bases.

**Theorem 14.** Let \( A \) be the union of all right bases of a ternary semigroup \( T \). If \( L = T \setminus A \) is non-empty, then \( L \) is a left ideal of \( T \).

**Proof.** Let \( x, y \in T \) and \( a \in L = T \setminus A \). Suppose that \( [xya] \notin L \). Then there exists \( b \in A \) such that \( b = [xya] \subseteq [T]a \). That is, \( (b)_l \subseteq (a)_l \). Then \( b \leq a \). Since \( b \in A \), so \( b \in B \) for some a right base \( B \) of \( T \). Since \( B \) is a right base, there exists \( b_1 \in B \) such that \( a \leq b_1 \). Then \( b \leq a \leq b_1 \), and thus \( b \leq b_1 \). This contradicts to the condition (2) of Lemma 8. Hence \( [xya] \in L \).

A proper left ideal \( M \) of a ternary semigroup \( T \) is said to be *maximal* if there is no a proper left ideal \( A \) of \( T \) such that \( M \subset A \).

**Theorem 15.** Let \( A \) be the union of all right bases of a ternary semigroup \( T \) such that \( A \neq \emptyset \). Then \( T \setminus A \) is a maximal proper left ideal of \( T \) if and only if \( A \neq T \) and \( A \subseteq (a)_l \) for all \( a \in A \).

**Proof.** Let \( L = T \setminus A \) be a maximal proper left ideal of a ternary semigroup \( T \). Then \( A \neq T \). Let \( a \in A \). Suppose that \( A \not\subseteq (a)_l \). Since \( A \not\subseteq (a)_l \), there exists \( x \in A \) such that \( x \notin (a)_l \). Thus \( x \notin T \setminus A \) and \( x \notin (a)_l \). This implies \( (T \setminus A) \cup (a)_l \neq T \). Then \( (T \setminus A) \cup (a)_l \neq T \) is a proper left ideal of \( T \) such that \( (T \setminus A) \subseteq (T \setminus A) \cup (a)_l \). This contradicts to the maximality of \( T \setminus A \). Hence \( A \subseteq (a)_l \).

Conversely, let \( A \subseteq (a)_l \) for all \( a \in A \), and \( A \neq T \). We have to prove that \( T \setminus A \) is a maximal proper left ideal of \( T \). By Theorem 14, \( T \setminus A \) is a proper left ideal of \( T \). Let \( L' \) be a left ideal of \( T \) such that \( T \setminus A \subseteq L' \). Then \( L' \cap A \neq \emptyset \). Let \( a \in L' \cap A \). Then \( a \in L' \), and it follows that \( (a)_l \subseteq L' \). Since \( A \subseteq (a)_l \), \( A \subseteq L' \). Consequently, \( A \subseteq L', T \setminus A \subseteq L' \). Therefore \( T = L' \).
Theorem 16. Let \( A \) be the union of all right base of a ternary semigroup \( T \) such that \( \emptyset \neq A \subset T \). Let \( M^* \) be a proper left ideal of \( T \) containing every proper left ideal of \( T \). The following statements are equivalent:

1. \( T \setminus A \) is a maximal left ideal of \( T \);
2. \( A \subseteq (a)_l \) for every \( a \in A \);
3. \( T \setminus A = M^* \);
4. every right base of \( T \) is a singleton set.

Proof. (1) \( \iff \) (2). This follows from Theorem 15.

(3) \( \iff \) (4). Assume that \( T \setminus A = M^* \). Then \( T \setminus A \) is a maximal left ideal of \( T \). Let \( a \in A \). By Theorem 15, we have \( A \subseteq (a)_l \). This implies \( T = (a)_l \). Hence \( \{a\} \) is a right base of \( T \). Let \( B \) be a right base of \( T \), and let \( a, b \in B \). Then \( B \subseteq A \), that is, \( a, b \in A \). Hence \( b \in T = (a)_l \). By Lemma 7, \( a = b \) (i.e., \( B \) is a singleton set).

Conversely, assume that every right base of \( T \) is a singleton set. Then \( T = (a)_l \) for all \( a \in A \). Let \( M \) be a left ideal of \( T \) such that \( M \) is not contained in \( T \setminus A \). Then there exists \( x \in A \cap M \). Since \( x \in M \), we have \( T = (x)_l \subseteq M \), and so \( T = M \).

(1) \( \iff \) (3). Assume that \( T \setminus A \) is a maximal left ideal of \( T \). Let \( M \) be a left ideal of \( T \) such that \( M \) is not contained in \( T \setminus A \). Then \( A \subseteq (x)_l \subseteq M \). Thus \( M = A \cup X \) for some \( X \subseteq T \setminus A \). Let \( y \in T \). Then there exists \( c \in A \) such that \( y \leq c \); hence \( y \in (y)_l \subseteq (c)_l \subseteq M \). Thus \( M = T \). Therefore, \( T \setminus A = M^* \).

The converse statement is obvious. \( \square \)

Theorem 17. Let \( A \) be the union of all right bases of a ternary semigroup \( T \). If \( \emptyset \neq A \neq T \) and \( T \setminus A \) is a maximal left ideal of \( T \), then every right base \( A \) of \( T \) from neither \([TTA] = T \) (i.e., \( A_l = [TTA] \)) or there is unique a right base \( A \) of \( T \) such that \( A \subseteq T \setminus [TTA] \).

Proof. Assume that \( T \setminus A \) is a maximal left ideal of \( T \) and \( A \) is a right base of \( T \). By Theorem 16, \( A = \{a\} \) for some \( a \in T \) and \( A \subseteq (x)_l \) for all \( x \in A \). If there exist \( x, y \in A \) such that \( x \in [TTx] \) and \( y \notin [TTy] \), then

\[ T \setminus A \subseteq [TTy] \subset T \]

where \( T \setminus A \neq [TTy] \), because \( x \in [TTy] \). Hence \( T \setminus A \neq [TTy] \). This is contradicts to the maximality of \( T \setminus A \). Hence, there are two cases to consider:

Case 1: \( x \in [TTx] \) for all \( x \in A \). We have \([TTA] = T \). That is, \( A_l = [TTA] \).
Case 2: \( x \notin [TTx] \) for all \( x \in A \). We have \( A \subseteq T \setminus [TTA] \). Suppose that \( T \) contains at least two right bases, \( A_1 = \{a_1\}, A_2 = \{a_2\} \), such that \( a_1 \notin [TTa_1], a_2 \notin [TTa_2] \) and \( a_1, a_2 \in A \). Thus

\[
T \setminus A \subseteq T \setminus \{a_1\} = [TTa_1].
\]

Since \( a_2 \in [TTa_1], T \setminus A \neq [TTa_1] \). This contradicts to the maximality of \( T \setminus A \). Hence, there is unique a right base \( A \) of \( T \) such that \( A \subseteq T \setminus [TTA] \).

\[\square\]

References


