INVERSE DOMINATION NUMBER OF JUMP GRAPH

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Abstract: Let $J(G) = (V, E)$ be a jump graph. Let D be a minimum dominating set in a jump Graph $J(G)$. If $V - D$ contains a dominating set D of $J(G)$, then D is called an inverse dominating set with respect to D . The minimum cardinality of an inverse dominating set of a Jump graph $J(G)$ is called the inverse domination number of $J(G)$. In this paper We study the graph theoretic properties of inverse domination of Jump graph and its exact values for some standard graphs. The relation between inverse domination of Jump graph with other parameters is also investigated.

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1. Introduction

Let $G(p, q)$ be a graph with $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G respectively. All the graphs considered here are finite,

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non-trivial, undirected and connected without loops or multiple edges. For basic terminology, we refer to Chartrand and Lesniak [7]. A set D of vertices in a graph $G = (V, E)$ is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D . The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G . Let D be a minimum dominating set of G . If $V - D$ contains a dominating set D' of G , then D is called an inverse dominating set with respect to D' . The inverse domination number $\gamma^{-1}(G)$ of G is the minimum cardinality of an inverse dominating set of G . This concept was first introduced by Kulli and Sigarkanthi [3] and it was studied by several graph theorists in [8, 9].

In general the degree of vertex v in a graph G is the number of edges of G incident with v and it is denoted by $\deg v$. The maximum (minimum) degree among the vertices of G is denoted by $\Delta(G)$ ($\delta(G)$). We denote the minimum number of edges in edge cover of G (i.e., edge cover number) by $\alpha_1(G)$ and the minimum number of edges in independent set of edges of G (i.e., edge independent set) by $\beta_1(G)$. The subgraph induced by $X \subseteq V$ is denoted by $\langle X \rangle$. A vertex of degree one is called a pendent vertex. A vertex adjacent to pendent vertex is called the support vertex. The maximum $d(u, v)$ for all u in G is eccentricity of v and the maximum eccentricity is the diameter $\text{diam}(G)$. The circumference of a graph G with atleast one cycle is the length of a longest cycle in G and it is denoted by $\text{circum}(G)$. As usual P_n , C_n and K_n are respectively, the path, cycle and complete graph of order n , $K_{r,s}$ is the complete bipartite graph with two partite sets containing r and s vertices. Any undefined term or notation in this paper can be found in [2].

Definition 1. The line graph $L(G)$ of G has the edges of G as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G . We call the complement of line graph $L(G)$ as the jump graph $J(G)$ of G , found in [4]. The jump graph $J(G)$ of a graph G is the graph defined on $E(G)$ and in which two vertices are adjacent if and only if they are not adjacent in G . Since both $L(G)$ and $J(G)$ are defined on the edge set of a graph G .

Remark 2. The isolated vertices of G (if G has) play no role in line graph and jump graph transformation. Here we assume that the graph G under consideration is non-empty and has no isolated vertices found in [4].

Definition 3. Let $G = (V, E)$ be a graph. Let D be a minimum dominating set in a Graph G . If $V - D$ Contains a dominating set D' of G , then D is called an inverse dominating set with respect to D' the minimum cardinality of an inverse dominating set of a graph G is called the inverse domination number of G and it is denoted by $\gamma^{-1}(G)$ studied in [3, 6].

Example 4.

Figure 1

$D_1 = \{2, 5\}$, $D_2 = \{2, 4\}$, $D_3 = \{3, 5\}$ are the minimum dominating sets. Their corresponding inverse dominating sets are $D_1 = \{1, 3, 4\}$, $D_2 = \{3, 5\}$, $D_3 = \{2, 4\}$ respectively. Thus $\gamma(G) = 2$ is the domination number of G . $\gamma^{-1}(G) = 2$ is the inverse domination number of G .

Remark 5. Every graph without isolated vertices contains an inverse dominating set, since the complement of any minimal dominating set is also a dominating set. Thus we consider a graph without isolated vertices.

Definition 6. Let $J(G) = (V, E)$ be a jump graph. Let D be a minimum dominating set in a jump Graph $J(G)$. If $V - D$ Contains a dominating set D of $J(G)$, then D is called an inverse dominating set with respect to D . The minimum cardinality of an inverse dominating set of a jump graph $J(G)$ is called the inverse domination number of $J(G)$ and it is denoted by $\gamma^{-1}[J(G)]$.

For any graph G , with $p \leq 4$, the jump graph $J(G)$ of G , is disconnected. Since we study only the connected jump graph, we choose $p > 4$ [5].

We recall following classical theorems to prove our results.

Theorem 7. [1] Let G be a graph without isolated vertices. If D is a minimal dominating set, then $V - D$ is a dominating set.

Theorem 8. [3] If a graph G has no isolated vertices, then $\gamma(G) \leq \gamma^{-1}(G)$.

Theorem 9. [3] If a graph G has no isolated vertices, then $\gamma(G) + \gamma^{-1}(G) \leq p$.

2. Main Results

Theorem 10.

1. For any path P_p with $p \geq 5$ $\gamma^{-1}[J(P_p)] = 2$

2. For any Cycle C_p , with $p \geq 5$ $\gamma^{-1}[J(C_p)] = 2$
3. For any complete graph K_p , with $p \geq 5$ $\gamma^{-1}[J(K_p)] \geq 2$
4. For any complete bipartite graph K_{mn}

$$\gamma^{-1}[J(K_{mn})] = \begin{cases} 2 & \text{for } k_{2,n} \text{ where } n \geq 2 \\ 3 & \text{for } k_{m,n} \text{ where } m, n \geq 3 \end{cases}$$

5. For any wheel W_p , $\gamma^{-1}[J(W_p)] = \begin{cases} 3 & \text{if } p = 5, 6 \\ 3 & \text{if } p \geq 7 \end{cases}$

Theorem 11. For any connected graph G $\gamma^{-1}[J(G)] \geq 2$.

Proof of the theorem is obvious

Theorem 12. For any connected (p, q) graph $\gamma^{-1}[J(G)] \leq q - \beta_1(G) + 1$

Proof. Let $V - D = \{v_1, v_2, v_3, \dots, v_n\}$ be the set of vertices in $J(G)$ corresponding to the set of independent edges $\{e_1, e_2, e_3, \dots, e_n\}$ of G . By the definition of $J(G)$, the elements of $V - D$ form an induced subgraph $< k_n >$ in $J(G)$. Further, let $S \cup \{v_1\}$, where $S \subset [V[J(G)] - (V - D)]$ be an inverse dominating set in $J(G)$. It follows that $|S \cup \{v_1\}| \leq |V[J(G)] - (V - D)| + |v_1|$ therefore $\gamma^{-1}[J(G)] \leq q - \beta_1(G) + 1$ $\square$

Theorem 13. For any connected graph G with diameter, $\text{diam}(G) \geq 2$, $\gamma^{-1}[J(G)] \geq 2$

Proof. Let uv be a path of maximum distance in G . Then $d(u, v) = \text{diam}(G)$ We can prove the theorem with the following cases.

Case (i): For $\text{diam}(G) = 2$, choose a vertex v_1 of eccentricity 2 with maximum degree among others. Let $V - D = \{v_1, v_2, \dots\}$ corresponding to the elements of $\{e_1, e_2, \dots\}$ forming an inverse dominating set in jump graph $J(G)$. Every vertex $u \notin V - D$ is adjacent to a vertex in $V - D$. Hence $V - D$ is a minimum inverse dominating set. So the inverse domination number of the jump graph will be equal to the degree of v_1 also by Theorem 11, we say $\gamma^{-1}[J(G)] > 2$

Case (ii): For $\text{diam}(G) > 2$. Let e_1 be any edge adjacent to u and e_2 be any edge adjacent to v . Let $\{e_1, e_2\} \subseteq E(G)$ form a corresponding vertex set $\{v_1, v_2\} \subseteq V(J(G))$. These two vertices form an inverse dominating set in jump graph. Since these vertices $\{v_1, v_2\}$ are adjacent to all other vertices of

$V - D(J(G))$, it follows that $\{v_1, v_2\}$ becomes a minimum inverse dominating set. Hence $\gamma^{-1}[J(G)] = 2$.

In view of above cases, we can conclude that for any connected graph G , $\gamma^{-1}[J(G)] \geq 2$ $\square$

Theorem 14. For any connected graph G with circumference, $\text{circum}(G) \geq 4$, $\gamma^{-1}[J(G)] \geq 2$

Proof. Let $u - v - u$ be a longest cycle in G . Then $C(G) = \text{Circum}(G)$. We can prove the theorem with the following cases.

Case (i): For $\text{Circum}(G) = 4$, choose a vertex v_1 that have longest cycle among others. Let $V - D = \{v_1, v_2, \dots\}$ corresponding to the elements of $\{e_1, e_2, \dots\}$ forming an inverse dominating set in jump graph $J(G)$. Every vertex $u \notin V - D$ is adjacent to a vertex in $V - D$. Hence $V - D$ is a minimum inverse dominating set. So the inverse domination number of the jump graph will be equal to the longest cycle also by Theorem 11, we say $\gamma^{-1}[J(G)] > 2$.

Case (ii): For $\text{Circum}(G) > 4$. Let e_1 be any edge adjacent to u and e_2 be any edge adjacent to v . Let $\{e_1, e_2\} \subseteq E(G)$ form a corresponding vertex set $\{v_1, v_2\} \subseteq V(J(G))$. These two vertices form an inverse dominating set in jump graph. Since these vertices $\{v_1, v_2\}$ are adjacent to all other vertices of $V - D(J(G))$, it follows that $\{v_1, v_2\}$ becomes a minimum inverse dominating set. Hence $\gamma^{-1}[J(G)] = 2$.

In view of above cases, we can conclude that for any connected graph G , $\gamma^{-1}[J(G)] \geq 2$ $\square$

Theorem 15. For any tree T with diameter greater than 3, $\gamma^{-1}[J(T)] \geq 2$

Proof. If the diameter is less than or equal to 3, then the jump graph will be disconnected.

Let uv be a path of maximum length in a tree T where diameter is greater than 3. Let e_i be the pendant edge adjacent to u and e_k be the pendant edge adjacent to v . The vertex set $\{v_i, v_k\}$ of $J(T)$ corresponding to the edges of $\{e_i, e_k\}$ in T will form an inverse dominating set in $J(T)$. Since all the other vertices of $V[J(T)]$ are adjacent with $\{v_i, v_k\}$, it forms a minimum inverse dominating set. Hence $\gamma^{-1}[J(T)] \geq 2$. $\square$

Theorem 16. For any connected (p, q) graph G , $\gamma^{-1}[J(G)] < q - \Delta(G)$ where $\Delta(G)$ is the maximum degree of G .

Proof. Let $V - D = \{v_1, v_2, v_3, \dots, v_n\}$ be the set of vertices in G and $V_1 = (V - D) - v_1$ where v_1 is one of the vertices with maximum degree. By definition of jump graph, $E(G) = V[J(G)]$. Consider $I = \{e_1, e_2, e_3, \dots, e_k\}$ as the set of edges adjacent to v_1 in G . Let $H \subseteq V(J(G))$ be the set of vertices of $J(G)$ such that, $H \subseteq E - I$. Then H itself forms a minimally inverse dominating set. Therefore $\gamma^{-1}[J(G)] \leq |E| - |I|$. Hence $\gamma^{-1}[J(G)] < q - \Delta(G)$. $\square$

Theorem 17. For any connected graph G without pendant vertex

$$\gamma^{-1}[J(G)] \leq \delta(G).$$

Proof. Let $V - D = \{v_1, v_2, v_3, \dots, v_n\}$ be the set of vertices in G and v_1 be one among the vertices with minimum degree. Let $\{e_1, e_2, e_3, \dots, e_k\}$ be the set of edges adjacent to v_1 in G . Then $E_1 \subseteq V[J(G)]$ will form the inverse dominating set in $J(G)$. So $|E_1| = \delta(G)$ obviously it becomes the minimum inverse dominating set. Therefore $\gamma^{-1}[J(G)] \leq \delta(G)$. $\square$

3. Conclusion

Thus we conclude the inverse domination number and the parameters of inverse domination number of jump graphs like k_p , C_p , P_p , W_p and $K_{m,n}$.

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