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NEW CASES OF RECONSTRUCTIBILITY OF SBT GRAPHS

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Abstract: In earlier papers the author showed that all graphs which are not single-block trunk (SBT) graphs are reconstructible, and that two families of SBT graphs are reconstructible. Here some further families of SBT graphs are shown to be reconstructible.

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1. Introduction

Let G be a graph. A limb in G is a maximal induced subgraph, which is a tree. If G is not a tree each limb L has a root r, which attaches it to the rest of the graph. Removing the subgraphs $L - r$ for the limbs L, the remaining induced subgraph is called the trunk.

If G is not a tree, and the trunk is inseparable, G will be called a single-block trunk (SBT) graph. In [2] it is shown that if G is not an SBT graph then G is reconstructible. Let $n_v(G)$ ($n_e(G)$) denote the number of vertices (edges) of G. If G is an SBT graph let $n_p(G) = n_e(G) - n_p(G)$; letting B denote the block comprising the trunk of G, this equals $n_e(B) - n_v(B)$. In [3] it is shown that an SBT graph is reconstructible if $n_p \leq 1$.

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In this paper some results will be proved which imply that an SBT graph is reconstructible if $n_p \leq 2$. There are errors in [3], and new proofs will be given for $n_p = 0$ and $n_p = 1$ as well.

A graph G with a vertex v deleted is denoted G_v.

It has long been known ([1]) that if a graph G has limbs, then the multiset of limbs with the attachment point marked, and the trunk, are reconstructible. By the size of a limb will be meant its number of edges, or the number of vertices other than the root. A limb of size 1 will be called a 1-limb.

The proof of lemma 11 of [2] is essentially a proof that the limbs are determined; it has an error, which will be corrected here. Namely, the claim is made that if in G_v for all degree 1 vertices v the limbs are n 1-limbs for some $n \geq 1$, then in G the limbs are $n+1$ 1-limbs. This is true if $n \geq 2$; suppose $n = 1$. If there is a vertex w of degree 2 such that in G_w there are two components, one of which is an isolated vertex and the other of which has no degree 1 vertex, then the limbs of G are a single path of length 2. If there is a vertex w such that G_w has 2 components which are isolated vertices, then then the limbs of G are a single complete binary tree of height 1. If neither of these hold, the limbs are 2 1-limbs.

Let L_{ds} be the multiset of limbs of size s attached to vertices of degree d in the trunk.

Theorem 1. *The multiset of L_{ds} is reconstructible.*

Proof. Except for the case of a single 1-limb, this is theorem 2 of [3]. In the case of a single 1-limb, G_v where v is the root of the 1-limb may be determined, and the degree of v is determined. □

Theorem 2. *Let G be an SBT graph. For given d, unless the set of sizes of the limbs attached to degree d vertices of B is an interval $[0, m]$ for some m, G is reconstructible.*

Proof. If there is a limb of size s but none of size $s-1$ let v be a leaf of a tree of size s. In G_v there is a single limb of size $s-1$, and this may be replaced by the limb missing from L_{ds}. □

2. Automorphisms

Suppose G is an SBT graph and α is an automorphism of B. Let G^α be the result of α's action on G. That is, for a vertex r of B, let T_r (resp. T_r^α) denote the tree rooted at r in G (resp. G^α); then $T_r^\alpha = T_{\alpha(r)}$.

Suppose v is a degree 1 vertex of a limb of G. G "lies over" G_v, meaning that there is a vertex r_0 (namely the root of the limb containing v), such that T_{rv} equals T_r, except if $r = r_0$ it has v deleted. G lies over G_v in some other way iff G^α lies over G_v, where α is a nontrivial automorphism of B. Say that α is allowable (for v) in this case.

Clearly, an automorphism α of G is allowable. Although they will not be used, some additional facts will be stated. Suppose G^α lies over G_v, where r_1 is the vertex of B such that $T_{r_1}^\alpha$ does not equal $T_{r_1,v}$. It is possible that $r_1 = r_0$.

Suppose $r_0, \alpha(r_0), \ldots, \alpha^{l-1}(r_0)$ is the orbit of r_0. For $0 < i < l$, as long as $\alpha^i(r_0) \neq r_1$, $T_{\alpha^i(r_0)} = T_{\alpha^{i-1}(r_0)}$. This is so for all such i iff $r_1 = r_0$. In any case, r_1 is in the orbit of r_0. For r in some other orbit, $T_{\alpha(r)} = T_r$. α is an automorphism of G iff $r_1 = r_0$. Otherwise, $T_{\alpha^j(r_0)} = T_{r_0,v}$ for $i \leq j < l$ where $\alpha^i(r_0) = r_1$.

Theorem 3. *Suppose G is an SBT graph. If B is rigid and there are at least 2 degree 1 vertices then G is reconstructible.*

Proof. Let v range over the degree 1 vertices. The G_v may be "aligned" along B. G is then the union of the G_v. □

3. L-paths

Recalling some definitions of [4], an inseparable graph A is said to be a subdivision of an inseparable graph B if A is obtained from B by dividing edges by adding degree 2 vertices. Note that $n_p(A) = n_p(B)$. B is said to be S-minimal if it is not a subdivision of another graph. It is shown in [4] that an inseparable graph B is a subdivision of a unique S-minimal graph B_r.

If G is a graph let $n_v^d(G)$ denote the number of vertices of degree d, and $n_v^{\geq d}(G)$ the number of degree at least d. For an inseparable graph B $n_v^{\geq 3}(B) = n_v^{\geq 3}(B_r)$.

$n_v^{\geq 3}(B) = 0$ iff B is a cycle iff $n_p(B) = 0$ iff B_r is a 3-cycle. If $n_v^{\geq 3}(B) \neq 0$ then $n_v^{\geq 3}(B) \geq 2$. In this case, by a T-path in B is meant a maximal path all of whose interior vertices have degree 2 in B. A T-path in an SBT graph G is a T-path in its trunk B. An L-path in G is a T-path, with the limbs attached to its interior vertices adjoined, and its end points marked. The length of an L-path P is the length of its T-path, and the size of P is the sum of the sizes of the limbs, or the number of vertices not on the T-path.

An L-path P of length l may be oriented by labeling the vertices of its T-path from 0 to l, from one end to the other. If P is an oriented L-path let P^r denote P with the opposite orientation, and for a degree 1 vertex v of a limb let P_v have the inherited orientation. Let $r_0 < \cdots < r_t$ be the roots of the limbs, let T_r denote the limb rooted at vertex r, and let T_{r_i} be the limb containing v. Let ρ denote reversal of P. Say that ρ is allowable if P^ρ lies over P_v. If ρ is allowable say that it is type 1 if T_{r_i} equals T_{l-r_i}, and type 2 if $T_{r_i,v}$ equals T_{l-r_i}.

Theorem 4. *Suppose P is an L-path of size $s \geq 2$ and v is a degree 1 vertex of a limb. Suppose ρ is allowable. If ρ is type 1 then $P^\rho = P$, otherwise ρ is not allowable for w where w is the end of a degree 1 vertex of T_{l-r_i}.*

Proof. Straightforward. □

4. B a cycle

Suppose G is an SBT graph and its trunk B is a cycle.

Lemma 5. *If G has limbs of size ≥ 2 then G is reconstructible.*

Proof. Let u be a degree 1 vertex of a 1-limb T, and let v be a degree 1 vertex of a limb of size ≥ 2. Let R be B with the roots marked; this may be determined from G_v. Let R_u be B with the roots marked, other than that of the limb T containing u; this may determined from G_u.

If there is a rotation which is an automorphism of R let α be the one of highest order. G may be reconstructed from G_v, by locating the orbit with T, and then the location of T, and adding a 1-limb.

If R is rigid than R fits over R_u in only one way, whence the location in G_v of T may be determined and G reconstructed.

In the remaining case the automorphism group of R is generated by a single reflection, whose axis may be determined in any G_v. G may be divided into two subgraphs P_1 and P_2, each containing the vertices on one side of the axis, where a vertex on the axis and its attached limb if any are in both P_i. Let r denote the number of limbs of size ≥ 2. If $r \geq 3$ then P_1 and P_2 in G may be determined by considering the G_v.

Indeed, let $r_1 \leq r_2$ be the number of limbs on each P_i. By considering the G_v these may be determined, and also the side P_2 with r_2 limbs. The other side P_1 is also easily determined, unless $r_2 = r_1 + 1$. In this case, let Q be the degree 1 vertices of limbs of size ≥ 2 along P_2. Among the P_i (two each) of the G_v where $v \in Q$, only one subgraph occurs $|Q|$ times, and this is P_1.

In the remaining cases, $r \leq 2$; the reflection symmetry of R will not be used.

Suppose $r = 2$. Let T_1, T_2 be the two limbs of size ≥ 2. By considering the G_u the distance from T_1 to T_2, and the number of 1-limbs in between, may be determined. If all 1-limbs are between, G may be reconstructed from G_v where v is a degree 1 vertex of T_i where the size of T_i is as small as possible.

Otherwise, the L-path P_0 between T_1 and T_2 may be determined from the G_u. Let S be the 1-limb roots not on P_0. P_0 may be located in G_u for any u in a 1-limb with root in S. For $i = 1, 2$ let P_i be the paths from the root of T_i, away from P_0, containing the 1-limb roots closest to T_i. For $i = 1, 2, 3$ let p_i be the number of 1-limbs along P_i. Say that u is on P_i if its 1-limb is.

If $p_0 \geq 2$ G may be reconstructed from the G_u where u is on P_0, by theorem 4.

Suppose $p_0 \leq 1$. Suppose $p_1 + p_2 \geq 3$. By considering the G_u it may be determined if all of S lies along a single P_i. If this is the case G may be reconstructed from the G_u for u in a 1-limb with root in S, by taking the union. Otherwise, $P_0 \cup P_1$ and $P_0 \cup P_2$ may be determined, and G reconstructed from these. This is clear except for the case $p_2 = p_1 + 1$. In this case, among the paths (two each) of the G_u where u is on P_2, only one path occurs p_2 times, and this is P_1.

Suppose $1 \leq p_1 + p_2 \leq 2$. If $p_0 = 1$ then P_1 and P_2 may be determined. There are 4 cases $p_0, p_1 + p_2$ as follows.

1,2. Whether the 1-limbs other than the one on P_0 are on the same P_i may be determined. G may then be reconstructed as when $p_1 + p_2 \geq 3$.

1,1. Let d_0 be the distance from T_1 to T_2. In G_v where v is in the T_i of least size, there is exactly 1 limb at distance d from the T_i of greatest size, with a 1-limb in between. G is now readily reconstructed.

0,2. Let d_0 be the distance from T_1 to T_2, d_1, d_2 the distance from the 1-limbs to the nearest T_i, and b the cycle length. The possibilities $d_0, d_1, d_2 - d_1, b - d_2 - d_0$ and $d_0, d_1, b - d_2 - d_1 - d_0, d_2$ may be compared to the cyclic distances determined from R. Trying all 8 possible orderings of the second possibility, there is an ambiguity in only 3 cases, and in all of them $d_2 = (b - d_0)/2$. Whether this occurs is readily determined, and G reconstructed in this case as well.

0,1. Let d_0 be the distance from T_1 to T_2. Suppose v is in a T_i of least size. If there is a single 1-limb at distance d_0 from T_{3-i} in T_v then G may be reconstructed from G_v. Suppose there are 2 such. If the distance between them is d_0 G may be reconstructed from G_u. Otherwise G may be reconstructed from G_v where v is in T_{3-i}.

Suppose $r = 1$. Defined P_1, P_2, p_1, p_2 as in the case $r = 2$. If $p_1 + p_2 \geq 3$ G may be reconstructed as in the case $r = 2$.

If $p_1 + p_2 = 2$ proceed as in the 0,2 subcase of the case $r = 2$. The possibilities are $d_1, d_2 - d_1, b - d_2$ and $d_1, b - d_2 - d_1, d_2$. Trying all 6 possible orderings of the second possibility, there is an ambiguity in only 2 cases, and in both of them $d_2 = b/2$. Whether this occurs is readily determined, and G reconstructed in this case as well.

If $p_1 + p_2 = 1$ G may be reconstructed from G_v. □

Lemma 6. *Suppose the limbs of G are 1 or more 1-limbs; then G is reconstructible.*

Proof. Let b be the length of the cycle. Let v range over the vertices of B. If for any G_v the maximum path length is b or $b - 2$, G may be reconstructed from such a G_v. Otherwise, $b = 4t$ where $t > 0$, and numbering B circularly from 0, there are 1-limbs attached to vertices numbered i where $i \equiv 0, 1 \bmod 4$. □

It is a question of interest whether, when there are at least 3 1-limbs, G can be reconstructed from the G_u. The author has verified this for 3 or 4 1-limbs, and conjectures that it is true in general.

Theorem 7. *If B is a cycle then G is reconstructible.*

Proof. If there are no limbs the claim is readily verified. Otherwise it follows by lemmas 5 and 6. □

5. $n_v^{\geq 3}(B) = 2$

From hereon, for an SBT graph G with trunk B, say that a vertex of B of degree ≥ 3 is an E-vertex. Let n_v^E denote the number of them, i.e., $n_v^{\geq 3}(B)$. Call a limb attached to an E-vertex an E-limb. Call the remaining limbs I-limbs. A vertex v will be said to be determining if G_v can be determined, and G reconstructed from G_v.

If $n_v^E(B) = 2$ then $n_p(B) \geq 1$ and B consists of two vertices joined by $n_p(B) + 2$ T-paths, at most one of which is an edge; and B_r consists of two vertices joined by an edge and $n_p(B)+1$ T-paths of length 2. Let $d_E = n_p(B)+2$ denote the E-vertex degree. A vertex v is an E-vertex iff G_v is acyclic.

Theorem 8. *If $n_v^E = 2$ then G is reconstructible.*

Proof. Let $s_1 \leq s_2$ be the sizes of the E-limbs. If $\{s_1, s_2\}$ is not $\{0\}$ or $\{0, 1\}$ the claim follows by theorem 2. If the L-paths all have size 0 then an E-vertex with an E-limb if any attached is determining. Suppose neither of these cases holds. Say that an augmented L-path is an L-path, with the E-limb if any attached to its root. Let u range over degree 1 vertices.

The L-path sizes may be determined. If v is the degree 1 vertex of an E-limb they may be determined from G_v; suppose there are no E-limbs, and let n be the number of degree 1 vertices. If $n \geq 3$ the size multiset may be determined from those for the G_u, by a standard method (see e.g. the proof of lemma 11 of [2]). If $n = 1$, or if there is a single limb of size 2, the claim is trivial. If there are 2 1-limbs let v be the root of a limb; the size list is $\{0^{d_E-2}, 1, 1\}$ if G_v has a cycle with a limb, else $\{0^{d_e-1}, 2\}$.

Let m be the minimum nonzero size, and let r be the number of L-paths of size m. There is more than 1 limb iff either $r > 1$, or $r = 1$ and there are L-paths of size $> m$. It will next be shown that in this case, the multiset of augmented L-paths is determined.

Indeed, it is determined if a vertex v is a non-root vertex of a limb of an L-path of size m; let Q denote the set of these. Choose a $v \in Q$; the augmented L-paths of size $> m$ are those for G_v. If $r > 1$, consider the size m augmented L-paths for the G_u, where u ranges over Q. Each size m augmented L-path appears $(r-1)m$ times.

If $r = 1$ and there are L-paths of size greater than m, choose a vertex v in the L-path of size m. Let u range over the non-root limb vertices of the L-paths of size greater than m. For each such u, let A_u be the multiset of size m augmented L-paths in G_u. For each such u we can also determined the multiset

B_u of size m augmented L-paths in G_{vu}. The size m augmented L-path C of G may be determined by taking the multiset union of the A_u and of the B_u; the former will equal the latter, with some number of additional elements which are all copies of C.

Let v be a degree 1 vertex of an L-path of size m. If $m > 1$ the size 0 augmented L-paths are those of G_v. Otherwise the augmented L-path C of size 1 missing from G is determined, and the length 0 augmented L-paths of G are those of G_v, with C_v deleted.

If there is an E-limb, G is determined by its multiset of augmented L-paths; suppose not.

Suppose there is an $s \geq 2$ such that there is an L-path P of size s but none of size $s-1$. Then it is determined whether v is a degree 1 vertex of a limb of P, and if so P_v may be found in G_v. G may be reconstructed using theorem 4.

Suppose $d_E \geq 4$. Since there are I-limbs there must be an L-path of size 1; the root of its 1-limb is determining.

Suppose there is a single L-path P of size $m \geq 2$, and there is an E-vertex. Then P may be found in G_v where v is the degree 1 vertex of the E-limb. G may be reconstructed using theorem 4.

If P has a single limb of size ≥ 2 a degree 1 vertex of the limb is determining.

Suppose there is a single L-path P of size $m \geq 3$, and there is no E-limb. Let P_0 be the sub-path of P containing the limbs, with limbs at the ends. The length of P_0 may be determined from the G_v where v ranges over the degree 1 vertices. P_0 may then be determined from G_w where w is a degree 2 vertex on a size 0 L-path of length ≥ 2. G may be reconstructed using theorem 4 applied to P_0.

The remaining cases are as follows, where ijk are the L-path sizes, and E indicates an E-limb also.

002, 2 1-limbs. Let v be a vertex such that G_v is a cycle with a single limb attached. Since the lengths of the T-paths are known the length of the T-path P is determined from the length of the cycles; and also the location on the cycle of the other E-vertex. Since it is known whether v has a 1-limb attached, G may be reconstructed from G_v.

001. Let v be a vertex such that G_v is a cycle with a single limb attached, or no limbs. The argument is similar to the preceding case; if there is no 1-limb the E-vertices may be spaced on the cycle.

001E. Let v be a vertex such that G_v is a cycle with a single limb attached. The argument is similar to the preceding case.

In the remaining cases, $d_E = 3$ and the L-paths are determined. If an L-path of length 2 has a limb let s be the smallest size of such; an end vertex of a size s limb on an L-path of length 2 is determining. Thus it may be assumed that the L-paths have length ≥ 3.

A vertex u will be specified, as that where G_u has a cycle with certain limbs. One of these will be recognizable as having an E-vertex x as its root. Let P_1 be the path containing u, and l_1 its length; l_1 equals the sum of the lengths of the T-paths, minus the length the cycle. Knowing l_1 and the size of P_1, P_1 may be reconstructed by connecting an end of a new edge up to a uniquely determined vertex of the limb rooted at x, and possibly also an isolated vertex. Knowing P_1 the other two paths P_2 and P_3 are known, and the other E-vertex y can be located on the cycle. G may be reconstructed by attaching the other end of the new edge of P_1 to the cycle at y. Let l_2, l_3 be the lengths of P_2, P_3; y may be either at distance l_2 from x counterclockwise around the cycle, or at distance l_3.

011. Let u be such that G_u has 1 1-limb and a limb of size ≥ 2, x being the root of the latter. Let P_2 be the size 0 path, let d, d' be the distances from the root of the 1-limb of P_3 to the ends, and let e be the distance from x on the cycle in G_u, to the 1-limb on the cycle. There are two possibilities for the location of the other E-vertex in G_u; for both possibilities for y to be allowable, $l_2 < e$ must hold, and both e and $e - l_2$ must be among $\{d, d'\}$; it follows that $e = (l_2 + l_3)/2$ must hold, and the two possibilities are isomorphic.

012. Let u be such that G_u has 1 1-limb and a limb of size ≥ 2, x being the root of the latter. This case differs from case 011 only in that either the limb at x is of size ≥ 3, or there is an isolated vertex.

111. Let u be such that G_u has 2 1-limb and a 1 limb of size ≥ 2, x being the root of the latter. Let e, e' be the circular distance from the roots r, r' of the 1-limbs of G_u to x, along the arc not containing the other 1-limb. For a possibility for y to be allowed, it must lie on the arc between r and r', not containing x. If $e = e'$ the possibilities yield isomorphic graphs. If either P_2 or P_3 is symmetric the degree 1 vertex of its 1-limb is determining. Otherwise an ambiguity can occur only if $l_3 - e = e'$ and $l_2 - e' = e$, whence $l_2 = l_3$ and y is determined.

112. Let u be such that G_u has 2 1-limbs and a 1 limb of size ≥ 2, x being the root of the latter. This case differs from case 111 only as case 012 differs from case 011.

122. Let u be such that G_u has 3 1-limbs and a 1 limb of size ≥ 2, or 1 1-limb, 1 limb of size 2, and 1 limb of size ≥ 3, x being the root of the largest. In the latter case the size 2 path is readily located on the cycle. In the former let P_2 be the L-path with 2 1-limbs and P_3 the L-path with 1 1-limb. Write $l_2 = b_1 + b_2 + b_3$ where b_1, b_3 are the distances from the roots of the 1-limbs to

the ends and b_2 is the distance between them. Similarly write $l_3 = a_1 + a_2$. One direction of traversing the cycle may be written (renumbering as necessary) as $a_1, a_2 + b_1, b_2, b_3$. An ambiguity can occur only if this sequence equals one of 4 possibilities with the b's first. These may be seen to all be impossible.

123. Let P_1 be as in case 122; the argument differs from case 122 only in how P_1 is determined. It may be assumed that there is at most one isolated vertex in G_u. □

6. L-Multipaths

An L-multipath in an SBT graph G is defined to be the subgraph consisting of all L-paths between a pair of E-vertices. The number of L-paths is called the multiplicity. The size of an L-multipath is the sum of the sizes of its limbs.

If G is a graph with no degree 1 vertices let MPG(G) (the "multipath graph" of G) be the graph whose vertices are the E-vertices of G; two such are joined by an edge iff they are joined by one or more T-paths in G. If B is an inseparable graph, MPG(B) equals MPG(B_r). If B is an S-minimal inseparable graph then MPG(B) is obtained from B by deleting all T-paths of length 2. For an inseparable graph B, write B_m for MPG(B). B_m is inseparable, and any inseparable graph may occur as B_m (replace each edge by a triangle, or add 2 length 2 paths to an edge).

Note that an automorphism of B acts on B_m.

For an inseparable graph B B_m is an edge iff $n_{\bar{v}}^{\geq 3}(B) = 2$. Otherwise B_m has no degree 1 vertices, else B would be separable. When $n_p = 1$ $n_{\bar{v}}^{\geq 3}(B) = 2$ and B_m is an edge. By theorem 8, G is reconstructible in this case.

An L-multipath may be oriented by labeling the vertices of each T-path consecutively starting at an E-vertex. If M is an oriented L-multipath let M^r denote M with the opposite orientation; and for a degree 1 vertex v of a limb let M_v have the inherited orientation.

Theorem 9. *Suppose M is an oriented L-multipath of size $s \geq 2$, and for every degree 1 vertex v of a limb, M_v is an oriented subgraph of M^r. Then $M = M^r$.*

Proof. If there is only one path with limbs the claim follows by theorem 4; suppose there is more than 1. Let m be the smallest nonzero size of an L-path. Let P_j for $1 \leq j \leq k$ be the L-paths. Let v be a degree 1 vertex of a limb on a path P_i of size m. Let π be a permutation of $[1,k]$ such that $P_{jv} \subseteq P^r_{\pi(j)}$ for all j. Let S be the indices of the L-paths of size $> m$; then $\pi[S] = S$. Thus, for any $j \in S$ there is a cycle $j, \pi(j), \cdots, \pi^{t-1}(j), \pi^t(j) = j$ such that $P_{\pi^s(j)} \subseteq P^r_{\pi^{s+1}(j)}$ for $1 \leq s < t$. It follows that $P_{\pi^s(j)} = P^r_{\pi^{s+1}(j)}$ for $1 \leq s < t$, and so all $P_{\pi^s(j)}$ are equal, or t is even and $P_{\pi^s(j)}$ equals P_j for $t/2$ values of s, and P^r_j for $t/2$ values of s. In either case, the multiset of L-paths of size $> m$ is symmetric.

Let T be the indices of the size m L-paths. Renumbering, v is in P_0 and one of the cycles of π is $0, 1, \ldots, t-1$. The L-path multiset of each other cycle is symmetric. In the 0 cycle, P_0 appears $\lceil (t+1)/2 \rceil$ times and P^r_0 appears $\lfloor (t+1)/2 \rfloor$ times. If t is even or P_0 is symmetric then M is symmetric. If all L-paths have size m P_0 must be symmetric.

If there is an L-path of size $\geq m+2$, let v be a vertex of such. Let π be the permutation for G_v. Then $\pi[T] = T$, so the L-path multiset for T is symmetric, so M is symmetric. In the remaining case there is an L-path of size $m+1$, and for any such Q, and any vertex v of a limb of Q, $Q_v = P^r_0$; this is clearly impossible. $\square$

Theorem 10. *Let G be an SBT graph with $n^E_v(B) \geq 3$.*

a. *If G has limbs the L-multipath sizes are determined.*
b. *If G has more than one L-multipath of nonzero size then the L-multipaths are determined.*

Suppose the L-multipaths have been determined.

c. *If there is an L-multipath of size $s \geq 2$ but none of size $s-1$, G is determined.*
d. *For each multiplicity m, if there is an L-multipath of multiplicity m and size $s \geq 2$, but none of multiplicity m and size $s-1$, then G is determined.*

Proof. Part a is proved similarly to the claim for L-paths in theorem 8, Indeed, the argument for the L-multipath sizes is virtually identical, except for the case of 2 1-limbs. Let x, y be the degree 1 vertices. If the multipaths with an I-limb attached in G_x or G_y differ, or if they are both a path of length 2, then there are 2 size 1 L-multipaths. If they are a path of length ≥ 3, there is a single size 2 L-multipath iff there is a vertex w such that in G_w there is a single limb. If they are a multipath of multiplicity ≥ 2, there is a single size 2 L-multipath iff there is a vertex w on a path of a multipath of multiplicity ≥ 2, such that there is an L-multipath with 2 I-limbs of size 1.

The argument for part b is readily adapted from an argument in the proof of theorem 8.

For part c, let v range over degree 1 vertices of limbs of the L-multipaths of size s. For each v both the L-multipath M containing v, and M_v, are determined. If for some v M_v is not a subgraph of M^r, M may be laid over M_v. Otherwise, by theorem 9, for any v, M_v may be replaced by M.

Part d follows as part c, letting v range over degree 1 vertices of L-multipaths of multiplicity m and size s. □

7. B_m a Cycle

Lemma 11. *Suppose G is an inseparable graph. It is determined whether B_m is a cycle.*

Proof. There must be ≥ 3 E-vertices. B_m is a cycle iff, for any E-vertex v, $\mathrm{MPG}(G_v^T)$ is a path, where for a graph G G^T denotes the trunk of G. □

Suppose for the rest of the section that G is an SBT graph with B_m a cycle. Say that a multipath is fat (resp. thin) if it has multiplicity > 1 (resp. 1). Say that it is T_i if it is thin and has length i; and F_{ij} for $1 \leq i \leq j$ if it has multiplicity 2 and path lengths i, j. The same terminology applies to L-multipaths.

The automorphism group of B is either trivial, generated by a rotation, generated by a reflection, or generated by a reflection and a rotation. In cases 2 and 4 say that B admits a rotation. If B admits a rotation then there is a subgraph Q of B, which is a concatenation of q multipaths, such that B is the circular concatenation of q' copies of Q. Further, if q is as small as possible then for any Q, these are the only copies of Q around B.

Lemma 12. *If G has no limbs then G is reconstructible.*

Proof. It may be assumed that all T-paths have length at most 3, since if v is a degree 2 vertex of distance ≥ 2 from both ends then G may be reconstructed from G_v.

If there are any T_i multipaths for $i = 2, 3$ let w be an interior vertex of such; G is determined from G_w, except that an end cycle of G_w of length 4 might be either an F_{13} or F_{22} multipath. It is determined whether a vertex v is an interior vertex of the path of length 3 of an F_{13} multipath, and so the number of F_{13} multipaths is determined. It follows that if w is an interior vertex of a T_3 multipath then G is determined from G_w.

Suppose there are no T_3 multipaths but there are T_2 multipaths. Let w be the interior vertex of a T_2 multipath P. Unless one end of P belongs to an F_{13} multipath and the other to an F_{22} multipath, G is determined from G_w. In particular it may be assumed that there is a degree 3 vertex v such that in G_v, one end is a 3 vertex complete binary tree and the other is a path of length 1; G may be reconstructed from G_v.

Suppose there are no T_2 or T_3 multipaths, and there are T_1 multipaths. A vertex v is the end vertex of a T_1 path iff one end D of G_v is a fat multipath or a cycle; let E be the other end, which is a tree. The multipath from which E is derived may be determined. This is clear if E is not a path, or if E is a path of length 1 or 3. If E is a path of length 2 the choice between a T_1 multipath followed by an F_{12} multipath, and an F_{13} multipath, may be made on the basis of the number of T_1 multipaths. If D is not a cycle of length 4 G is readily determined. If D is a cycle of length 4 G is determined also, since the number of F_{13} multipaths is known.

Letting v be the other end of the T_1 multipath, E must be an F_{13} also. Thus, every T_1 multipath must have its ends in an F_{13} multipath, and so G may be reconstructed from G_v where v is an end of a T_1 multipath.

Suppose all multipaths are fat. Let v be an E-vertex. Add a new vertex to G_v and connect it to the degree 1 vertices. If either 0 or 2 additional edges are required, these may be added as needed to reconstruct G from G_v. Otherwise, in every adjacent pair of multipaths, one contains a length 1 path and the other does not. Which root of the trees in G_v to connect to the new vertex is readily determined in G_v. □

Lemma 13. *Let q be the number of limbs of size ≥ 2. If $q \geq 2$ then G is reconstructible.*

Proof. Let u be a degree 1 vertex of a 1-limb T, and let v be a degree 1 vertex of a limb of size ≥ 2. Let R be B with the limb roots marked, and let R_u be B with the roots marked, other than that of T.

The cases where R has a rotation which is an automorphism, and where R is rigid, may be proved as in the case of lemma 5.

In the remaining case, the automorphism group of R is generated by a single reflection, and the axis may be determined in any G_v. G may be divided into two subgraphs P_1 and P_2, each containing the vertices on one side of the axis, where a vertex on the axis and its attached limb if any are in both P_i.

If $q \geq 3$ then P_1 and P_2 in G may be determined by considering the G_v, as in the proof of lemma 5.

Suppose $q = 2$. The roots r_1 and r_2 of the size ≥ 2 limbs may be located in P_1, so it only remains to determine if in G they are located in opposite sides.

If $r_1 = r_2$ then they are on opposite sides. If one of r_1, r_2 is on the axis the question is irrelevant. If one of the limbs (say the one at r_1) has size 3 then this can be noted in the marked positions in P_1; whether r_1, r_2 are on the same side can then be determined from G_v where v is a degree 1 vertex of the limb of size 3.

Suppose r_1, r_2 are in the same L-multipath M in P_1. It may be determined whether they are in the same multipath in G from any G_u. Unless M is on the axis, G may clearly be reconstructed. If M is on the axis, the axis is that of M with its limbs removed, so whether r_1, r_2 are on the same side is determined from any G_u.

Suppose there is no L-multipath containing r_1, r_2.

Suppose there are L-multipaths M_1, M_2 on the axis with r_1 in M_1 and r_2 in M_2. If either limb is an I-limb, the axis is known. Otherwise, the distance d between r_1 and r_2 in B_m can be determined from any G_u; r_1, r_2 are on opposite sides iff $d = l/2$ where l is the length of B_m.

Suppose r_1 is on M_1 where M_1 is on the axis, but r_2 is not on the other L-multipath on the axis, if any. In G_u, let w be the E-vertex of M_1 closest to r_2, and let P be the T-path of M_1 containing r_2, and let l be its length. Then r_1 and r_2 are on the same side iff the distance from r_1 to w is less than $l/2$.

In the remaining case, let C be a cycle of length $2l$, with alternate positions denoting E-vertices and L-multipaths. Number the nodes counterclockwise from 0, where node 0 is an E-vertex or L-multipath according to which lies at the top of the axis. The locations of r_1 and r_2 on one side of C are determined; let these be $0 < i < j < l$. If r_1, r_2 are on the same side the distance between them in C is $j - i$. If they are on opposite sides, if $i + j \leq l$ the distance is $i + j$, and if $i + j > l$ the distance is $2l - i - j$; in either case it may be determined if r_1, r_2 are on the same side. $\square$

In the remaining cases, B is known.

Lemma 14. *Suppose G has 1 limb; then G is reconstructible.*

Proof. By theorem 2 the limb may be assumed to be a 1-limb. Let x be its root.

Suppose the limb is an I-limb. The L-multipath M containing x may be determined from G_x. Indeed, the multipath is known from B, and the location of x may be determined from the limbs of G_x.

Suppose there is a thin L-multipath P of length ≥ 4. Let z be an interior vertex of P which is at distance 2 from an end whose neighbor interior vertex has no limb. G may be readily reconstructed from G_z, no matter where x is located.

If M is a T_3 L-multipath there is a vertex y such that in G_y one end E_1 is a multipath with a path of length 2 attached, and the other end E_2 is a multipath or cycle. The multipath M_1 and M_2 giving rise to E_1 and E_2 may be determined, and G reconstructed from G_y.

If M is a T_2 L-multipath the ends E_1 and E_2 in G_x are each a multipath or cycle. Unless both are 4-cycles, G is readily reconstructed; this is also true if the multipaths missing from B are both F_{22} or F_{13} multipaths. In the remaining case there is a vertex y such that in G_y there is a path of length ≥ 4, with an F_{13} multipath at one end with a 1-limb attached to the closest vertex to that end of the path; G is readily reconstructed from G_y.

In the remaining cases M is a fat multipath.

If M is adjacent to a T_3 L-multipath there is a vertex y such that in G_y one end E_1 is a cycle or multipath with 2 1-limbs attached, and the other end E_2 is a cycle or multipath. E_1 is derived from M, so the multipath from which E_2 is derived is also known. If E_1 is a multipath G is readily reconstructed from G_u. Otherwise M is an F_{ij} L-multipath. If $i = j$ G is readily reconstructed, so suppose $i < j$. Let $l_1 \leq l_2$ be the distances from the attachment vertex of E_1 to the roots of the limbs; if $l_1 = l_2$ G is readily reconstructed, so suppose $l_1 < l_2$. It is readily verified that if the 1-limb is on the path of length i then $j = l_2$, and if it is on the path of length j then $i = l_1$; G may thus be reconstructed.

If M is adjacent to a T_2 L-multipath there is a vertex y such that in G_y one end E_1 is a cycle or multipath with a 1-limb attached, and the other end E_2 is a cycle or multipath. E_1 may be replaced by M. The multipath missing from B is then known, and G may be reconstructed.

If M is adjacent to a T_1 L-multipath there is a vertex y such that in G_y one end E_1 is a cycle or multipath with a 1-limb attached, and the other end E_2 is a tree. E_1 may be replaced by M. The multipath M_2 missing from B may then be determined, and also the part of it to be replaced may be determined and G reconstructed.

Suppose there is a vertex y such that in G_y there is an L-multipath with an I-limb and an E-limb (both 1-limbs). Since the missing multipath is known from B, a new vertex may be connected to the end of the E-limb, and to a second vertex, to reconstruct G from G_y.

If there is no such y, let w be such that in G_w the number of length 2 paths in a multipath adjacent to M is as small as possible; G may be reconstructed from G_w.

Now suppose that the limb is an E-limb, with root v. At least 1 multipath M incident to v is fat. If the other T is thin, the argument is similar to arguments already given. Namely, if T has length ≥ 4 let u on T have distance at least 2 from both ends; if T has length 3 let u such that E_1 in G_u has a limb of size 2; if T has length 2 let u be such that E_1 has a 1 limb; and if T has length 1 let u be such that E_1 has a 1-limb.

In the remaining case, the other multipath N incident to v is also fat. If one of M, N has a path of length 3, suppose w.l.g. that M is one of smallest multiplicity. Let u be on M such that in G_u there is a single limb, of size ≥ 2. Unless $M_u = N$ G may be reconstructed from G_u. If $N = M_u$ G is reconstructible from G_w where w is on a length 2 path of M. If neither M, N has a path of length 3 suppose w.l.g. that M has smallest multiplicity between M, N; G is reconstructible from G_w where w is on a length 2 path of M. □

Lemma 15. *Suppose B admits a rotation; then G is reconstructible.*

Proof. Let x be a vertex on an L-path of an L-multipath M of multiplicity ≥ 2 in G. With q as at the start of the section, let Q be a run of q multipaths, not including the multipath M_0 missing from G_x. Using Q, the location of M_x in G_x may be found. x may be chosen so that the E-limbs of G_x are those of G, with one of them having an extra subtree A of its root; further the root is an E-vertex of M.

G may be reconstructed from G_x by adding a new vertex x, connecting it to the E-vertex of M_x other than the root of A, connecting it as necessary to any disconnected component(s), and connecting it to the correct vertex of A. The latter may be determined, since the length of the L-path containing x is known, and I-limbs are 1-limbs, with the possible exception of a single I-limb of size 2. □

Lemma 16. *Suppose the automorphism group of B is generated by a reflection; then G is reconstructible.*

Proof. Similarly to the proof of lemma 13 when the automorphism group of R is generated by a single reflection, the axis may be determined in any G_u where u is a degree 1 vertex. G may be divided into two parts P_1 and P_2. If the number q of limbs is ≥ 3 G may be reconstructed as in lemma 13.

Suppose $q = 2$. The roots r_1 and r_2 of the limbs may be located in P_1, so it only remains to determine if in G they are located in opposite sides.

If $r_1 = r_2$ then they are on opposite side. If one of r_1, r_2 is on the axis the question is irrelevant. If one of the limbs (say the one at r_1) has size 2 then this can be noted in the marked positions in P_1; whether r_1, r_2 are on the same side can then be determined from G_v where v is a degree 1 vertex of the limb of size 2.

Suppose r_1, r_2 are in the same L-multipath M in P_1. Suppose M is thin. If M is on the axis there is a vertex w on the path of M such that in G_w there is only one limb, and G may be reconstructed from G_w. If M is not on the axis there is a vertex w on the path of M such that in G_w there is only one limb, iff r_1, r_2 are on the same side. Suppose M is fat. If M is on the axis let w be the root of an I-limb of M; G may be reconstructed from G_w. If M is not on the axis let w be the root of an I-limb of M if there is one, else any vertex on a T-path of M; it can be determined from G_w whether r_1, r_2 are on the same side.

Suppose there is no L-multipath containing r_1, r_2. Let C be the cycle of length $2l$ as in the proof of lemma 13; it suffices to determine the distance in C between the nodes corresponding to r_1, r_2.

Suppose r_1, r_2 are I-vertices of L-multipaths M_1, M_2. If there is a fat multipaths M_3 distinct from these let w be a vertex on a path of M_3; d may be determined from G_w. Otherwise $l = 3$ and one l-multipath is thin; all possibilities have already been covered.

Suppose r_1 is an E-vertex with a 1-limb attached, with degree 1 vertex u. If $l = 3$ G may be reconstructed from G_u, since r_2 must be an Suppose $l = 4$. If there is an E-vertex on the axis, or if r_2 is an E-vertex, G is reconstructible from G_u. Otherwise whether the r_2 L-multipath is parallel or perpendicular to the axis is known from G_u, and G is reconstructible from G_t where t is the other degree 1 vertex.

Thus, it may be assumed that $l \geq 5$. Let l_1 be the number of L-multipaths in G_{r_1}, including the end multipaths or cycles, which will be denoted M_1, M_2 ($M_1 = M_2$ iff $l = 5$). Let $l_2 = l_l - 1$, so that $2 \leq l_2 \leq 4$. Let v_1, v_2 be the E-vertices of M_1, M_2 in G_{r_1} (these exists iff $l \geq 6$ and are unequal iff $l \geq 7$, and say that $P(r_2)$ holds if r_2 is v_1, v_2, or lies between them; if this is the case let d_i be the distance from r_2 to v_i, $i = 1, 2$. Let j be the number of fat multipaths incident to r_1.

The rest of the proof is broken into the cases $l_2 j$.

Case 42. If $P(r_2)$ then $d = \min(d_1, d_2) + 6$. Otherwise, $d = 5$ if r_2 is an I-vertex of a fat multipath, $d = 4$ if r_2 is an E-vertex, and $d = 3$ if r_2 is an I-vertex of a thin multipath.

Case 32. W.l.g. the L-multipaths from M_1 to M_2 may be written $M_1 F_1 F_2 T_2 M_2$

where F_i is fat and T_2 is thin. Unless for some $p \geq 2, q \geq 1$ F_1 is $\mathrm{F}_{1,p+q}$, F_2 is $\mathrm{F}_{1,p}$, and T_2 is T_q, M_1 is known in G_{r_1}; suppose this is the case. If $P(r_2)$ then $d = \min(d_1 + 4, d_2 + 6)$. Otherwise, if r_2 is an I-vertex of a thin multipath then $d = 3$, if r_2 is an E-vertex then $d = 4$, and if r_2 is an I-vertex of a fat multipath then $d = 3$ if M_1 has 2 limbs, else $d = 5$. In the remaining case there is a vertex w such that in G_w there is an E-vertex with attached branches of size 1 and $p + q - 2$, with missing $\mathrm{F}_{1,p+q}$ multipath, etc.; G is reconstructible from G_w.

Case 22. If $P(r_2)$ then $d = \min(d_1, d_2) + 4$. Otherwise, $d = 3$.

Case 31. W.l.g. the L-multipaths from M_1 to M_2 may be written $M_1T_1F_2T_2M_2$ where F_2 is fat and T_i is thin. Unless for some $p \geq 2, q \geq 1$ T_1 is T_{p+q}, F_2 is $\mathrm{F}_{1,p}$, and T_2 is T_q, M_1 is known in G_{r_1}; suppose this is the case. If $P(r_2)$ then $d = \min(d_1 + 4, d_2 + 6)$. Otherwise, if r_2 is an I-vertex of a thin multipath then $d = 3$, if r_2 is an E-vertex then $d = 4$, and if r_2 is an I-vertex of a fat multipath then $d = 5$ if M_2 has 2 limbs, else $d = 3$. In the remaining case there is a vertex w such that in G_w there is an end multipath or cycle with a vertex with attached branches of size 1 and $p + q - 2$, with missing T_{p+q} multipath, etc.; G is reconstructible from G_w.

Case 21. Same as case 22.

□

Theorem 17. *If B_m is a cycle then G is reconstructible.*

Proof. This follows by lemmas 12 to 16, and theorem 3. □

8. B_m K_4

In this section, G will denote an SBT graph with B_m equaling K_4. In two cases, a fact is shown for all but a finite number of graphs, and the fact verified for them by a computer program. The "Nauty" library [5] was used for graph canonicalization in these cases.

Lemma 18. *If G has no limbs then G is reconstructible.*

Proof. If for some $m \geq 2$ there is a multipath of multiplicity m but none of multiplicity $m - 1$, let v be a vertex on a path of such a multipath; then G is reconstructible from G_v. Thus, the multipath multiplicities may be assumed to form an interval $[1, m]$.

Suppose all thin multipaths are edges, and there is at least 1 fat multipath. Let $m = 2$. If a multipath M of multiplicity m has all paths of length ≥ 2 let v be an interior vertex of a path of M; G is reconstructible from G_v. Proceeding inductively on m and using the same argument, every fat multipath must contain an edge. It now follows that G is reconstructible from G_v where v is any E-vertex.

Suppose there is a thin multipath P of length ≥ 2 and let v be an interior vertex. Unless for one of the end vertices of P both other incident multipaths are thin, G may be reconstructed from G_v. Thus, there must be an E-vertex w with all 3 incident multipaths thin. If in the triangle of remaining multipaths two of them are fat then the E-vertices in G_v may be found and G reconstructed.

Suppose there is a fat multipath M and 5 thin multipaths, at least one of which has length ≥ 2. If the multipath opposite M has length ≥ 2 then G is reconstructible from G_w, else G is reconstructible from G_v, where v and w are as above.

Suppose all multipaths are thin. Let q be the number of multipaths which are edges; this is half the sum of the numbers at the E-vertices. If $q \geq 5$ there is a triangle of edges. If there is a triangle of edges let v be the E-vertex not on it; then G is reconstructible from G_v.

Suppose it is known that there is a parallel pair of edges; G is reconstructible from G_v where v is an interior vertex of a thin multipath of length ≥ 2. If $3 \leq q \leq 4$ and there is no triangle then there is a parallel pair of edges.

Say that an E-vertex is connectible if the 3 incident thin multipaths have length ≥ 2; if v is a connectible E-vertex then G is reconstructible from G_v. If $q \leq 1$ or $q = 2$ and the edges are adjacent then there is a connectible E-vertex. □

For the rest of the section it will be assumed that G has limbs. If i, j are E-vertices call the L-multipath joining i and j the ij multipath.

Lemma 19. *If G has more than 1 E-limb then G is reconstructible.*

Proof. Let G^I be G, with the E-limbs removed; G^I is known from G_u where u is a degree 1 vertex of an E-limb. Consider the orbits of the automorphism group of G^I.

By a typical argument, for each orbit the E-limbs attached to the nodes of the orbit may be determined. Let r be the number of E-limbs of size 1; then either $r > 1$, or $r = 1$ and there are E-limbs of size > 1. The E-limbs of size > 1 belonging to each orbit may be determined from G_u where u is the degree 1 vertex of a size 1 E-limb. If $r > 1$ let u range over the degree 1 vertices of size 1 E-limbs; the orbits containing size 1 E-limbs may be determined, and then the number in each such orbit, from the G_u. If $r = 1$ let u be a degree 1 vertex of an E-limb of size 2; the orbit containing the size 1 E-limb may be determined from G_u.

The proof of the lemma is divided into cases, according to the orbit size list.

Case 1111. The vertex to which each E-limb is attached is known.

Case 112. The automorphism group acts transitively on the size 2 orbit, so the trees assigned to it may be assigned to the nodes of the orbit arbitrarily.

Case 22. Both orbits may be assumed to have a single E-limb, of size 1, since otherwise G can be reconstructed from the G_u where u is a degree 1 vertex of a size 1 E-limb. If the automorphism group is $Z_2 \times Z_2$ (with generators (01),(23), say), the trees assigned to each orbit may be assigned to the nodes of the orbit arbitrarily.

Suppose the automorphism group is Z_2, w.l.g. with generator (01)(23). The 01 L-multipath A and the 23 L-multipath A' are symmetric. Let B denote the 02 and 13 L-multipaths and C the 03 and 12. $B \neq C$, else the automorphism group is not Z_2.

If at least 1 of B, C (w.l.g. B) is asymmetric let u be a degree 1 vertex of an I-limb. In G_u there is a unique copy of B, and it is readily determined whether the E-limbs are at the ends of a copy of B, and G may be reconstructed from G^I. If B, C are symmetric then $A \neq A'$. If at least 1 of B, C (say B) has I-limbs let u be a degree 1 vertex of an I-limb. The pair of C L-multipaths may be found in G_u and it may be determined whether the E-limbs are at the ends of a copy of C and G may be reconstructed from G^I. The argument if either A or A' has an I-limb is similar.

If at least one of B, C is fat (say B) let w be a vertex on a path of B such that G_w has only 2 limbs. The two parallel copies of C may be found in G_w and it may be determined whether the E-limbs are at the ends of a copy of C. The argument if either A or A' is fat is similar.

In the remaining case let w be a vertex on the longer of A, A' such that in G_w there are two limbs. It may be determined from G_w whether the limbs are at the ends of a copy of B or C, and G may be reconstructed from G_w.

Case 13. If the automorphism group is S_3 the 1-limbs on the size 3 orbit may be assigned arbitrarily to the nodes of the orbit. Otherwise, numbering so that the size 3 orbit is 012, there is an asymmetric L-multipath A such that the 01, 12, and 20 L-multipaths all equal A. A must have I-limbs, and G is reconstructible from G_u where u is a degree 1 vertex of an I-limb if A.

Case 4. If the automorphism group is $Z_2 \times Z_2$ the 01,23 multipaths are both A, the 02,13 multipaths are both B, and the 03,12 multipaths are both C, where A, B, and C are symmetric and all distinct. The proof is divided into subcases, according to the E-limb size list S. If S is 11 the proof is similar to the previous case of orbit sizes 22 and automorphism group $Z_2 \times Z_2$. If there are any I-limbs let u be a degree 1 vertex of one; then which of A, B, C join the 2 E-limbs may be determined from G_u and G reconstructed from G^I. If there is a fat multipath let w be a vertex on a path of one such that G_w has only 2 limbs; which of A, B, C joins them may be determined from G_w. In the remaining case let w be a vertex along the longest of A, B, C such that G_w has 2 limbs,

If S is 12 the L-multipath joining the two E-limbs can be determined from G_u where u is a degree 1 vertex of the size 2 E-limb. If S is 111 the E-limbs may be assigned to E-vertices arbitrarily. If S is 112 the two L-multipaths joining the size 2 E-limb to a size 1 E-limb are determined from the G_u where u is the degree 1 vertex of a size 1 E-limb. If S is 122 the L-multipath joining the 2 size 2 E-limbs can be determined from G_u where u is the degree 1 vertex of the size 1 E-limb. If S is 123 the L-multipath joining the size 2 and size 3 E-limb may be determined from G_u where u is the degree 1 vertex of the size 1 E-limb; and the L-multipath joining the size 1 and size 2 E-limb may be determined from G_u where u is a degree 1 vertex of the size 2 E-limb.

If the automorphism group is Z_4 the nodes may be numbered so that the 01, 12, 23, and 30 L-multipaths are all equal to an asymmetric L-multipath A. G is readily reconstructed from G_u where u is a degree 1 vertex of an I-limb of A.

If the automorphism group is D_8 the graph consists of 3 parallel classes of L-multipaths, two of which are equal. The argument is essentially the same as in the case of 3 distinct parallel classes given above.

In the remaining case the automorphism group is S_4 and the E-limbs may be assigned arbitrarily to the E-vertices. □

Lemma 20. *If G has 1 E-limb and the sum of the sizes of the I-limbs is ≥ 2 then G is reconstructible.*

Proof. Let v denote the E-vertex with the E-limb attached. Let Σ_1 denote the set of L-multipaths incident to v, with v marked. Let Σ_2 denote the remaining 3 L-multipaths. For $i = 1, 2$ let s_i denote the sum of the sizes of the limbs of Σ_i; by a standard argument s_1 and s_2 may be determined, and then Σ_1 and Σ_2, from the G_u where u ranges over the degree 1 vertices of the I-limbs. By an argument as in theorem 10 the L-multipath sizes in both Σ_1 and Σ_2 must form an interval starting at 0. If $s_1 = 0$ then v is the only E-vertex with 3 incident size 0 L-multipaths, else there is only 1 L-multipath of nonzero size and G is reconstructible by theorem 10. Likewise, $s_2 \neq 0$ may be assumed.

Let G^E be G with I-limbs deleted. If G^E is rigid G may be reconstructed by taking the union of the G_u where u ranges over degree 1 vertices of I-limbs. Otherwise, number E-vertices so that the E-vertex with the E-limb attached is 0, and (12) is an automorphism of G^E.

Suppose the 03 multipath B differs from the 01 and 02 multipaths A. If the 12 multipath has any I-limbs then G is reconstructible from G_u where u is a degree 1 vertex of one; likewise for the 03 L-multipath. If the 01 and 02 L-multipaths both have nonzero size then G is reconstructible from G_u where u is a degree 1 vertex of an I-limb on a copy of A of smallest size. If the 03 and 13 multipaths differ from B, likewise for them. If they equal B there is only one vertex in G^I with 3 incident equal multipaths, and G is reconstructible from G^I. Thus, there are 2 nonzero size L-multipaths. From G^I it is known whether they are vertex disjoint, and so G is reconstructible from G_u where u is the degree 1 vertex of the I-limb on a copy of A.

Suppose $B = A$. If there is only 1 E-vertex with 3 incident multipaths equaling A then G is reconstructible from G^I. Otherwise, all multipaths are A, except possibly the 02 multipath. If it differs, similarly to arguments just given, it must be size 0, whence the 02 multipath must be size 0, and the argument proceeds as before.

Suppose all 6 multipaths equal A. If there are 2 nonzero size L-multipaths in Σ_2, w.l.g. the 12 L-multipath may be assumed to have size 0. As usual the 03 L-multipath has size 0, and so only one L-multipath in Σ_1 has nonzero size, say the 01 L-multipath. G may be reconstructed from G_u where u is a degree 1 vertex of an I-limb of the 13 L-multipath. In the remaining case of two nonzero size multipaths the argument is as before. □

Lemma 21. *If G has 1 E-limb and a single I-limb of size 1 then G is reconstructible.*

Proof. Let v, Σ_1, Σ_2 be as in the previous proof. Let A be the L-multipath of nonzero size. A is in Σ_1 iff there is a vertex w on a path, such that G_w has only 1 limb.

Suppose A is in Σ_1. Let w be the root of the I-limb. From G_w the L-multipaths in Σ_1, with v marked, may be determined, and also the multipath joining the other ends of the two such other than A. If Σ_1 contains only 1 copy of A then G may be reconstructed from G_u where u is the degree 1 vertex of the I-limb. Otherwise, let w be a vertex on a path of a second copy of A such that in G_w there is only 2 limbs; v can be located in G_w, and also the 4th E-vertex since B is known. Thus, G may be reconstructed from G_w. □

Lemma 22. *If G has 1 E-limb and no I-limbs then G is reconstructible.*

Proof. The argument in the proof of lemma 18 that G is reconstructible if there is a fat multipath carries through if there is a single E-limb, with some modifications.

Suppose the E-vertices are numbered so that 0 has 3 incident thin multipaths, and the 12 and 23 multipaths are fat. G is reconstructible from G_{v_0} unless the vertex with the E-limb attached is (w.l.g.) v_1 and the 03 multipath has length 2. In this case G is reconstructible from G_w where w is the vertex on the 03 path.

Suppose there is a fat multipath M and 5 thin multipaths, at least one of which has length ≥ 2. The length l of the path opposite M is known since B is. If there is a path P of length ≥ 2 incident to an E-vertex of M let w be a vertex of P such that in G_w there are as few limbs as possible. G is reconstructible from G_w since l is known. In the remaining case the paths incident to the E-vertices of M are edges, and the remaining path P has length ≥ 2. G may be reconstructed from G_w where w is an interior vertex of P such that in G_w the number of limbs is as small as possible.

Suppose all multipaths are thin. Let v, Σ_1, Σ_2 be as in the previous proof. Σ_1, and hence Σ_2, are known from G_v. If all paths in Σ_1 have length 1 then G is reconstructible because Σ_2 is known. If some path in Σ_1 has length ≥ 4 then G is reconstructible from G_w where w is on such a path, at distance ≥ 2 from each end. If some path in Σ_2 has length ≥ 6 then G is reconstructible from G_w where w is on such a path, at distance ≥ 3 from each end. There remain only a finite number of graphs; a computer program was written to verify that these all have distinct decks. □

Lemma 23. *If G has a fat L-multipath then the L-multipaths are determined.*

Proof. This is trivial if G has an E-limb; suppose not. By theorem 10 it suffices to consider the case of a single L-multipath M of nonzero size. If M is thin let v be a vertex on a path of a fat multipath. If M is fat let v be an E-vertex not incident to M. In either case M is an L-multipath of G_v. □

Lemma 24. *Suppose P is an L-path of length ≥ 2 with its ends marked and v is a vertex on P. Then either P lies over P_v in only one way, or P is symmetric.*

Proof. Orient P and let $T_1, \ldots, T_{l-1}$ be the trees along P, where l is the length of P. Say that P can proceed along P_v to i if it can proceed to $i-1$ (vacuous if $i=1$), and a copy of T_i can be cut out of the remaining subtree of P. It is readily verified that if P in both orientations can proceed along one of the two trees of P_v of maximum length to $\lfloor (l-1)/2 \rfloor$ then P is symmetric, proving the lemma. □

Lemma 25. *If G has no E-limbs and at least 1 fat L-multipath then G is reconstructible.*

Proof. The proof is a modification of statements in the proof lemma 18. If for some $m \geq 2$ there is a multipath of multiplicity m but none of multiplicity $m-1$, let v be a vertex on a path of such a multipath. Using lemmas 23 and 24, G is reconstructible from G_v. Thus, the multipath multiplicities may be assumed to form an interval $[1, m]$.

Suppose there is a thin L-multipath P length ≥ 2. Again using lemmas 23 and 24, G is reconstructible from G_v where v is on P, unless there is an E-vertex v with 3 incident thin L-multipaths.

If G has 2 fat multipaths, using lemma 23 and subtracting the L-multipaths of G_v, the multiset Σ of L-multipaths incident to v is known. If these all have size 0 G is reconstructible from G_v. If Σ contains an L-multipath of length ≥ 3 G is reconstructible from G_w where w is on such a path and at a distance ≥ 2 from v. Otherwise, G is reconstructible from G_v.

Suppose there is a fat multipath M and 5 thin multipaths, at least one of which has length ≥ 2. The argument is as in this case of the proof of lemma 22, except that w is arbitrary and lemmas 23 and 24 are used.

Thus, it may be assumed that all thin L-multipath are edges.

If there is an E-vertex v such that G_v is a triangle with some limbs attached, using lemma 23, G may be reconstructed from G_v.

If the L-multipaths which are edges form a square let v be an interior vertex of a path on an L-multipath of multiplicity 2; then G is reconstructible from G_v.

If the L-multipaths which are edges form a path of length 2 let v be an E-vertex such that G_v is a triangle of fat L-multipaths with one E-vertex having a limb. Then G is reconstructible from G_v.

If the L-multipaths which are edges form a parallel pair let v be an interior vertex of a path on an L-multipath of multiplicity 2; then G is reconstructible from G_v.

Suppose there is a single L-multipaths which is an edge, with E-vertices v_1, v_2, and v_3, v_4 the remaining E-vertices. By an argument in the proof of lemma 18, adapted using lemmas 23 and 24, every fat multipath must contain an edge. By considering G_v for $v \in \{v_3, v_4\}$, the multiset Σ of L-multipaths incident to v_1 or v_2 may be determined. If an L-multipath of Σ has a path of length ≥ 3, let M be one of least multiplicity among such, let P be a path of M of length ≥ 3, and let w be the vertex of P such that in G_w there is only one limb, and it is attached to v_3 or v_4. M_v may be found in G_v, and P may be determined, whence by lemma 24 G may be reconstructed.

If there is no such M in Σ, let M be the L-multipath with ends v_3, v_4. If M has a path P of length ≥ 3 proceed similarly. Otherwise, all paths which are not edges have length 2, and G is reconstructible from G_v where $v \in \{v_3, v_4\}$. $\square$

Lemma 26. *If G has no E-limbs, all L-multipaths are thin, and there are ≥ 2 L-multipath of nonzero size, then G is reconstructible.*

Proof. By theorem 10.b the L-paths are known. Let l_0 be the minimum length of a size 0 L-path. Let F denote the set of size 0 length l_0 paths.

If F contains a triangle, i.e., there is an E-vertex v such that in G_v there is a cycle of length $3l_0$ with 3 limbs attached, G is reconstructible from G_v.

If F is a square let u be the degree 1 vertex of the I-limb of an L-path of size 1; G is reconstructible from G_u.

Suppose F is a path of length 2. Number the E-vertices so that along the path they are 102, and 3 is the remaining one. Let P be the 12 L-path; P is determined from any G_u where u is a degree 1 vertex of a limb not on P. If P has size ≥ 2 then G is reconstructible using lemma 4. Let Σ be the L-paths incident to vertex 3, with 3 marked. If P has size 1 Σ is known from G_u where u is the degree 1 vertex of the limb of P; otherwise it may be determined by a standard argument from the G_u for u a degree 1 vertex. As noted in earlier arguments, it may be assumed that the 03 L-path has size 0, and only one L-path in Σ has nonzero size, say the 01 L-path. G may be reconstructed from G_u where u is the degree 1 vertex of the I-limb of the 01 L-path.

Suppose F is a parallel pair. If there is a path of length ≥ 3 in F^{cmpl}, the complement of F, let w be a vertex on such a path P such that G_w has as few limbs as possible. In G_w, 3 of the E-vertices of G may be found from degree considerations, and the 4th using the fact that the two length l_0 L-paths are parallel; G may now be reconstructed using lemma 24.

Suppose all L-paths in F^{cmpl} have length 2. If $l_0 \geq 2$ there cannot be an L-path not in F of size 0, whence G is reconstructible from G_u for u the degree 1 vertex of the limb on an L-path of size 1. If $l_0 = 1$ there must be an L-path not in F of size 0; let w be its midpoint. G is reconstructible from G_w, since on at least 1 path of length 3 the E-vertex may be located, and the other may then be by finding the parallel edge.

Suppose F is a single L-path; let $\{0,1\}$ be the its incident E-vertices, and $\{2,3\}$ the E-vertices incident to the path P parallel to the edge. If P has size ≥ 2 then G is reconstructible using lemma 4.

Suppose P has size 1. If P is symmetric then G is reconstructible from G_u where u is the degree 1 vertex of the limb of P. Suppose P is not symmetric. For $i \in \{2,3\}$ let Q_i be the path between 0 and 1 containing i, with attached limbs, and let d_{ij} be the distance from i to j, $j \in \{0,1\}$. If $\{d_{20}, d_{21}\} \neq \{d_{30}, d_{31}\}$, let u be the degree 1 vertex of the limb on P and let t be any other degree 1 vertex; the location of the limb on P may be determined from G_t, and G then reconstructed from G_u. Let s_i be the size of Q_i, and assume $s_2 \leq s_3$; if $0 < s_2 < s_3$ the argument is similar. The remaining cases are as follows, noting that $s_2 \leq 2$ may be assumed.

01. The limb on Q_3 is between j and 3 for exactly one $j \in \{0,1\}$, and the distance from its root to j is determined from G_u.

0,≥ 2. G may be reconstructed using lemma 4 for Q_3.

11. Considering Q_i as oriented, unless $Q_2 = Q_3$ or $Q_2 = Q_3^r$, the location of the limb on P may be determined from the G_t. In the remaining cases the two possibilities are isomorphic.

22. Using lemma 4 it may be assumed that both Q_i are symmetric. Unless $Q_2 = Q_3$ the location of the limb on P may be determined from the G_t, and in the remaining case the two possibilities are isomorphic.

Suppose P has size 0, and let 0,1 etc. be as above. if P has length 2, which can occur only if $l_0 = 1$, then $Q_2.Q_3$ are known from G_w where w is the midpoint of P. Unless $s_2 = 1$ and $s_3 \leq 2$ G is reconstructible by lemma 10.

In case 11 of s_2, s_3, unless $Q_2 = Q_3$ or $Q_2 = Q_3^r$, G is reconstructible from either G_t using G_w, and in the remaining cases the two possibilities are isomorphic.

In case 12, unless Q_3 is symmetric and Q_2 is not Q_{3t} where t is a degree 1 vertex of a limb on Q_3, G is reconstructible from G_t using G_w, where t is the degree 1 vertex of the limb on Q_2. Unless $\{d_{20}, d_{21}\}$ equals $\{d_{30}, d_{31}\}$, G may be reconstructed from G_t for t the degree 1 vertex of a limb of Q_3. If $d_{20} = d_{30}$ G is reconstructible from G_t where in G_t $Q_2 = Q_3$ and if if $d_{20} = d_{31}$ G is reconstructible from G_t where in G_t $Q_2 = Q_3^r$.

Suppose P has length ≥ 3. If $2 \leq s_2 < s_3$, Q_2 is known from G_w where w is a vertex on P such that in G_w Q_2 has size 2; G may be reconstructed using lemma 4. If $2 \leq s_2 = s_3$, Q_2, Q_3 are known from the G_w where w is a vertex on P such that in G_w either Q_2 or Q_3 has size 2; G is reconstructible by lemma 10. If $s_2 = 1$ and $s_3 \geq 3$, Q_3 is known from G_t where t is the degree 1 vertex of the limb on Q_2; G may be reconstructed using lemma 10.

In case 11 of s_2, s_3, unless $\{d_{20}, d_{21}\}$ equals $\{d_{30}, d_{31}\}$, G may be reconstructed from the G_t. This is also true unless $d_{20} = d_{21}$, and in that case G is reconstructible from G_w where w is a vertex along P.

In case 12, again $\{d_{20}, d_{21}\} = \{d_{30}, d_{31}\}$ may be assumed. Letting w be the vertex on P such that in G_w Q_3 has 3 limbs, and t the degree 1 vertex of the limb on Q_2, G_w lies over G_t in only one way, unless Q_3 is symmetric and v_3 is the midpoint. In that case, $\{d_{20}, d_{21}\}$ is known from G_t where t is a degree 1 vertex of a limb on Q_3. G may then be reconstructed from G_w where w is a vertex along P such that in G_w Q_3 has 2 limbs. □

Lemma 27. *If G has no E-limbs, all L-multipaths are thin, and there is only 1 L-multipath of nonzero size, then G is reconstructible.*

Proof. Let P be the path with limbs. Suppose there is a limb of size ≥ 2. If there is only one limb G is reconstructible. Otherwise let R be P with the roots of limbs marked; R is determined from G_u where u is the degree 1 vertex of a limb of size ≥ 2. Let t be the degree 1 vertex of a limb of size 1. Either R fits over P_t in only one way, or R is symmetric; in either case G is reconstructible from G_t.

Suppose P has length ≥ 4. Let w be on P, such that in G_w there are 2 limbs, one of size 1, or one of size 2 if there is no w with one of size 1. G is reconstructible from G_w. If P has length 3 and there are 2 limbs, G is reconstructible from G_u where u is a degree 1 vertex of a 1-limb.

If a size 0 L- path has length ≥ 6 G is reconstructible from G_w where w is at distance ≥ 3 from both ends. There remain only a finite number of graphs; a computer program was written to verify that these all have distinct decks. □

Theorem 28. *If G is an SBT graph with $B_m = K_4$ then G is reconstructible.*

Proof. This follows by lemmas 18 to 27. □

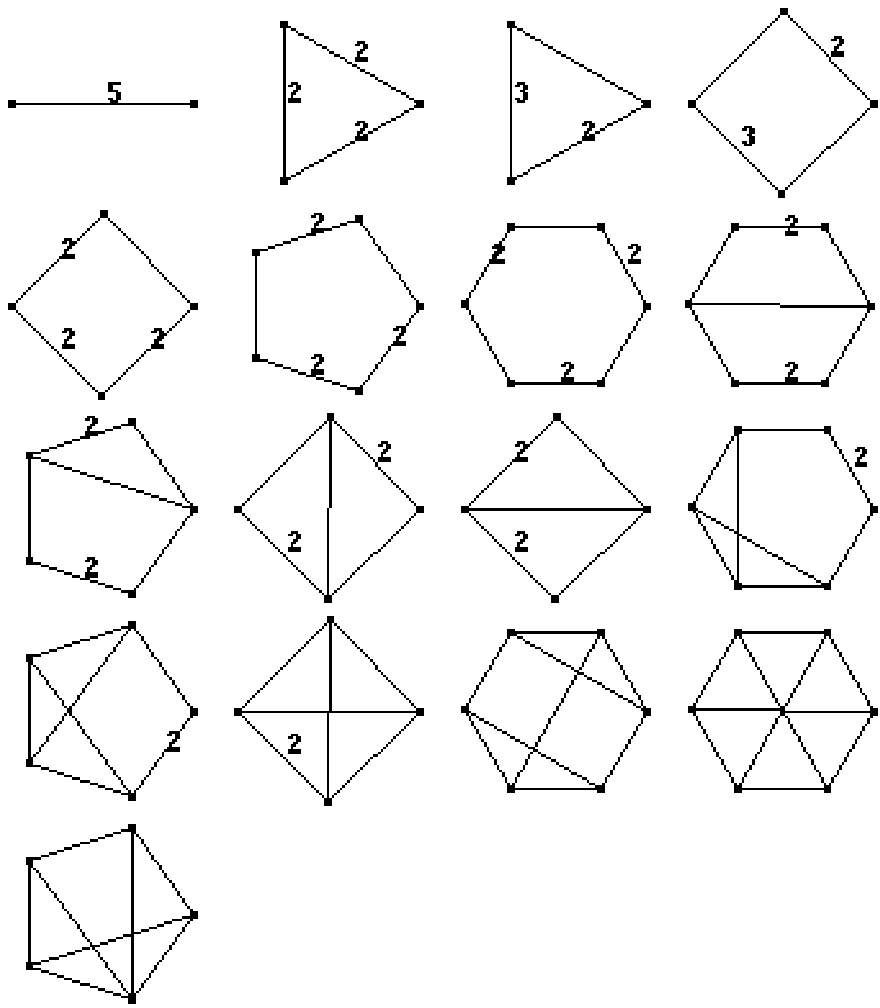

Figure 1: S-minimal graphs for $n_p = 3$

9. $n_p = 2$

Theorem 29. *If $n_p = 2$ then G is reconstructible.*

Proof. When $n_p = 2$ there are 4 S-minimal graphs; see Figure 2 of [4]. From left to right the multipath graphs are an edge, a triangle, a square, and K_4. In the first case, G is reconstructible by theorem 8. In the second and third cases

G is reconstructible by theorem 17. In the fourth case G is reconstructible by theorem 28. □

10. Remarks on $n_p = 3$

Figure 1 shows the 17 S-minimal graphs for $n_p = 3$, each given as its multipath graph, with multipath multiplicities indicated. Numbering from 1, left to right, top to bottom, SBT graphs with B_m equaling graph 1 are reconstructible by theorem 8, those with B_m equaling graph 2-7 are reconstructible by theorem 17, and those with B_m equaling graph 14 are reconstructible by theorem 28.

As noted earlier, any inseparable graph of size ≥ 2 can occur as B_m. The simplest B_m yet to be considered is that of graphs 10-11 of figure 1. It is clear, though, that even though the graph reconstruction conjecture seems likely to be true, the case-by-case methods of this paper are inadequate to make more substantial progress.

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