International Journal of Pure and Applied Mathematics
Volume 103 No. 4 2015, 625-633
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v103i4.3

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ONE MODULO THREE MEAN LABELING OF CYCLE RELATED GRAPHS

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Abstract: The concept of one modulo three mean labeling was introduced in [2]. In this paper, we prove that the graphs EJ_n, $P_{4m}(+)\overline{K_n}, K_{1,2n} \times P_2, NA(Q_m), S'(P_{2n}), D(C_n, v')$ and $D(C_n, e')$ are one modulo three mean graphs.

AMS Subject Classification: 05C78
Key Words: one modulo three mean labeling, one modulo three mean graphs

1. Introduction

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminologies of graph theory as in [1]. Swaminathan and Sekar introduced the notion of one modulo three graceful labeling in [4]. Motivated by the work of these authors Jeyanthi and Maheswari

Received: March 12, 2015

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[2] introduced the concept of one modulo three mean labeling. A graph G is said to be one modulo three mean graph if there is an injective function ϕ from the vertex set of G to the set $\{a/0 \leq a \leq 3q-2$ and either $a \equiv 0(mod\ 3)$ or $a \equiv 1(mod\ 3)\}$ where q is the number of edges of G and ϕ induces a bijection ϕ^* from the edge set of G to $\{a/1 \leq a \leq 3q-2$ and either $a \equiv 1(mod\ 3)\}$ given by $\phi^*(uv) = \left\lceil \frac{\phi(u)+\phi(v)}{2} \right\rceil$ and the function ϕ is called one called one modulo three mean labeling of G. In [2], they proved that P_{2n}, comb, bistar $B_{n,n}$, T_p-tree with even number of vertices, C_{4n+1}, ladder $L_{n+1}, K_{1,2n} \times K_2$ are one modulo three mean graphs. Also they proved that $B_{m,n}, K_{1,n}, K_n, n > 3$ are not one modulo three mean graphs. In [3], it is proved that $DA(Q_n)$, $DA(Q_2) \odot nK_1, DA(Q_m) \odot nK_1, DA(T_2) \odot nK_1, DA(T_m) \odot nK_1$, $\overline{S}(DA(T_n)), \overline{S}(DA(Q_n)), mP_n, m \geq 1, C_m *_e C_n(m, n \equiv 1(mod\ 4))$ graphs are one modulo three mean graphs. In this paper we extend our study on one modulo three mean labeling and prove that $EJ_n, P_{4m}(+)\overline{K}_n$, $K_{1,2n} \times P_2$, $NA(Q_m)$, $S'(P_{2n})$, $D(C_n, v')$, $D(C_n, e')$ are one modulo three mean graphs.

We use the following known theorems and definitions in the subsequent section.

Theorem 1. [4] *Let $G_1(p_1, q_1), G_2(p_2, q_2), \ldots, G_m(p_m, q_m)$ be a one modulo three mean graphs with $q_i(1 \leq i \leq m)$ is odd and u_i, v_i be the vertices of $G_i(1 \leq i \leq m)$ labeled with 0 and $3q_i - 2$. Then the graph G obtained by joining v_1 with u_2 and v_2 with u_3 and v_3 with u_4 and so on until we join v_{m-1} with u_m by an edge is also a one modulo three mean graph.*

Definition 2. Let G be a graph. For each point v of a graph G, take a new point v' and join v' to the vertices of G which are adjacent to v. The graph thus obtained is called the splitting graph of G and is denoted by $S'(G)$.

Definition 3. Let G be a graph and v be any vertex of G. A new vertex v' is said to be duplication of v if all the vertices which are adjacent to v are adjacent to v'. The graph obtained by duplication v is denoted by $D(G, v')$.

Definition 4. Let G be a graph and e be any edge of G. A new edge e' is said to be duplication of an edge $e$if all the edges which are incident to e in G are incident to e'. The graph obtained by duplication e is denoted by $D(G, e')$.

Definition 5. An n^{th} alternate quadrilateral snake $NA(Q_m)$ consists of n alternate quadrilateral snakes that have a common path. That is, a n^{th} alternate quadrilateral snake is obtained from a path $u_1, u_2, \ldots, u_m$ by joining u_{2i-1} and $u_{2i}(1 \leq i \leq \frac{m}{2})$ to the n new vertices v_{ij} and w_{ij} respectively and then joining $v_{ij}, w_{ij}(1 \leq i \leq \frac{m}{2}, 1 \leq j \leq n)$.

Definition 6. A cartesian product of two graphs G_1 and G_2 is the graph $G_1 \times G_2$ such that its vertex set is a cartesian product of $V(G_1)$ and $V(G_2)$. That is $V(G_1 \times G_2) = V(G_1) \times V(G_2) = \{(x, y)/x \in V(G_1), y \in V(G_2)\}$ and its edge set is defined as $E(G_1 \times G_2) = \{((x_1, x_2), (y_1, y_2))/x_1 = y_1$ and $(x_2, y_2) \in E(G_2)$ or $x_2 = y_2$ and $(x_1, y_1) \in E(G_1)\}$.

Definition 7. A triangular ladder $TL_n, n \geq 2$ is a graph obtained from L_n by adding the edges $u_iv_{i+1}, 1 \leq i \leq n-1$ where u_i and v_i are the vertices of L_n such that $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ are two paths of length n in the graph L_n.

Definition 8. A friendship graph F_n is a one point union of n copies of cycle C_3.

Definition 9. The extend jewel graph EJ_n is a graph with vertex set $V(EJ_n) = \{u, v, x, y, w, z, u_i : 1 \leq i \leq n\}$ and edge set

$$E(EJ_n) = \{uv, ux, xy, yz, vw, wz, vu_i, zu_i : 1 \leq i \leq n\}.$$

Definition 10. The composition of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph $G_1[G_2]$ with a vertex set $V = V_1 \times V_2$ and an edge set $E = \{uv|u = (u_1, u_2), v = (v_1, v_2)$ and either $u_1v_1 \in E_1$ or $u_1 = v_1$ and $u_2v_2 \in E_2\}$.

Definition 11. Let G be a graph with two or more vertices. A total graph $T(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ and the two vertices are adjacent in $T(G)$ whenever they are either adjacent or incident in G.

2. Main Results

Theorem 12. *If G is a graph in which every edge is an edge of a triangle, then G is not a one modulo three mean graph.*

Proof. Let G be a graph in which every edge is an edge of a triangle. Suppose G is a one modulo three mean graph. To get 1 on edge label, there must be two adjacent vertices u and v such that $f(u) = 0$ and $f(v) = 1$. Let $uvwu$ be a triangle in which the edge uv lies. To get 4 on edge label, there must be $f(w) = 7$, then uw and vw get the same edge label. This is a contradiction to the fact of one modulo three mean labeling. Hence G is not a one modulo three mean labeling graph. ∎

Corollary 13. *The wheel graph W_n, flower graph FL_n, triangular snakes, double triangular snakes, triangular ladders, fans $P_n + K_1, n \geq 2$, Double fans $P_n + K_2, n \geq 2$, friendship graph C_3^n, windmills $K_m^n, m > 3$, square graph $B_{n,n}^2$, total graph $T(P_n)$ and composition graph $P_n[P_2]$ are not one modulo three mean graph.*

Theorem 14. *The extend jewel graph EJ_n is a one modulo three mean graph.*

Proof. Let vertex set $V(EJ_n) = \{u, v, x, y, w, z, u_i : 1 \leq i \leq n\}$ and edge set $E(EJ_n) = \{uv, ux, xy, yz, vw, wz, vu_i, zu_i, xw : 1 \leq i \leq n\}$. Then EJ_n has $n + 6$ vertices and $2n + 7$ edges. Define a vertex labeling $\phi : V(G) \to \{0, 1, 3, \ldots, 6n + 19\}$ as follows. $f(u_i) = 6i + 7$ if $1 \leq i \leq n, f(u) = 1, f(v) = 0, f(w) = 7, f(x) = 6n + 13, f(y) = 6n + 18, f(z) = 6n + 19$. It can be verified that the induced edge labels of EJ_n are $1, 4, \ldots, 6n + 19$. Hence ϕ is a one modulo three mean labeling of EJ_n. □

An example for the one modulo three mean labeling of EJ_4 is given in Figure 1.

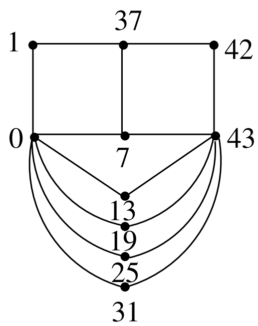

Figure 1

Theorem 15. *Let $G = P_m(+)\overline{K}_n$ be the graph with the vertex set $V(G) = \{u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and the edge set $E(G) = \{u_iu_{i+1}, u_1v_j, u_mv_j : 1 \leq i \leq m-1, 1 \leq j \leq n\}$. Then G is a one modulo three mean graph if $m \equiv 0 (mod\ 4)$.*

Proof. Let vertex set $V(G) = \{u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and the edge set $E(G) = \{u_iu_{i+1}, u_1v_j, u_mv_j : 1 \leq i \leq m-1, 1 \leq j \leq n\}$. Here $|V(G)| = m + n, |E(G)| = m + 2n - 1$. Define a vertex labeling $\phi : V(G) \to \{0, 1, 3, \ldots, 3m + 6n - 5\}$ as follows. $f(v_{n-i+1}) = 6(n - i) + 3m + 1$ if $1 \leq i \leq n$,

$$f(u_{2i-1}) = \begin{cases} 6(i-1) & \text{if } 1 \leq i \leq \frac{m}{4} \\ 6i - 5 & \text{if } \frac{m}{4} + 1 \leq i \leq \frac{m}{2}, \end{cases}$$

$$f(u_{2i}) = \begin{cases} 6i-5 & \text{if } 1 \le i \le \frac{m}{4} \\ 6(i-1)+6n & \text{if } \frac{m}{4}+1 \le i \le \frac{m}{2} \end{cases}$$

It can be verified that the induced edge labels of G are $1, 4, \ldots, 3m+6n-5$. Hence ϕ is a one modulo three mean labeling of $P_m(+)\overline{K_n}$. Hence $P_m(+)\overline{K_n}$ is a one modulo three mean graph if $m \equiv 0(mod\ 4)$. □

An example for the one modulo three mean labeling of $P_8(+)\overline{K}_3$ is shown in Figure 2.

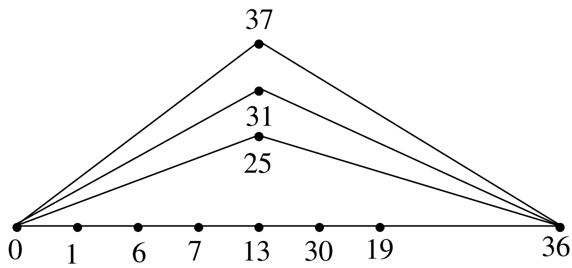

Figure 2

Remark 16. [4] *The graph $DA(Q_n)$ is a one modulo three mean graph.*

Theorem 17. *The book graph $K_{1,n} \times P_2 (n > 2)$ is a one modulo three mean graph if and only if n is even.*

Proof. Let $G = K_{1,n} \times P_2$. The vertex set $V(G) = \{u, v, u_i, v_i : 1 \le i \le n\}$ and the edge set $E(G) = \{uv, uu_i, vv_i, u_iv_i : 1 \le i \le n\}$. Here $|V(G)| = 2n+2, |E(G)| = 3n+1$. Define a vertex labeling $\phi : V(G) \to \{0, 1, 3, \ldots, 9n+1\}$ as follows. $f(u) = 0, f(v) = 9n+1, f(u_i) = 6i-5$ if $1 \le i \le n$.

$$f(v_{\frac{n}{2}-i+1}) = \begin{cases} 9n-6i+6 & \text{if } 1 \le i \le \lfloor \frac{n}{4} \rfloor \\ 9n-6i+1 & \text{if } \lfloor \frac{n}{4} \rfloor + 1 \le i \le \frac{n}{2} \end{cases}$$

$f(v_{n-i+1}) = 6n + 6\lfloor \frac{n-1}{4} \rfloor + 7$ if $i = \lfloor \frac{n-1}{4} \rfloor + 1$,

$f(v_3) = 18$ if $n = 4$.

If $n > 4$, $f(v_{n-i+1}) = \begin{cases} 6n-6i & \text{if } 1 \le i \le \lfloor \frac{n-1}{4} \rfloor \\ 6n-6i+6 & \text{if } \lfloor \frac{n-1}{4} \rfloor + 2 \le i \le \frac{n}{2} \end{cases}$

It can be verified that the induced edge labels of G are $1, 4, \ldots, 9n+1$. Hence ϕ is a one modulo three mean labeling of $K_{1,n} \times P_2$. Hence $K_{1,n} \times P_2$ is a one modulo three mean graph if n is even. Conversely, assume that $n \equiv 1(mod\ 2)$ and take $n = 2k+1$. Then $|V(G)| = 4k+4$ and $|E(G)| = 6k+4$. Let ϕ be a one modulo three mean labeling of $K_{1,n} \times P_2$, then 0,1, $18k+10$ and $18k+9$ must be the vertex labels of one modulo three mean graphs. If $f(u) = 0, f(v) = 1, f(u_i) = 18k+10, f(v_i) = 18k+9 (1 \le i \le n)$, then the induced edge label of uu_i and vv_i get the same label $9k+5$ which is not possible. Also this is a contradiction to the fact that the edge labels are congruent to one modulo three. If $f(u) = 0, f(v_i) = 1, f(v) = 18k + 9$ (or) $f(u) = 1, f(v_i) = 0, f(v) = 18k + 10$ (or) $f(v) = 0, f(u_i) = 1, f(u) = 18k + 9$ (or) $f(u) = 18k + 10, f(v) = 1, f(u_i) = 0 (1 \le i \le n)$, then the induced edge label is $9k + 5$ which is not possible. This is a contradiction to the fact that the edge labels are congruent to one modulo three. Therefore, 0 and $18k + 10$ cannot be the labels of the adjacent vertices. Hence the book graph $K_{1,n} \times P_2$ is not a one modulo three mean graph if n is odd. □

Theorem 18. *The n^{th} alternate quadrilateral snake $NA(Q_m)$ is a one modulo three mean graph if n is even.*

Proof. By Theorem 17, $NA(Q_2)$ is a one modulo three mean graph. Let $G_i = NA(Q_2)$ for $1 \le i \le m-1$. Since each G_i has $3n+1$ edges, by Theorem 1, $NA(Q_m)$ admits one modulo three mean labeling. □

An example for the one modulo three mean labeling of $4A(Q_4)$ is shown in Figure 3.

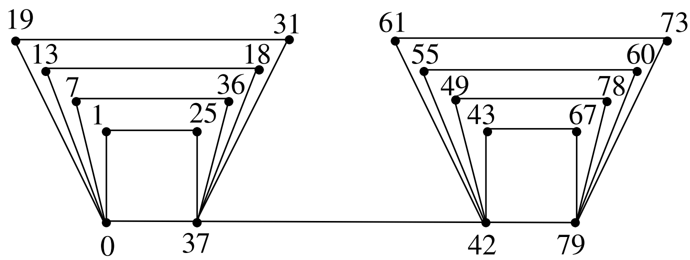

Figure 3.

Theorem 19. *The splitting graph $S'(P_n)$ is a one modulo three mean graph if n is even.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n and $v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n$ be the vertices of $S'(P_n)$. Here $S'(P_n)$ has $2n$ vertices and $3(n-1)$ edges. Define a vertex labeling $\phi : V(S'(P_n)) \to \{0, 1, 3, \ldots, 9n-11\}$ as follows.

$$f(v_i) = \begin{cases} 9(i-1) & \text{if } i \text{ is odd} \\ 9i - 11 & \text{if } i \text{ is even} \end{cases},$$

$$f(v'_i) = \begin{cases} 9i - 3 & \text{if } i \text{ is odd} \\ 9i - 17 & \text{if } i \text{ is even.} \end{cases}$$

It can be verified that the induced edge labels of $S'(P_n)$ are $1, 4, \ldots, 9n-11$. Clearly ϕ is a one modulo three mean labeling of $S'(P_n)$. Hence $S'(P_n)$ is a one modulo three mean graph if n is even. □

An example for the one modulo three mean labeling of splitting graph of a path $S'(P_6)$ is shown in Figure 4.

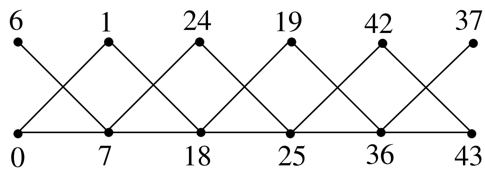

Figure 4.

Theorem 20. *The graph obtained by duplicating an arbitrary vertex of a cycle $D(C_n, v')$ admits one modulo three mean labeling if $n \equiv 1, 3(mod\ 4)$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle C_n. Let $G = D(C_n, v')$ be the graph obtained by duplicating an arbitrary vertex v of C_n. Without loss of generality take $v = v_n$ and the duplication of v_n be v'_n. Hence $|V(D(C_n, v'_n))| = n + 1$ and $|E(D(C_n, v'_n))| = n + 2$. Define a vertex labeling $\phi : V(G) \to \{0, 1, 3, \ldots, 3n + 4\}$ by considering the following two cases.

Case (i). $n \equiv 1(mod\ 4)$.

$f(v_1) = 1, f(v_n) = 13, f(v'_n) = 7$ and

$$f(v_{n-2i+1}) = \begin{cases} 3n - 6i + 9 & \text{if } 1 \le i \le \lfloor \frac{n}{4} \rfloor \\ 3n - 6i - 3 & \text{if } \lceil \frac{n}{4} \rceil \le i \le \lfloor \frac{n}{2} \rfloor \end{cases},$$

$f(v_{n-2i}) = 3n - 6i + 10$ if $1 \le i \le \lfloor \frac{n}{2} \rfloor - 1$.

Case (ii). $n \equiv 3(mod\ 4)$.

$f(v_1) = 3n + 4, f(v'_n) = 3n + 3,$

$$f(v_{2i}) = \begin{cases} 6(i-1) & \text{if } 1 \le i \le \lceil \frac{n}{4} \rceil \\ 6i - 5 & \text{if } \lceil \frac{n}{4} \rceil + 1 \le i \le \lfloor \frac{n}{2} \rfloor \end{cases},$$

$$f(v_{2i+1}) = \begin{cases} 6i - 5 & \text{if } 1 \le i \le \lceil \frac{n}{4} \rceil \\ 6i & \text{if } \lceil \frac{n}{4} \rceil + 1 \le i \le \lfloor \frac{n}{2} \rfloor \end{cases}.$$

It can be verified that the induced edge labels of $D(C_n, v')$ are $1, 4, \ldots, 3n + 4$. Clearly ϕ is a one modulo three mean labeling of $D(C_n, v')$. Hence $D(C_n, v')$ is one modulo three mean graph if $n \equiv 1, 3(mod\ 4)$. □

An example for the one modulo three mean labeling of duplicating an arbitrary vertex of a cycle $D(C_{11}, v')$ is shown in Figure 5.

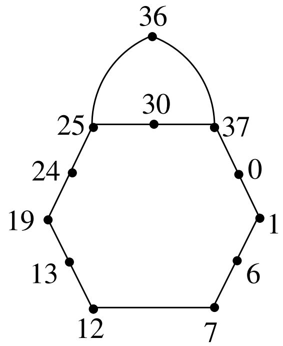

Figure 5.

Theorem 21. *The graph obtained by duplicating an arbitrary edge in cycle $D(C_n, e')$ is a one modulo three mean graph if $n \equiv 0, 2(mod\ 4)$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n. Let $G = D(C_n, e')$ be a graph obtained by duplicating an arbitrary edge e of C_n. Without loss of generality take $e = v_1v_2$ and the duplication of e be edge is $e' = v_1'v_2'$. Hence $|V(D(C_n, e'))| = n + 2$ and $|E(D(C_n, e'))| = n + 3$. Define a vertex labeling $\phi : V(G) \to \{0, 1, 3, 4, \ldots, 3n + 7\}$ by considering the following two cases.

Case (i). $n \equiv 2(mod\ 4)$.

$f(v_1) = \begin{cases} 12 & \text{if } n = 6 \\ 19 & \text{if } n > 6, \end{cases}$ $f(v_2) = 7, f(v_1') = 3n + 6, f(v_2') = 1.$

$f(v_{2i+1}) = 6(i - 1)$ if $1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor$, $f(v_{\frac{n}{2}+2}) = \begin{cases} 3n & \text{if } n = 6 \\ \frac{3n}{2} + 3 & \text{if } n > 6, \end{cases}$

$f(v_{2i+1}) = 6i + 6$ if $\left\lfloor \frac{n}{4} \right\rfloor + 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$,

$f(v_{n-2i+2}) = \begin{cases} 3n - 6i + 13 & \text{if } 1 \leq i \leq \frac{n}{2} - 2 \\ 13 & \text{if } i = \frac{n}{2} - 1 \end{cases}$

Case (ii). $n \equiv 0(mod\ 4)$.

$f(v_1) = 12, f(v_2) = 1, f(v_1') = 3n + 6, f(v_2') = 7,$

$f(v_{2i+1}) = \begin{cases} 6(i - 1) & \text{if } 1 \leq i \leq \frac{n}{4} \\ 6i + 6 & \text{if } \frac{n}{4} + 1 \leq i \leq \frac{n}{2} - 1 \end{cases}$

$f(v_{n-2i+2}) = 3n - 6i + 13$ if $1 \leq i \leq \frac{n}{2} - 1$. It can be verified that the induced edge labels of $D(C_n, e')$ are $1, 4, \ldots, 3n + 4$. Hence ϕ is a one modulo three mean labeling of $D(C_n, e')$. Hence $D(C_n, e')$ is a one modulo three mean graph if $n \equiv 0, 2(mod\ 4)$. □

An example for the one modulo three mean labeling of duplicating an arbitrary edge of a cycle $D(C_{10}, e')$ is shown in Figure 6.

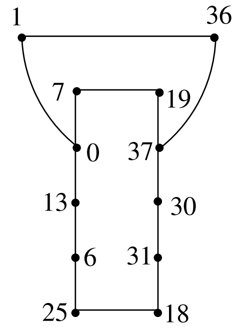

Figure 6.

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