ONE MODULO THREE MEAN LABELING OF CYCLE RELATED GRAPHS

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Abstract: The concept of one modulo three mean labeling was introduced in [2]. In this paper, we prove that the graphs $EJ_n$, $P_{4m}(+)K_{n_1, n_2}$, $P_2, NA(Q_m), S'(P_{2n}), D(C_n, v')$ and $D(C_n, e')$ are one modulo three mean graphs.

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1. Introduction

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminologies of graph theory as in [1]. Swaminathan and Sekar introduced the notion of one modulo three graceful labeling in [4]. Motivated by the work of these authors Jeyanthi and Maheswari

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[2] introduced the concept of one modulo three mean labeling. A graph $G$ is said to be one modulo three mean graph if there is an injective function $\phi$ from the vertex set of $G$ to the set $\{a/0 \leq a \leq 3q - 2 \text{ and either } a \equiv 0(\text{mod } 3) \text{ or } a \equiv 1(\text{mod } 3)\}$ where $q$ is the number of edges of $G$ and $\phi$ induces a bijection $\phi^*$ from the edge set of $G$ to $\{a/1 \leq a \leq 3q - 2 \text{ and either } a \equiv 1(\text{mod } 3)\}$ given by $\phi^*(uv) = \left\lfloor \frac{\phi(u) + \phi(v)}{2} \right\rfloor$ and the function $\phi$ is called one modulo three mean labeling of $G$. In [2], they proved that $P_{2n}$, comb, bistar $B_{n,n}$, $T_p$-tree with even number of vertices, $C_{4n+1}$, ladder $L_{n+1}, K_{1,2n} \times K_2$ are one modulo three mean graphs. Also they proved that $B_{m,n}, K_{1,n}, K_n, n > 3$ are not one modulo three mean graphs. In [3], it is proved that $DA(Q_n), DA(Q_2) \circ nK_1, DA(Q_m) \circ nK_1, DA(T_2) \circ nK_1, DA(T_m) \circ nK_1,$ $\overline{S}(DA(T_n)), \overline{S}(DA(Q_m)), mP_n, m \geq 1, C_m * e C_n(m, n \equiv 1(\text{mod } 4))$ graphs are one modulo three mean graphs. In this paper we extend our study on one modulo three mean labeling and prove that $EJ_n, P_{4m}(+)\overline{K}_n, K_{1,2n} \times P_2, NA(Q_m), S'(P_{2n}), D(C_n, v'), D(C_n, e')$ are one modulo three mean graphs.

We use the following known theorems and definitions in the subsequent section.

**Theorem 1.** [4] Let $G_1(p_1, q_1), G_2(p_2, q_2), \ldots, G_m(p_m, q_m)$ be a one modulo three mean graphs with $q_i(1 \leq i \leq m)$ is odd and $u_i, v_i$ be the vertices of $G_i(1 \leq i \leq m)$ labeled with 0 and $3q_i - 2$. Then the graph $G$ obtained by joining $v_1$ with $u_2$ and $v_2$ with $u_3$ and $v_3$ with $u_4$ and so on until we join $v_{m-1}$ with $u_m$ by an edge is also a one modulo three mean graph.

**Definition 2.** Let $G$ be a graph. For each point $v$ of a graph $G$, take a new point $v'$ and join $v'$ to the vertices of $G$ which are adjacent to $v$. The graph thus obtained is called the splitting graph of $G$ and is denoted by $S'(G)$.

**Definition 3.** Let $G$ be a graph and $v$ be any vertex of $G$. A new vertex $v'$ is said to be duplication of $v$ if all the vertices which are adjacent to $v$ are adjacent to $v'$. The graph obtained by duplication $v$ is denoted by $D(G, v')$.

**Definition 4.** Let $G$ be a graph and $e$ be any edge of $G$. A new edge $e'$ is said to be duplication of an edge $e$ if all the edges which are incident to $e$ in $G$ are incident to $e'$. The graph obtained by duplication $e$ is denoted by $D(G, e')$.

**Definition 5.** An $n^{th}$ alternate quadrilateral snake $NA(Q_m)$ consists of $n$ alternate quadrilateral snakes that have a common path. That is, a $n^{th}$ alternate quadrilateral snake is obtained from a path $u_1, u_2, \ldots, u_m$ by joining $u_{2i-1}$ and $u_{2i}(1 \leq i \leq \frac{m}{2})$ to the $n$ new vertices $v_{ij}$ and $w_{ij}$ respectively and then joining $v_{ij}, w_{ij}(1 \leq i \leq \frac{m}{2}, 1 \leq j \leq n)$.  

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Definition 6. A cartesian product of two graphs $G_1$ and $G_2$ is the graph $G_1 \times G_2$ such that its vertex set is a cartesian product of $V(G_1)$ and $V(G_2)$. That is $V(G_1 \times G_2) = V(G_1) \times V(G_2) = \{(x, y) | x \in V(G_1), y \in V(G_2)\}$ and its edge set is defined as $E(G_1 \times G_2) = \{((x_1, x_2), (y_1, y_2)) | x_1 = y_1$ and $(x_2, y_2) \in E(G_2)$ or $x_2 = y_2$ and $(x_1, y_1) \in E(G_1)\}$.

Definition 7. A triangular ladder $TL_n, n \geq 2$ is a graph obtained from $L_n$ by adding the edges $u_i v_{i+1}, 1 \leq i \leq n - 1$ where $u_i$ and $v_i$ are the vertices of $L_n$ such that $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ are two paths of length $n$ in the graph $L_n$.

Definition 8. A friendship graph $F_n$ is a one point union of $n$ copies of cycle $C_3$.

Definition 9. The extend jewel graph $EJ_n$ is a graph with vertex set $V(EJ_n) = \{u, v, x, y, w, z, u_i : 1 \leq i \leq n\}$ and edge set $E(EJ_n) = \{uv, ux, xy, yz, vw, wz, vu_i, zu_i : 1 \leq i \leq n\}$.

Definition 10. The composition of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph $G_1[G_2]$ with a vertex set $V = V_1 \times V_2$ and an edge set $E = \{uv | u = (u_1, u_2), v = (v_1, v_2)\}$ and either $u_1 v_1 \in E_1$ or $u_1 = v_1$ and $u_2 v_2 \in E_2$.

Definition 11. Let $G$ be a graph with two or more vertices. A total graph $T(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ and the two vertices are adjacent in $T(G)$ whenever they are either adjacent or incident in $G$.

2. Main Results

Theorem 12. If $G$ is a graph in which every edge is an edge of a triangle, then $G$ is not a one modulo three mean graph.

Proof. Let $G$ be a graph in which every edge is an edge of a triangle. Suppose $G$ is a one modulo three mean graph. To get 1 on edge label, there must be two adjacent vertices $u$ and $v$ such that $f(u) = 0$ and $f(v) = 1$. Let $uwvu$ be a triangle in which the edge $uv$ lies. To get 4 on edge label, there must be $f(w) = 7$, then $uw$ and $vw$ get the same edge label. This is a contradiction to the fact of one modulo three mean labeling. Hence $G$ is not a one modulo three mean labeling graph. \qed
Corollary 13. The wheel graph $W_n$, flower graph $F_{L_n}$, triangular snakes, double triangular snakes, triangular ladders, fans $P_n + K_1, n \geq 2$, Double fans $P_n + K_2, n \geq 2$, friendship graph $C_n^n$, windmills $K_n^m, m > 3$, square graph $B_{2n,n}^2$, total graph $T(P_n)$ and composition graph $P_n[P_2]$ are not one modulo three mean graph.

Theorem 14. The extend jewel graph $EJ_n$ is a one modulo three mean graph.

Proof. Let vertex set $V(EJ_n) = \{u, v, x, y, z, u_i | 1 \leq i \leq n\}$ and edge set $E(EJ_n) = \{uv, ux, xy, yz, vw, wz, vu_i, zu_i, xw | 1 \leq i \leq n\}$. Then $EJ_n$ has $n + 6$ vertices and $2n + 7$ edges. Define a vertex labeling $\phi : V(G) \rightarrow \{0, 1, 3, \ldots , 6n + 19\}$ as follows. $f(u_i) = 6i + 7$ if $1 \leq i \leq n$, $f(u) = 1$, $f(v) = 0$, $f(w) = 7$, $f(x) = 6n + 13$, $f(y) = 6n + 18$, $f(z) = 6n + 19$. It can be verified that the induced edge labels of $EJ_n$ are $1, 4, \ldots , 6n + 19$. Hence $\phi$ is a one modulo three mean labeling of $EJ_n$. \hfill \Box

An example for the one modulo three mean labeling of $EJ_4$ is given in Figure 1.

Theorem 15. Let $G = P_m(+)K_n$ be the graph with the vertex set $V(G) = \{u_i, v_j | 1 \leq i \leq m, 1 \leq j \leq n\}$ and the edge set $E(G) = \{u_iu_{i+1}, u_1v_j, u_mv_j | 1 \leq i \leq m - 1, 1 \leq j \leq n\}$. Then $G$ is a one modulo three mean graph if $m \equiv 0(\text{mod } 4)$.

Proof. Let vertex set $V(G) = \{u_i, v_j | 1 \leq i \leq m, 1 \leq j \leq n\}$ and the edge set $E(G) = \{u_iu_{i+1}, u_1v_j, u_mv_j | 1 \leq i \leq m - 1, 1 \leq j \leq n\}$. Here $|V(G)| = m + n, |E(G)| = m + 2n - 1$. Define a vertex labeling $\phi : V(G) \rightarrow \{0, 1, 3, \ldots , 3m + 6n - 5\}$ as follows. $f(u_{n+1}) = 6(n-i)+3m+1$ if $1 \leq i \leq n$, $f(u_{2i-1}) = \begin{cases} 6(i - 1) & \text{if } 1 \leq i \leq \frac{m}{4} \\ 6i - 5 & \text{if } \frac{m}{4} + 1 \leq i \leq \frac{m}{2} \end{cases}$.
\[ f(u_{2i}) = \begin{cases} 
6i - 5 & \text{if } 1 \leq i \leq \frac{m}{4} \\
6(i - 1) + 6n & \text{if } \frac{m}{4} + 1 \leq i \leq \frac{m}{2} 
\end{cases} \]

It can be verified that the induced edge labels of \( G \) are 1, 4, \ldots, \( 3m + 6n - 5 \). Hence \( \phi \) is a one modulo three mean labeling of \( P_m(+)K_n \). Hence \( P_m(+)K_n \) is a one modulo three mean graph if \( m \equiv 0 \pmod{4} \). \( \square \)

An example for the one modulo three mean labeling of \( P_3(+)K_3 \) is shown in Figure 2.

![Figure 2](image_url)

**Remark 16.** [4] *The graph DA\( (Q_n) \) is a one modulo three mean graph.*

**Theorem 17.** *The book graph \( K_{1,n} \times P_2(n > 2) \) is a one modulo three mean graph if and only if \( n \) is even.*

**Proof.** Let \( G = K_{1,n} \times P_2 \). The vertex set \( V(G) = \{u, v, u_i, v_i : 1 \leq i \leq n\} \) and the edge set \( E(G) = \{uv, uu_i, vv_i, u_iv_i : 1 \leq i \leq n\} \). Here \( |V(G)| = 2n + 2, |E(G)| = 3n + 1 \). Define a vertex labeling \( \phi : V(G) \to \{0, 1, 3, \ldots, 9n + 1\} \) as follows. \( f(u) = 0, f(v) = 9n + 1, f(u_i) = 6i - 5 \) if \( 1 \leq i \leq n \).

\[
\begin{align*}
f(v_{2i+1}) &= \begin{cases} 
9n - 6i + 6 & \text{if } 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor \\
9n - 6i + 1 & \text{if } \left\lfloor \frac{n}{4} \right\rfloor + 1 \leq i \leq \frac{n}{2} 
\end{cases} \\
f(v_{n-i+1}) &= 6n + 6 \left\lfloor \frac{n-1}{4} \right\rfloor + 7 \text{ if } i = \left\lfloor \frac{n-1}{4} \right\rfloor + 1, \\
f(v_3) &= 18 \text{ if } n = 4.
\end{align*}
\]

If \( n > 4 \), \( f(v_{n-i+1}) = \begin{cases} 
6n - 6i & \text{if } 1 \leq i \leq \left\lfloor \frac{n-1}{4} \right\rfloor \\
6n - 6i + 6 & \text{if } \left\lfloor \frac{n-1}{4} \right\rfloor + 2 \leq i \leq \frac{n}{2} 
\end{cases} \)

It can be verified that the induced edge labels of \( G \) are 1, 4, \ldots, \( 9n + 1 \). Hence \( \phi \) is a one modulo three mean labeling of \( K_{1,n} \times P_2 \). Hence \( K_{1,n} \times P_2 \) is a one modulo three mean graph if \( n \) is even. Conversely, assume that \( n \equiv 1 \pmod{2} \) and take \( n = 2k + 1 \). Then \( |V(G)| = 4k + 4 \) and \( |E(G)| = 6k + 4 \). Let \( \phi \) be a one modulo three mean labeling of \( K_{1,n} \times P_2 \), then 0, 1, 18k + 10 and 18k + 9 must be the vertex labels of one modulo three mean graphs. If \( f(u) = 0, f(v) = 1, f(u_i) = 18k + 10, f(v_i) = 18k + 9(1 \leq i \leq n) \), then the induced edge label of \( uu_i \) and \( vv_i \) get the same label \( 9k + 5 \) which is not possible. Also this is a
contradiction to the fact that the edge labels are congruent to one modulo three. If $f(u) = 0, f(v) = 18k + 9$ (or) $f(u) = 1, f(v) = 18k + 10$ (or) $f(v) = 0, f(u) = 1, f(u) = 18k + 9$ (or) $f(u) = 18k + 10, f(v) = 1, f(u) = 0 (1 \leq i \leq n)$, then the induced edge label is $9k + 5$ which is not possible. This is a contradiction to the fact that the edge labels are congruent to one modulo three. Therefore, 0 and $18k + 10$ cannot be the labels of the adjacent vertices. Hence the book graph $K_{1,n} \times P_2$ is not a one modulo three mean graph if $n$ is odd.

**Theorem 18.** The $n^{th}$ alternate quadrilateral snake $NA(Q_m)$ is a one modulo three mean graph if $n$ is even.

**Proof.** By Theorem 17, $NA(Q_2)$ is a one modulo three mean graph. Let $G_i = NA(Q_2)$ for $1 \leq i \leq m - 1$. Since each $G_i$ has $3n + 1$ edges, by Theorem 1, $NA(Q_m)$ admits one modulo three mean labeling.

An example for the one modulo three mean labeling of $4A(Q_4)$ is shown in Figure 3.

![Figure 3](image)

**Theorem 19.** The splitting graph $S'(P_n)$ is a one modulo three mean graph if $n$ is even.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices of $P_n$ and $v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n$ be the vertices of $S'(P_n)$. Here $S'(P_n)$ has $2n$ vertices and $3(n - 1)$ edges. Define a vertex labeling $\phi : V(S'(P_n)) \to \{0, 1, 3, \ldots, 9n - 11\}$ as follows.

$$f(v_i) = \begin{cases} 9(i - 1) & \text{if } i \text{ is odd} \\ 9i - 11 & \text{if } i \text{ is even} \end{cases}$$

$$f(v'_i) = \begin{cases} 9i - 3 & \text{if } i \text{ is odd} \\ 9i - 17 & \text{if } i \text{ is even} \end{cases}$$

It can be verified that the induced edge labels of $S'(P_n)$ are $1, 4, \ldots, 9n - 11$. Clearly $\phi$ is a one modulo three mean labeling of $S'(P_n)$. Hence $S'(P_n)$ is a one modulo three mean graph if $n$ is even.
An example for the one modulo three mean labeling of splitting graph of a path $S'(P_6)$ is shown in Figure 4.

![Figure 4. An example for the one modulo three mean labeling of splitting graph of a path $S'(P_6)$](image)

**Theorem 20.** The graph obtained by duplicating an arbitrary vertex of a cycle $D(C_n, v')$ admits one modulo three mean labeling if $n \equiv 1, 3(\text{mod } 4)$.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle $C_n$. Let $G = D(C_n, v')$ be the graph obtained by duplicating an arbitrary vertex $v$ of $C_n$. Without loss of generality take $v = v_n$ and the duplication of $v_n$ be $v'_n$. Hence $|V(D(C_n, v'_n))| = n + 1$ and $|E(D(C_n, v'_n))| = n + 2$. Define a vertex labeling $\phi : V(G) \rightarrow \{0, 1, 3, \ldots, 3n + 4\}$ by considering the following two cases.

**Case (i).** $n \equiv 1(\text{mod } 4)$.

- $f(v_1) = 1$, $f(v_n) = 13$, $f(v'_n) = 7$ and
- $f(v_{n-2i+1}) = \begin{cases} 3n - 6i + 9 & \text{if } 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor \\ 3n - 6i - 3 & \text{if } \left\lceil \frac{n}{4} \right\rceil \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor , \end{cases}$
- $f(v_{n-2i}) = 3n - 6i + 10$ if $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$.

**Case (ii).** $n \equiv 3(\text{mod } 4)$.

- $f(v_1) = 3n + 4$, $f(v'_n) = 3n + 3$.
- $f(v_{2i}) = \begin{cases} 6i - 1 & \text{if } 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor \\ 6i & \text{if } \left\lceil \frac{n}{4} \right\rceil + 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor , \end{cases}$
- $f(v_{2i+1}) = \begin{cases} 6i - 5 & \text{if } 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor \\ 6i & \text{if } \left\lceil \frac{n}{4} \right\rceil + 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor . \end{cases}$

It can be verified that the induced edge labels of $D(C_n, v')$ are $1, 4, \ldots, 3n + 4$. Clearly $\phi$ is a one modulo three mean labeling of $D(C_n, v')$. Hence $D(C_n, v')$ is one modulo three mean graph if $n \equiv 1, 3(\text{mod } 4)$.

An example for the one modulo three mean labeling of duplicating an arbitrary vertex of a cycle $D(C_{11}, v')$ is shown in Figure 5.
Theorem 21. The graph obtained by duplicating an arbitrary edge in cycle $D(C_n, e')$ is a one modulo three mean graph if $n \equiv 0, 2(\text{mod} 4)$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle $C_n$. Let $G = D(C_n, e')$ be a graph obtained by duplicating an arbitrary edge $e$ of $C_n$. Without loss of generality take $e = v_1v_2$ and the duplication of $e$ be edge is $e' = v'_1v'_2$. Hence $|V(D(C_n, e'))| = n + 2$ and $|E(D(C_n, e'))| = n + 3$. Define a vertex labeling $\phi : V(G) \to \{0, 1, 3, 4, \ldots, 3n + 7\}$ by considering the following two cases.

Case (i). $n \equiv 2(\text{mod} 4)$.

$$
\begin{align*}
  f(v_1) &= \begin{cases} 12 & \text{if } n = 6 \\ 19 & \text{if } n > 6, \end{cases} \\
  f(v_2) &= 7, \\
  f(v'_1) &= 3n + 6, \\
  f(v'_2) &= 1, \\
  f(v_{2i+1}) &= 6(i - 1) \text{ if } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
  f(v_{2i+2}) &= \begin{cases} 3n & \text{if } n = 6 \\ \frac{3n}{2} + 3 & \text{if } n > 6, \end{cases} \\
  f(v_{2i+1}) &= 6i + 6 \text{ if } \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\
  f(v_n - 2i) &= \begin{cases} 3n - 6i + 13 & \text{if } 1 \leq i \leq \frac{n}{2} - 2 \\ 13 & \text{if } i = \frac{n}{2} - 1. \end{cases}
\end{align*}
$$

Case (ii). $n \equiv 0(\text{mod} 4)$.

$$
\begin{align*}
  f(v_1) &= 12, f(v_2) = 1, f(v'_1) = 3n + 6, f(v'_2) = 7, \\
  f(v_{2i+1}) &= \begin{cases} 6(i - 1) & \text{if } 1 \leq i \leq \frac{n}{4} \\ 6i + 6 & \text{if } \frac{n}{4} + 1 \leq i \leq \frac{2n}{4} - 1 \end{cases}, \\
  f(v_n - 2i) &= 3n - 6i + 13 \text{ if } 1 \leq i \leq \frac{n}{2} - 1. \end{align*}
$$

It can be verified that the induced edge labels of $D(C_n, e')$ are $1, 4, \ldots, 3n + 4$. Hence $\phi$ is a one modulo three mean labeling of $D(C_n, e')$. Hence $D(C_n, e')$ is a one modulo three mean graph if $n \equiv 0, 2(\text{mod} 4)$.

An example for the one modulo three mean labeling of duplicating an arbitrary edge of a cycle $D(C_{10}, e')$ is shown in Figure 6.
References


