International Journal of Pure and Applied Mathematics
Volume 103 No. 4 2015, 653-666
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v103i4.5

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BIOPERATIONS ON α-OPEN SETS IN TOPOLOGICAL SPACES

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Abstract: In this paper, we introduce the concept of $\alpha_{[\gamma,\gamma']}$-open sets in topological spaces and study some of their properties. Furthermore, we offer a new class of functions called ($\alpha_{[\gamma,\gamma']}$, $\alpha_{[\beta,\beta']}$)-continuous functions and investigate their fundamental properties.

AMS Subject Classification: 54A05, 54A10
Key Words: bioperation, α-open set, $\alpha_{[\gamma,\gamma']}$-open set

1. Introduction

Njastad [3] introduced α-open sets in a topological space and studied some of its properties. Kasahara [2] defined the concept of an operation on topological spaces and introduced α-closed graphs of an operation. Ogata [4] called the operation α as γ operation and introduced the notion of τ_γ which is the collection

Received: April 3, 2015

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of all γ-open sets in a topological space (X, τ). H. Z. Ibrahim [6] defined the concept of an operation on $\alpha O(X, \tau)$ and introduced α_γ-open sets in topological spaces and studied some of their basic properties. In this paper, we have introduced and studied the notion of $\alpha O(X, \tau)_{[\gamma,\gamma']}$ which is the collection of all $\alpha_{[\gamma,\gamma']}$-open sets by using operations γ and $\gamma^{'}$ on $\alpha O(X, \tau)$. We also introduced the class of $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-continuous functions and investogated some of its important properties.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) represent non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A of a topological space (X, τ) is said to be α-open [3] if $A \subseteq Int(Cl(Int(A)))$. The complement of an α-open set is said to be α-closed. The intersection of all α-closed sets containing A is called the α-closure of A and is denoted by $\alpha Cl(A)$. The family of all α-open (resp. α-closed) sets in a topological space (X, τ) is denoted by $\alpha O(X, \tau)$ (resp. $\alpha C(X, \tau)$). An operation γ [2] on a topology τ is a mapping from τ into power set $P(X)$ of X such that $V \subseteq V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V. A subset A of X with an operation γ on τ is called γ-open [4] if for each $x \in A$, there exists an open set U of X containing x such that $U^\gamma \subseteq A$. Clearly $\tau_\gamma \subseteq \tau$. Complements of γ-open sets are called γ-closed. An operation $\gamma : \alpha O(X, \tau) \to P(X)$ [6] is a mapping satisfying the condition, $V \subseteq V^\gamma$ for each $V \in \alpha O(X, \tau)$. We call the mapping γ an operation on $\alpha O(X, \tau)$. A subset A of X is called an α_γ-open set [6] if for each point $x \in A$, there exists an α-open set U of X containing x such that $U^\gamma \subseteq A$. We denote the set of all α_γ-open sets of (X, τ) by $\alpha O(X, \tau)_\gamma$. An operation γ on $\alpha O(X, \tau)$ is said to be α-regular [6] if for every α-open sets U and V containing $x \in X$, there exists an α-open set W of x such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$. An operation γ on $\alpha O(X, \tau)$ is said to be α-open [6] if for every α-open set U of $x \in X$, there, exists an α_γ-open set V of X such that $x \in V$ and $V \subseteq U^\gamma$.

3. $\alpha_{[\gamma,\gamma']}$-Open Sets

Definition 3.1. A non-empty subset A of (X, τ) is said to be $\alpha_{[\gamma,\gamma']}$-open if for each $x \in A$ there exist α-open sets U and V of X containing x such that $U^{\gamma} \cap V^{\gamma'} \subseteq A$. The set of all $\alpha_{[\gamma,\gamma']}$-open sets of (X, τ) is denoted by $\alpha O(X, \tau)_{[\gamma,\gamma']}$. We suppose that the empty set is $\alpha_{[\gamma,\gamma']}$-open for any operations γ and γ'.

Proposition 3.2. *If A_i is $\alpha_{[\gamma,\gamma']}$-open for every $i \in I$, then $\cup\{A_i : i \in I\}$ is $\alpha_{[\gamma,\gamma']}$-open.*

Proof. Let $x \in \cup_{i\in I} A_i$, then $x \in A_i$ for some $i \in I$. Since A_i is an $\alpha_{[\gamma,\gamma']}$-open set, so there exist α-open sets U and V of (X, τ) such that $x \in U \cap V \subseteq U^{\gamma} \cap V^{\gamma'} \subseteq A_i \subseteq \cup_{i\in I} A_i$. Therefore $\cup_{i\in I} A_i$ is an $\alpha_{[\gamma,\gamma']}$-open set of (X, τ). □

If A and B are two $\alpha_{[\gamma,\gamma']}$-open sets in (X, τ), then the following example shows that $A \cap B$ need not be $\alpha_{[\gamma,\gamma']}$-open.

Example 3.3. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$ be a topology on X. For each $A \in \alpha O(X, \tau)$, we define two operations γ and γ', by

$$A^{\gamma} = \begin{cases} Cl(A) & \text{if } c \in A, \\ X & \text{if } c \notin A, \end{cases}$$

and

$$A^{\gamma'} = \begin{cases} A & \text{if } A \neq \{b\}, \\ X & \text{if } A = \{b\}. \end{cases}$$

Then, it is obvious that the sets $\{a, b\}$ and $\{b, c\}$ are $\alpha_{[\gamma,\gamma']}$-open, however their intersection $\{b\}$ is not $\alpha_{[\gamma,\gamma']}$-open.

A subfamily τ of the power set $P(X)$ of a non-empty set X is called a supratopology [1] on X if τ satisfies the following conditions:
1. τ contains ϕ and X,
2. τ is closed under the arbitrary union.
The pair (X, τ) is called a supratopological space.

From the above example we notice that the family of all $\alpha_{[\gamma,\gamma']}$-open subsets of a space X is a supratopology and need not be a topology in general.

In the following proposition, the intersection of two $\alpha_{[\gamma,\gamma']}$-open sets is also $\alpha_{[\gamma,\gamma']}$-open.

Proposition 3.4. *Let γ and γ' be α-regular operations. If A and B are $\alpha_{[\gamma,\gamma']}$-open, then $A \cap B$ is $\alpha_{[\gamma,\gamma']}$-open.*

Proof. Let $x \in A \cap B$, then $x \in A$ and $x \in B$. Since A and B are $\alpha_{[\gamma,\gamma']}$-open sets, there exist α-open sets U, V, W and S such that $x \in U \cap V \subseteq U^{\gamma} \cap V^{\gamma'} \subseteq A$ and $x \in W \cap S \subseteq W^{\gamma} \cap S^{\gamma'} \subseteq B$. Since γ and γ' are α-regular operations, then there exist an α-open sets K and L containing x such that $K^{\gamma} \cap L^{\gamma^{'}} \subseteq (U^{\gamma} \cap W^{\gamma}) \cap (V^{\gamma^{'}} \cap S^{\gamma^{'}}) = (U^{\gamma} \cap V^{\gamma^{'}}) \cap (W^{\gamma} \cap S^{\gamma^{'}}) \subseteq A \cap B$. This implies that $A \cap B$ is $\alpha_{[\gamma,\gamma']}$-open set. □

Remark 3.5. By the above propositon, if γ and $\gamma^{'}$ are α-regular operations, then $\alpha O(X,\tau)_{[\gamma,\gamma']}$ forms a topology on X.

Proposition 3.6. *The set A is $\alpha_{[\gamma,\gamma']}$-open in X if and only if for each $x \in A$, there exists an $\alpha_{[\gamma,\gamma']}$-open set B such that $x \in B \subseteq A$.*

Proof. Suppose that A is an $\alpha_{[\gamma,\gamma']}$-open set in the space X. Then for each $x \in A$, put $B = A$ which is an $\alpha_{[\gamma,\gamma']}$-open set such that $x \in B \subseteq A$. Conversely, suppose that for each $x \in A$, there exists an $\alpha_{[\gamma,\gamma']}$-open set B such that $x \in B \subseteq A$. Thus $A = \cup_{x\in A} B_x$, where $B_x \in \alpha O(X,\tau)_{[\gamma,\gamma']}$. Therefore, A is an $\alpha_{[\gamma,\gamma']}$-open set. □

Proposition 3.7. *If A is $\alpha_{[\gamma,\gamma']}$-open, then A is α-open.*

The converse of the above proposition need not be true in general as it is shown below.

Example 3.8. Let (X,τ), γ and $\gamma^{'}$ be the same space and the same operations as in Example 3.3. Then $\{b\}$ is α-open but not $\alpha_{[\gamma,\gamma']}$-open.

Remark 3.9. A subset A is an $\alpha_{[id,id']}$-open set of (X,τ) if and only if A is α-open in (X,τ). The operation $id : \alpha O(X,\tau) \to P(X)$ is defined by $V^{id} = V$ for any set $V \in \alpha O(X,\tau)$. This operation is called the identity operation on $\alpha O(X,\tau)$. Therefore $\alpha O(X,\tau)_{[id,id']} = \alpha O(X,\tau)$.

Remark 3.10. From Remark 3.9 and [[6]; Remark 2.3] we have $\alpha O(X,\tau)_{[id,id']} = \alpha O(X,\tau) = \alpha O(X,\tau)_{id} = \alpha O(X,\tau)_{id'}$.

Remark 3.11. The following example shows that the concept of $\alpha_{[\gamma,\gamma']}$-open and open are independent.

Example 3.12. Let $X = \{a,b,c\}$ and $\tau = \{\phi, X, \{a\}\}$ be a topology on X. For each $A \in \alpha O(X,\tau)$ we define two operations γ and $\gamma^{'}$, respectively, by $A^{\gamma} = X$ and

$$A^{\gamma^{'}} = \begin{cases} A & \text{if } A = \{a,b\}, \\ X & \text{if } A \neq \{a,b\}. \end{cases}$$

Then, $\alpha_{[\gamma,\gamma']}$-open sets are ϕ, X, and $\{a,b\}$.

Definition 3.13. [5] A non-empty subset A of (X, τ) is said to be $[\gamma, \gamma^{'}]$-open if for each $x \in A$ there exist open sets U and V of X containing x such that $U^{\gamma} \cap V^{\gamma^{'}} \subseteq A$.

Proposition 3.14. *If A is $[\gamma, \gamma^{'}]$-open, then A is $\alpha_{[\gamma,\gamma']}$-open.*

The converse of the above proposition need not be true in general as it is shown below.

Example 3.15. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{b\}\}$ be a topology on X. For each $A \in \alpha O(X, \tau)$, we define two operations γ and $\gamma^{'}$, respectively, by $A^{\gamma} = A^{\gamma^{'}} = A$. Then $\{a, b\}$ is $\alpha_{[\gamma,\gamma']}$-open but not $[\gamma, \gamma^{'}]$-open.

Remark 3.16. If A is $(\gamma, \gamma^{'})$-open [7], then A is $\alpha_{[\gamma,\gamma']}$-open.

Proposition 3.17. *If A is α_{γ}-open and B is $\alpha_{\gamma'}$-open, then $A \cap B$ is $\alpha_{[\gamma,\gamma']}$-open.*

Proposition 3.18. *If A is α_{γ}-open, then A is $\alpha_{[\gamma,\gamma']}$-open for any operation $\gamma^{'}$.*

The converse of Proposition 3.18 need not be true in general as it is shown below.

Example 3.19. Let $X = \{a, b, c\}$ and τ be a discrete topology on X. For each $A \in \alpha O(X, \tau)$ we define two operations γ and $\gamma^{'}$, respectively, by

$$A^{\gamma} = A^{\gamma^{'}} = \begin{cases} A & \text{if } A = \{a, b\} \text{ or } \{b, c\}, \\ X & \text{otherwise.} \end{cases}$$

Then $\{b\}$ is $\alpha_{[\gamma,\gamma']}$-open but not α_{γ}-open.

Remark 3.20. If A is γ-open, then A is $\alpha_{[\gamma,\gamma']}$-open for any operation $\gamma^{'}$.

Proposition 3.21. *Let $X : \alpha O(X, \tau) \to P(X)$ be an operation defined by $U^X = X$ for every $U \in \alpha O(X, \tau)$. Then A is α_{γ}-open if and only if A is $\alpha_{[\gamma,X]}$-open.*

Definition 3.22. A topological space (X, τ) is said to be α_{γ}-regular if for each $x \in X$ and for each α-open set V in X containing x, there exists an α-open set U in X containing x such that $U^{\gamma} \subseteq V$.

In the following proposition, we give a condition under which the family of α-open sets is identical to the family of α_{γ}-open sets.

Proposition 3.23. *A topological space (X, τ) with an operation γ on $\alpha O(X, \tau)$ is α_{γ}-regular if and only if $\alpha O(X, \tau) = \alpha O(X, \tau)_{\gamma}$.*

Remark 3.24. If a topological space (X, τ) is α_γ-regular, then $\tau \subseteq \alpha O(X,\tau)_\gamma$.

Definition 3.25. A topological space (X, τ) is said to be $\alpha_{[\gamma,\gamma']}$-regular if for each point x in X and every α-open set U containing x there exist α-open sets W and S of x such that $W^\gamma \cap S^{\gamma'} \subseteq U$.

Proposition 3.26. *A topological space (X, τ) with operations γ and γ' on $\alpha O(X, \tau)$ is $\alpha_{[\gamma,\gamma']}$-regular if and only if $\alpha O(X,\tau) = \alpha O(X,\tau)_{[\gamma,\gamma']}$.*

Proof. Let (X, τ) be $\alpha_{[\gamma,\gamma']}$-regular and $A \in \alpha O(X, \tau)$. Since (X, τ) is $\alpha_{[\gamma,\gamma']}$-regular, then for each $x \in A$, there exist α-open sets W and S of x such that $W^\gamma \cap S^{\gamma'} \subseteq A$. This implies that $A \in \alpha O(X,\tau)_{[\gamma,\gamma']}$. But we have $\alpha O(X,\tau)_{[\gamma,\gamma']} \subseteq \alpha O(X,\tau)$. Therefore $\alpha O(X,\tau) = \alpha O(X,\tau)_{[\gamma,\gamma']}$.
Conversely, let $\alpha O(X,\tau) = \alpha O(X,\tau)_{[\gamma,\gamma']}$, $x \in X$ and V be α-open of x. Then by assumption V is $\alpha_{[\gamma,\gamma']}$-open set. This implies that there exist α-open sets W and S of x such that $W^\gamma \cap S^{\gamma'} \subseteq V$. Therefore (X, τ) is $\alpha_{[\gamma,\gamma']}$-regular. □

Remark 3.27. If a space X is $\alpha_{[\gamma,\gamma']}$-regular, then $\tau \subseteq \alpha O(X,\tau)_{[\gamma,\gamma']}$.

Proposition 3.28. *For α_γ-regularlity, $\alpha_{\gamma'}$-regularlity and $\alpha_{[\gamma,\gamma']}$-regularlity of a space (X, τ), the following properties hold.*

1. *(X, τ) is $\alpha_{[\gamma,X]}$-regular if and only if it is α_γ-regular.*
2. *If (X, τ) is α_γ-regular and $\alpha_{\gamma'}$-regular, then it is $\alpha_{[\gamma,\gamma']}$-regular.*

Definition 3.29. A subset F of (X, τ) is said to be $\alpha_{[\gamma,\gamma']}$-closed if its complement $X \setminus F$ is $\alpha_{[\gamma,\gamma']}$-open.
We denote the set of all $\alpha_{[\gamma,\gamma']}$-closed sets of (X, τ) by $\alpha C(X,\tau)_{[\gamma,\gamma']}$.

Definition 3.30. Let A be a subset of a topological space (X, τ). The intersection of all $\alpha_{[\gamma,\gamma']}$-closed sets containing A is called the $\alpha_{[\gamma,\gamma']}$-closure of A and denoted by $\alpha_{[\gamma,\gamma']}\text{-}Cl(A)$.

Proposition 3.31. *For a point $x \in X$, $x \in \alpha_{[\gamma,\gamma']}\text{-}Cl(A)$ if and only if $V \cap A \neq \phi$ for every $\alpha_{[\gamma,\gamma']}$-open set V containing x.*

Proposition 3.32. *Let A and B be subsets of (X, τ). Then the following hold:*

1. *$A \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(A)$.*
2. *If $A \subseteq B$, then $\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(B)$.*

3. *$A \in \alpha C(X,\tau)_{[\gamma,\gamma']}$ if and only if $\alpha_{[\gamma,\gamma']}$-$Cl(A) = A$.*
4. *$\alpha_{[\gamma,\gamma']}$-$Cl(A) \in \alpha C(X,\tau)_{[\gamma,\gamma']}$.*
5. *$\alpha_{[\gamma,\gamma']}$-$Cl(A \cap B) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(A) \cap \alpha_{[\gamma,\gamma']}$-$Cl(B)$.*
6. *If γ and γ' are α-regular, then $\alpha_{[\gamma,\gamma']}$-$Cl(A \cup B) = \alpha_{[\gamma,\gamma']}$-$Cl(A) \cup \alpha_{[\gamma,\gamma']}$-$Cl(B)$.*

We introduce the following definition of $\alpha Cl_{[\gamma,\gamma']}(A)$.

Definition 3.33. For a subset A of (X,τ), we define $\alpha Cl_{[\gamma,\gamma']}(A)$ as follows: $\alpha Cl_{[\gamma,\gamma']}(A) = \{x \in X : (U^{\gamma} \cap W^{\gamma'}) \cap A \neq \phi$ holds for every α-open sets U and W containing $x\}$.

Remark 3.34. In Definitions 3.29, 3.30 and 3.33, put $\gamma' = X$. Then, for any subset A of X, the following hold:

1. $\alpha_{[\gamma,X]}$-$Cl(A) = \alpha_{\gamma}$-$Cl(A)$.
2. $\alpha C(X,\tau)_{[\gamma,X]} = \{F : F$ is α_{γ}-closed$\}$.
3. $\alpha Cl_{[\gamma,X]}(A) = \alpha Cl_{\gamma}(A)$.

Proposition 3.35. *For a subset A of (X,τ), we have $A \subseteq \alpha Cl(A) \subseteq \alpha Cl_{[\gamma,\gamma']}(A) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(A)$.*

Theorem 3.36. *Let A be a subset of a topological space (X,τ), the following properties are equivalent:*

1. *$A \in \alpha O(X,\tau)_{[\gamma,\gamma']}$.*
2. *$\alpha Cl_{[\gamma,\gamma']}(X \setminus A) = X \setminus A$.*
3. *$\alpha_{[\gamma,\gamma']}$-$Cl(X \setminus A) = X \setminus A$.*
4. *$X \setminus A \in \alpha C(X,\tau)_{[\gamma,\gamma']}$.*

Theorem 3.37. *For a subset A of (X,τ), the following properties hold:*

1. *If (X,τ) is $\alpha_{[\gamma,\gamma']}$-regular, then $\alpha Cl(A) = \alpha Cl_{[\gamma,\gamma']}(A) = \alpha_{[\gamma,\gamma']}$-$Cl(A)$.*
2. *$\alpha Cl_{[\gamma,\gamma']}(A)$ is an α-closed subset of (X,τ).*

Theorem 3.38. *Let γ and $\gamma^{'}$ be α-open operations and A a subset of (X,τ). Then, the following statements hold:*

1. $\alpha Cl_{[\gamma,\gamma']}(A) = \alpha_{[\gamma,\gamma']}\text{-}Cl(A)$.
2. $\alpha Cl_{[\gamma,\gamma']}(\alpha Cl_{[\gamma,\gamma']}(A)) = \alpha Cl_{[\gamma,\gamma']}(A)$.

Proof. 1. By Proposition 3.35, it suffices to prove that $\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \subseteq \alpha Cl_{[\gamma,\gamma']}(A)$. Let $x \in \alpha_{[\gamma,\gamma']}\text{-}Cl(A)$ and W and S be α-open sets of X containing x. By the α-openness of γ and $\gamma^{'}$, there exist an α_γ-open set $W^{'}$ and an $\alpha_{\gamma'}$-open set $S^{'}$ such that $x \in W^{'} \subseteq W^\gamma$ and $x \in S^{'} \subseteq S^{\gamma^{'}}$. By Propositions 3.17 and 3.31, $(S^{'} \cap W^{'}) \cap A \neq \phi$ and hence $(S^\gamma \cap W^{\gamma^{'}}) \cap A \neq \phi$. This implies that $x \in \alpha Cl_{[\gamma,\gamma']}(A)$.

2. This follows immediately from (1) and Proposition 3.32 (3).

□

Remark 3.39. The below example shows that the equalities of Theorem 3.38 are not true without the assumption that both operations are α-open.

Example 3.40. Let $X = \{a,b,c\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$. For each $A \in \alpha O(X,\tau)$ we define two operations γ and $\gamma^{'}$, respectively, by $A^\gamma = Cl(A)$ and $A^{\gamma^{'}} = X$. The operation γ is not α-open. However, $\alpha Cl_{[\gamma,\gamma']}\{a\} = \{a,c\} \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(\{a\}) = X$ and $\alpha Cl_{[\gamma,\gamma']}(\alpha Cl_{[\gamma,\gamma']}(\{a\})) = X \neq \{a,c\} = \alpha Cl_{[\gamma,\gamma']}(\{a\})$.

Theorem 3.41. *Let A and B be subsets of a topological space (X,τ) and $\gamma, \gamma^{'} : \alpha O(X,\tau) \to P(X)$ operations on $\alpha O(X,\tau)$. Then we have the following properties:*

1. $A \subseteq \alpha Cl_{[\gamma,\gamma']}(A)$.
2. $\alpha Cl_{[\gamma,\gamma']}(\phi) = \phi$ *and* $\alpha Cl_{[\gamma,\gamma']}(X) = X$.
3. $A \in \alpha C(X,\tau)_{[\gamma,\gamma']}$ *if and only if* $\alpha Cl_{[\gamma,\gamma']}(A) = A$.
4. *If* $A \subseteq B$, *then* $\alpha Cl_{[\gamma,\gamma']}(A) \subseteq \alpha Cl_{[\gamma,\gamma']}(B)$.
5. $\alpha Cl_{[\gamma,\gamma']}(A \cup B) \subseteq \alpha Cl_\gamma(A) \cup Cl_{\alpha_{\gamma'}}(B)$.
6. *If* γ *and* $\gamma^{'}$ *are* α*-regular, then* $\alpha Cl_{[\gamma,\gamma']}(A \cup B) = \alpha Cl_{[\gamma,\gamma']}(A) \cup \alpha Cl_{[\gamma,\gamma']}(B)$.

7. $\alpha Cl_{[\gamma,\gamma']}(A\cap B)\subseteq \alpha Cl_{[\gamma,\gamma']}(A)\cap \alpha Cl_{[\gamma,\gamma']}(B)$.

Proof. (1), (2) and (4). They are obtained from Definition 3.33.
(3). Suppose that A is $\alpha_{[\gamma,\gamma']}$-closed. Then $X\setminus A$ is $\alpha_{[\gamma,\gamma']}$-open in (X,τ). We claim that $\alpha Cl_{[\gamma,\gamma']}(A)\subseteq A$. Let $x\notin A$. There exist α-open sets U and V of (X,τ) containing x such that $U^\gamma\cap V^{\gamma'}\subseteq X\setminus A$, that is, $(U^\gamma\cap V^{\gamma'})\cap A=\phi$. Hence by Definition 3.33, we have that $x\notin \alpha Cl_{[\gamma,\gamma']}(A)$ and so $\alpha Cl_{[\gamma,\gamma']}(A)\subseteq A$. By (1), it is proved that $\alpha Cl_{[\gamma,\gamma']}(A)=A$. Conversely, suppose that $\alpha Cl_{[\gamma,\gamma']}(A)=A$. Let $x\in X\setminus A$. Since $x\notin \alpha Cl_{[\gamma,\gamma']}(A)$, there exist an α-open sets U and V containing x such that $(U^\gamma\cap V^{\gamma'})\cap A=\phi$, that is, $U^\gamma\cap V^{\gamma'}\subseteq X\setminus A$. Therefore, A is $\alpha_{[\gamma,\gamma']}$-closed.
(5), (7). They are obtained from (4).
(6). Let $x\notin \alpha Cl_{[\gamma,\gamma']}(A)\cup \alpha Cl_{[\gamma,\gamma']}(B)$. Then there exist α-open sets U,V,W and S of (X,τ) containing x such that $(U^\gamma\cap V^{\gamma'})\cap A=\phi$ and $(W^\gamma\cap S^{\gamma'})\cap B=\phi$. Since γ and γ' are α-regular, by definition of α-regular, there exist α-open sets K and L of (X,τ) containing x such that $k^\gamma\subseteq U^\gamma\cap W^\gamma$ and $L^{\gamma'}\subseteq V^{\gamma'}\cap S^{\gamma'}$. Thus, we have $(k^\gamma\cap L^{\gamma'})\cap(A\cup B)\subseteq((U^\gamma\cap W^\gamma)\cap(V^{\gamma'}\cap S^{\gamma'}))\cap(A\cup B)=((U^\gamma\cap V^{\gamma'})\cap(W^\gamma\cap S^{\gamma'}))\cap(A\cup B)=[((U^\gamma\cap V^{\gamma'})\cap(W^\gamma\cap S^{\gamma'}))\cap A]\cup[((U^\gamma\cap V^{\gamma'})\cap(W^\gamma\cap S^{\gamma'}))\cap B]=\phi$, that is, $(k^\gamma\cap L^{\gamma'})\cap(A\cup B)=\phi$. Hence, $x\notin \alpha Cl_{[\gamma,\gamma']}(A\cup B)$. This shows that $\alpha Cl_{[\gamma,\gamma']}(A)\cup \alpha Cl_{[\gamma,\gamma']}(B)\supseteq \alpha Cl_{[\gamma,\gamma']}(A\cup B)$. □

Remark 3.42. Example 3.3 shows that the inclusion of Theorem 3.41 (5) is a proper one in general. For a subset $\{a\}$, $\alpha Cl_{[\gamma,\gamma']}(\{a\})=\{a\}\subseteq \alpha Cl_\gamma(\{a\})\cup \alpha Cl_{\gamma'}(\{a\})=\{a,b\}$.

We define the $\alpha_{[\gamma,\gamma']}$-interior of a subset A of (X,τ) as follows:

Definition 3.43. Let A be a subset of a topological space (X,τ). The union of all $\alpha_{[\gamma,\gamma']}$-open sets contained in A is called the $\alpha_{[\gamma,\gamma']}$-interior of A and is denoted by $\alpha_{[\gamma,\gamma']}$-$Int(A)$.

Proposition 3.44. *For any subsets A,B of X, we have the following:*

1. *$\alpha_{[\gamma,\gamma']}$-$Int(A)$ is an $\alpha_{[\gamma,\gamma']}$-open set in X.*
2. *A is $\alpha_{[\gamma,\gamma']}$-open if and only if $A=\alpha_{[\gamma,\gamma']}$-$Int(A)$.*
3. *$\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Int(A))=\alpha_{[\gamma,\gamma']}$-$Int(A)$.*
4. *$\alpha_{[\gamma,\gamma']}$-$Int(A)\subseteq A$.*

5. *If* $A \subseteq B$, *then* $\alpha_{[\gamma,\gamma']}$-$Int(A) \subseteq \alpha_{[\gamma,\gamma']}$-$Int(B)$.

6. $\alpha_{[\gamma,\gamma']}$-$Int(A \cup B) \supseteq \alpha_{[\gamma,\gamma']}$-$Int(A) \cup \alpha_{[\gamma,\gamma']}$-$Int(B)$.

7. $\alpha_{[\gamma,\gamma']}$-$Int(A \cap B) \subseteq \alpha_{[\gamma,\gamma']}$-$Int(A) \cap \alpha_{[\gamma,\gamma']}$-$Int(B)$.

Proof. Obvious. □

Proposition 3.45. *Let* A *be any subset of a topological space* (X, τ). *Then the following statements are true:*

1. $X \setminus \alpha_{[\gamma,\gamma']}$-$Int(A) = \alpha_{[\gamma,\gamma']}$-$Cl(X \setminus A)$.

2. $X \setminus \alpha_{[\gamma,\gamma']}$-$Cl(A) = \alpha_{[\gamma,\gamma']}$-$Int(X \setminus A)$.

3. $\alpha_{[\gamma,\gamma']}$-$Int(A) = X \setminus \alpha_{[\gamma,\gamma']}$-$Cl(X \setminus A)$.

4. $\alpha_{[\gamma,\gamma']}$-$Cl(A) = X \setminus \alpha_{[\gamma,\gamma']}$-$Int(X \setminus A)$.

4. $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-Functions

Throughout this section, let $f : (X, \tau) \to (Y, \sigma)$ be a function and $\gamma, \gamma^{'} : \alpha O(X, \tau) \to P(X)$ be operations on $\alpha O(X, \tau)$ and $\beta, \beta^{'} : \alpha O(Y, \sigma) \to P(Y)$ be operations on $\alpha O(Y, \sigma)$.

Definition 4.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $(\alpha_{[\gamma,\gamma']}$, $\alpha_{[\beta,\beta']})$-continuous if for each point $x \in X$ and each α-open sets W and S of (Y, σ) containing $f(x)$ there exist α-open sets U and V of (X, τ) containing x such that $f(U^{\gamma} \cap V^{\gamma^{'}}) \subseteq W^{\beta} \cap S^{\beta'}$.

Theorem 4.2. *Let* $(1), (2), (3), (4), (5), (6)$ *and* (7) *be the following properties for a function* $f : (X, \tau) \to (Y, \sigma)$.

1. $f : (X, \tau) \to (Y, \sigma)$ *is* $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$*-continuous.*

2. $f(\alpha Cl_{[\gamma,\gamma']}(A)) \subseteq \alpha Cl_{[\beta,\beta']}(f(A))$ *for every subset* A *of* (X, τ).

3. $\alpha Cl_{[\gamma,\gamma']}(f^{-1}(B)) \subseteq f^{-1}(\alpha Cl_{[\beta,\beta']}(B))$ *for every subset* B *of* (Y, σ).

4. $f^{-1}(B)$ *is* $\alpha_{[\gamma,\gamma']}$*-closed for every* $\alpha_{[\beta,\beta']}$*-closed set* B *of* (Y, σ).

5. $f(\alpha_{[\gamma,\gamma']}$-$Cl(A)) \subseteq \alpha_{[\beta,\beta']}$-$Cl(f(A))$ *for every subset* A *of* (X, τ).

6. *$f^{-1}(V)$ is $\alpha_{[\gamma,\gamma']}$-open for every $\alpha_{[\beta,\beta']}$-open set V of (Y,σ).*

7. *for each point $x \in X$ and each $\alpha_{[\beta,\beta']}$-open W of (Y,σ) containing $f(x)$ there exist $\alpha_{[\gamma,\gamma']}$-open U of (X,τ) containing x such that $f(U) \subseteq W$.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) *hold.*

Proof. (1) $\Rightarrow$ (2). Let $x \in \alpha Cl_{[\gamma,\gamma']}(A)$ and W, S be α-open sets of (Y,σ) containing $f(x)$. There exist α-open sets U and V of (X,τ) containing x such that $f(U^{\gamma} \cap V^{\gamma'}) \subseteq W^{\beta} \cap S^{\beta'}$. Since $x \in \alpha Cl_{[\gamma,\gamma']}(A)$, then $(U^{\gamma} \cap V^{\gamma'}) \cap A \neq \phi$, implies that $f(U^{\gamma} \cap V^{\gamma'}) \cap f(A) \neq \phi$. Therefore, we have $f(A) \cap (W^{\beta} \cap S^{\beta'}) \neq \phi$. Therefore $f(x) \in \alpha Cl_{[\beta,\beta']}(f(A))$, which implies that $x \in f^{-1}(\alpha Cl_{[\beta,\beta']}(f(A)))$. Hence $\alpha Cl_{[\gamma,\gamma']}(A) \subseteq f^{-1}(\alpha Cl_{[\beta,\beta']}(f(A)))$, so that $f(\alpha Cl_{[\gamma,\gamma']}(A)) \subseteq \alpha Cl_{[\beta,\beta']}(f(A))$.
(2) $\Rightarrow$ (3). Let B be any subset of Y. Then $f^{-1}(B)$ is a subset of X. By (2), we have $f(\alpha Cl_{[\gamma,\gamma']}(f^{-1}(B))) \subseteq \alpha Cl_{[\beta,\beta']}(f(f^{-1}(B))) \subseteq \alpha Cl_{[\beta,\beta']}(B)$. Hence $\alpha Cl_{[\gamma,\gamma']}(f^{-1}(B)) \subseteq f^{-1}(\alpha Cl_{[\beta,\beta']}(B))$.
(3) $\Rightarrow$ (2). Let A be any subset of X. Then $f(A)$ is a subset of Y. By (3), we have $\alpha Cl_{[\gamma,\gamma']}(f^{-1}f((A))) \subseteq f^{-1}(\alpha Cl_{[\beta,\beta']}(f(A)))$. This implies that $\alpha Cl_{[\gamma,\gamma']}(A) \subseteq f^{-1}(\alpha Cl_{[\beta,\beta']}(f(A)))$. Hence $f(\alpha Cl_{[\gamma,\gamma']}(A)) \subseteq \alpha Cl_{[\beta,\beta']}(f(A))$.
(3) $\Rightarrow$ (4). Let B be an $\alpha_{[\beta,\beta']}$-closed set of (Y,σ). By (3) and Theorem 3.36, $\alpha Cl_{[\gamma,\gamma']}(f^{-1}(B)) \subseteq f^{-1}(B)$ and hence $f^{-1}(B)$ is $\alpha_{[\gamma,\gamma']}$-closed.
(4) $\Rightarrow$ (5). Let A be any subset of X. Then $f(A) \subseteq \alpha_{[\beta,\beta']}$-$Cl(f(A))$ and $\alpha_{[\beta,\beta']}$-$Cl(f(A))$ is an $\alpha_{[\beta,\beta']}$-closed set in Y. Hence $A \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$Cl(f(A)))$. By (4), we have $f^{-1}(\alpha_{[\beta,\beta']}$-$Cl(f(A)))$ which is an $\alpha_{[\gamma,\gamma']}$-closed set in X. Therefore, $\alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$Cl(f(A)))$. Hence $f(\alpha_{[\gamma,\gamma']}$-$Cl(A)) \subseteq \alpha_{[\beta,\beta']}$-$Cl(f(A))$.
(5) $\Rightarrow$ (4). Let B be an $\alpha_{[\beta,\beta']}$-closed set of (Y,σ). By (5), $\alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(B)) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$Cl(f(f^{-1}(B)))) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$Cl(B)) \subseteq f^{-1}(B)$. Therefore, by Proposition 3.32, $f^{-1}(B)$ is $\alpha_{[\gamma,\gamma']}$-closed.
(5) $\Leftrightarrow$ (6). This follows from Definition 3.29 and the equivalence of (4) $\Leftrightarrow$ (5).
(6) $\Rightarrow$ (7). Let W be any $\alpha_{[\beta,\beta']}$-open set in Y containing $f(x)$, so its inverse image is an $\alpha_{[\gamma,\gamma']}$-open set in X. Since $f(x) \in W$, then $x \in f^{-1}(W)$ and by hypothesis $f^{-1}(W)$ is an $\alpha_{[\gamma,\gamma']}$-open set in X containing x so that $f(f^{-1}(W)) \subseteq W$.
(7) $\Rightarrow$ (6). Let $V \in \alpha O(Y,\sigma)_{[\beta,\beta']}$. For each $x \in f^{-1}(V)$, by (7), there exists an $\alpha_{[\gamma,\gamma']}$-open set U_x containing x such that $f(U_x) \subseteq V$. Then we have $f^{-1}(V) =$

$\cup\{U_x \in \alpha O(X,\tau)_{[\gamma,\gamma']} : x \in f^{-1}(V)\}$ and hence $f^{-1}(V) \in \alpha O(X,\tau)_{[\gamma,\gamma']}$ using Proposition 3.2. □

Corollary 4.3. *If (Y,σ) is an $\alpha_{[\beta,\beta']}$-regular space, or operations β and β' are α-open on $\alpha O(Y,\sigma)$, then all properties of Theorem 4.2 are equivalent.*

Proof. By Theorem 4.2, it is sufficient to prove the implication (4) ⇒ (1), where (1) and (4) are the properties of Theorem 4.2.

First, we show the implication under the assumption that (Y,σ) is an $\alpha_{[\beta,\beta']}$-regular space. Let $x \in X$ and W, S be α-open sets of (Y,σ) containing $f(x)$. By Proposition 3.26, $Y \setminus (W \cap S)$ is $\alpha_{[\beta,\beta']}$-closed. Then, $f^{-1}(Y \setminus (W \cap S))$ is $\alpha_{[\gamma,\gamma']}$-closed by (4) and hence $f^{-1}(W \cap S)$ is $\alpha_{[\gamma,\gamma']}$-open set of (X,τ) containing x. Therefore, there exist α-open sets U and V of (X,τ) containing x such that $U^\gamma \cap V^{\gamma'} \subseteq f^{-1}(W \cap S)$ and so $f(U^\gamma \cap V^{\gamma'}) \subseteq W^\beta \cap S^{\beta'}$. This implies that f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-continuous.

Second, we suppose that the operations β and β' are α-open. Let $x \in X$ and W, S be α-open sets of (Y,σ) containing $f(x)$. By using α-openness of β and β', there exist an α_β-open set A and an $\alpha_{\beta'}$-open set B such that $f(x) \in A \cap B$ and $A \cap B \subseteq W^\beta \cap S^{\beta'}$. By Proposition 3.17, $Y \setminus (A \cap B)$ is $\alpha_{[\gamma,\gamma']}$-closed and hence $f^{-1}(Y \setminus (A \cap B))$ is $\alpha_{[\beta,\beta']}$-closed. Therefore, there exist α-open sets U and V of (X,τ) containing x such that $U^\gamma \cap V^{\gamma'} \subseteq f^{-1}(A \cap B)$ and so $f(U^\gamma \cap V^{\gamma'}) \subseteq W^\beta \cap S^{\beta'}$. This implies that f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-continuous. □

Remark 4.4. It is clear that the identity operation and the operation $X : \alpha O(X,\tau) \to (X,\tau)$ are α-open on $\alpha O(X,\tau)$. Therefore, by Corollary 4.3, if β and β' are chosen from mentioned operations above, then all properties of Theorem 4.2 are equivalent.

Remark 4.5. The converse of implication (1) ⇒ (4) in Theorem 4.2 is not true in general as shown by the following example.

Example 4.6. Let $X = \{1,2,3\}$ and $\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$ be a topology on X. Let $f : (X,\tau) \to (X,\tau)$ be the identity. Let $\gamma = \gamma' = \beta' = X : \alpha O(X,\tau) \to P(X)$ be the operations on $\alpha O(X,\tau)$ and β the closure operation on $\alpha O(X,\tau)$. Then, the condition (4) in Theorem 4.2 is true. It is shown that f is not $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-continuous.

Theorem 4.2 suggests the following.

Remark 4.7. If $f : (X,\tau) \to (Y,\sigma)$ is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-continuous, then the induced function $f : (X, \alpha O(X,\tau)_{[\gamma,\gamma']}) \to (Y, \alpha O(Y,\sigma)_{[\beta,\beta']})$ is continuous.

Remark 4.8. The converse of Remark 4.7 is not true in general as shown by the following example.

Example 4.9. Let (X,τ) as in Example 4.6 and $f:(X,\tau)\to(X,\tau)$ be a function defined by

$$f(x)=\begin{cases} 2 & \text{if } x=1,\\ 3 & \text{if } x=2,\\ 1 & \text{if } x=3,\end{cases}$$

moreover let $\gamma=\beta$ be the closure operation on $\alpha O(X,\tau)$ and $\gamma^{'}=\beta^{'}=X:\alpha O(X,\tau)\to P(X)$. Then, $\alpha O(X,\tau)_{[\gamma,X]}=\{\phi,X\}$ and it is shown that f is not $(\alpha_{[\gamma,X]},\ \alpha_{[\gamma,X]})$-continuous. However, $f:(X,\alpha O(X,\tau)_{[\gamma,\gamma']})\to(Y,\alpha O(Y,\sigma)_{[\beta,\beta']})$ is continuous.

Let (X,τ), (Y,σ) and (Z,η) be spaces and $\gamma,\gamma^{'}:\alpha O(X,\tau)\to P(X)$, $\beta,\beta^{'}:\alpha O(Y)\to P(Y)$ and $\delta,\delta^{'}:\alpha O(Z)\to P(Z)$, be operations on $\alpha O(X,\tau)$, $\alpha O(Y,\sigma)$ and $\alpha O(Z,\eta)$, respectively.

Theorem 4.10. *If $f:(X,\tau)\to(Y,\sigma)$ is $(\alpha_{[\gamma,\gamma']},\ \alpha_{[\beta,\beta']})$-continuous and $g:(Y,\sigma)\to(Z,\eta)$ is $(\alpha_{[\beta,\beta']},\ \alpha_{[\delta,\delta']})$-continuous, then its composition $gof:(X,\tau)\to(Z,\eta)$ is $(\alpha_{[\gamma,\gamma']},\ \alpha_{[\delta,\delta']})$-continuous.*

Proof. Let $x\in X$, K and L be α-open sets of Z containing $g(f(x))$. Since g is $(\alpha_{[\beta,\beta']},\ \alpha_{[\delta,\delta']})$-continuous, then there exist α-open sets W and S of Y containing $f(x)$ such that $g(W^{\beta}\cap S^{\beta^{'}})\subseteq K^{\delta}\cap L^{\delta^{'}}$. Also, since f is $(\alpha_{[\gamma,\gamma']},\ \alpha_{[\beta,\beta']})$-continuous, then there exist α-open sets U and V of X containing x such that $f(U^{\gamma}\cap V^{\gamma^{'}})\subseteq W^{\beta}\cap S^{\beta^{'}}$. This implies that $f(U^{\gamma}\cap V^{\gamma^{'}})\subseteq W^{\beta}\cap S^{\beta^{'}}\subseteq g^{-1}(K^{\delta}\cap L^{\delta^{'}})$. Then we obtain $(gof)(U^{\gamma}\cap V^{\gamma^{'}})\subseteq K^{\delta}\cap L^{\delta^{'}}$. Therefore, gof is $(\alpha_{[\gamma,\gamma']},\ \alpha_{[\delta,\delta']})$-continuous. □

Definition 4.11. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be $(\alpha_{[\gamma,\gamma']},\ \alpha_{[\beta,\beta']})$-closed if for $\alpha_{[\gamma,\gamma']}$-closed set A of X, $f(A)$ is $\alpha_{[\beta,\beta']}$-closed in Y.

Proposition 4.12. *Let $f:(X,\tau)\to(Y,\sigma)$ be an $(\alpha_{[\gamma,\gamma']},\ \alpha_{[\beta,\beta']})$-closed function. Then, for each subset B of (Y,σ) and each $\alpha_{[\gamma,\gamma']}$-open set U containing $f^{-1}(B)$, there exists an $\alpha_{[\beta,\beta']}$-open set V such that $B\subseteq V$ and $f^{-1}(V)\subseteq U$.*

Proof. Let $V=Y\setminus f(X\setminus U)$. Then V is $\alpha_{[\beta,\beta']}$-open. Thus $f^{-1}(B)\subseteq U$ implies $B\subseteq V$ and $f^{-1}(V)=f^{-1}(Y\setminus f(X\setminus U))=X\setminus f^{-1}(f(X\setminus U))\subseteq X\setminus(X\setminus U)=U$, or $f^{-1}(V)\subseteq U$. □

Proposition 4.13. *If $f : (X, \tau) \to (Y, \sigma)$ is bijective and $f^{-1} : (Y, \sigma) \to (X, \tau)$ is $(\alpha_{[\beta,\beta']}, \alpha_{[\gamma,\gamma']})$-continuous, then f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-closed.*

Proof. This follows from definitions and Theorem 4.2. □

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