THE SQUARE OF DIRAC OPERATOR ON HOMOGENEOUS SPACES

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Abstract: The solution of the eigenvalue problem of the square of Dirac operator with the normal metric induced by the Killing form on a general homogeneous space G/H is given.

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1. Introduction

In [1], we gave the solution of the eigenvalue problem of the Laplacian on a homogeneous space G/H , where G is a compact, semi-simple Lie group, H is a closed subgroup of G , and the rank of H is equal to the rank of G .

In this short paper, we give the solution of the eigenvalue problem of the square of Dirac operator with the normal metric induced by the Killing form on the same homogeneous space G/H .

2. The Square of Dirac Operator on G/H

We first review the Laplacian on a homogeneous space G/H . Here G is a compact, semi-simple Lie group, H is a closed subgroup of G , and the rank of H is equal to the rank of G .

Let $\mathfrak{g}$ and η be the Lie algebras of G and H , respectively.

We suppose that G/H is reductive, i.e. $\mathfrak{g}$ has an orthogonal decomposition $\mathfrak{g} = \eta \oplus \mathfrak{m}$ with $[\eta, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{g}$.

We can choose a common Cartan subalgebra

$$\mathfrak{h} \subset \eta \subset \mathfrak{g}.$$

The Killing form of Lie algebra $\mathfrak{g}$ (η) of G (H) induces a canonical metric $(,)$ on $G(H)$ and the dual space of $\mathfrak{g}$ (η). The metric $(,)$ of G induces a natural reductive metric $(,)_\mathfrak{m}$ on G/H .

Let $\Phi_\mathfrak{g}$ be the set of roots of $\mathfrak{g}$. The roots Φ_η of η form a subset of the roots of $\mathfrak{g}$, i.e.,

$$\Phi_\eta \subset \Phi_\mathfrak{g}.$$

Choosing a positive root system $\Phi_\mathfrak{g}^+$ for $\mathfrak{g}$ also determines a positive root system Φ_η^+ for η , where

$$\Phi_\eta^+ \subset \Phi_\mathfrak{g}^+.$$

Let $\rho_\mathfrak{g} = \frac{1}{2} \sum_{\alpha \in \Phi_\mathfrak{g}^+} \alpha$ and $\rho_\eta = \frac{1}{2} \sum_{\alpha \in \Phi_\eta^+} \alpha$ denote the Weyl vectors of $\mathfrak{g}$ and η respectively.

Let U_μ be a given irreducible representation of η with highest weight μ . Let $G \times_H U_\mu$ be an associated vector bundle of the principal bundle $P(G/H, H)$. The Hilbert space of square integrable sections of $G \times_H U_\mu$ decomposes into the direct sum of the eigenspaces of the Laplacian on G/H , which are irreducible representations V_λ of $\mathfrak{g}$ with highest weights λ 's. and this induces the following expression for the Laplacian on G/H which was discussed in [2], [3], [4], [5], [6], [7], [8], [9], and appears explicitly in [10].

Definition 1. The Laplacian on G/H is

$$\Delta = C_2(\mathfrak{g}, \cdot) - C_2(\eta, U). \quad (1)$$

Here $C_2(\mathfrak{g}, \cdot)$ is the quadratic Casimir element of $\mathfrak{g}$ calculated in an irreducible representation of $\mathfrak{g}$. $C_2(\eta, U)$ is the quadratic Casimir element of η calculated in a given irreducible representation U .

Lemma 2. *The value of the scalar curvature R of G/H is given by*

$$R = \frac{1}{4} \text{tr}_{\mathbf{m}} C_2 = 6[(\rho_{\mathbf{g}}, \rho_{\mathbf{g}}) - (\rho_{\boldsymbol{\eta}}, \rho_{\boldsymbol{\eta}})]. \quad (2)$$

Proof. Let $X_1, X_2, \dots, X_{\dim \mathbf{m}}$ be an orthonormal basis of $\mathbf{m}$. Let R_m be the Riemann curvature on $\mathbf{m}$.

$$\begin{aligned} R &= \sum_{i,j=1}^{\dim \mathbf{m}} R_m(X_i, X_j, X_j, X_i) \\ &= \frac{1}{4} \sum_{i,j=1}^{\dim \mathbf{m}} ([[X_i, X_j], X_j], X_i) \\ &= \frac{1}{4} \sum_{i,j=1}^{\dim \mathbf{m}} (\text{ad}_{X_j} \circ \text{ad}_{X_j}(X_i), X_i) \\ &= \frac{1}{4} \sum_{i=1}^{\dim \mathbf{m}} (\text{ad}(C_2)X_i, X_i) \\ &= \frac{1}{4} \text{tr}_{\mathbf{m}}(C_2). \end{aligned} \quad (3)$$

By a result of Kostant [11], $\frac{1}{24} \text{tr}_{\mathbf{g}}(C_2) = (\rho_{\mathbf{g}}, \rho_{\mathbf{g}})$, we have

$$\frac{1}{24} \text{tr}_{\mathbf{m}}(C_2) = (\rho_{\mathbf{g}}, \rho_{\mathbf{g}}) - (\rho_{\boldsymbol{\eta}}, \rho_{\boldsymbol{\eta}}). \quad (4)$$

This concludes the proof of the lemma.

Let D be the Dirac operator on G/H . In [9], Dolan gave the relation among the square of Dirac operator D^2 , the Laplacian Δ and the scalar curvature R on G/H

$$D^2 = \Delta + cR, \quad (5)$$

where c is a constant. In [9], c was taken to be $\frac{1}{8}$. Comparing the corresponding case of the Kostant type of Dirac operator [11] with ours, we adopt that

$$c = \frac{1}{6}. \quad (6)$$

It follows that the square of Dirac operator on G/H is

Proposition 3.

$$D^2 = C_2(\mathbf{g}, \cdot) - C_2(\boldsymbol{\eta}, U) + (\rho_{\mathbf{g}}, \rho_{\mathbf{g}}) - (\rho_{\boldsymbol{\eta}}, \rho_{\boldsymbol{\eta}}). \quad (7)$$

By Proposition 2 and Theorem 4 in [1], we have the following result.

Theorem 4. Given an irreducible representation U_μ of η with highest weight μ . The eigenvalue of D^2 labelled by a highest weight λ reads

$$E_\lambda = (\lambda + \rho_{\mathbf{g}}, \lambda + \rho_{\mathbf{g}}) - (\mu + \rho_\eta, \mu + \rho_\eta) \quad (8)$$

with

$$(\lambda + \rho_{\mathbf{g}}, \lambda + \rho_{\mathbf{g}}) \geq (\mu + \rho_\eta, \mu + \rho_\eta).$$

The multiplicity of the eigenvalue E_λ of D^2 is given by the Weyl dimension formula:

$$\dim V_\lambda = \frac{\prod_{\alpha \in \Phi_{\mathbf{g}}^+} (\lambda + \rho_{\mathbf{g}}, \alpha)}{\prod_{\alpha \in \Phi_{\mathbf{g}}^+} (\rho_{\mathbf{g}}, \alpha)}. \quad (9)$$

Moreover, If there exists an element $w \in W_{\mathbf{g}}$ in the Weyl group of $\mathbf{g}$ such that the weight $w(\mu + \rho_\eta) - \rho_{\mathbf{g}}$ is dominant for $\mathbf{g}$. Then The lowest eigenvalue of D^2 is

$$E_{w(\mu + \rho_\eta) - \rho_{\mathbf{g}}} = 0, \quad (10)$$

and the multiplicity of the lowest eigenvalue of D^2 is just the index of Dirac operator,

$$\text{Ind} D = \dim V_{w(\mu + \rho_\eta) - \rho_{\mathbf{g}}} = \frac{\prod_{\alpha \in \Phi_{\mathbf{g}}^+} (w(\mu + \rho_\eta), \alpha)}{\prod_{\alpha \in \Phi_{\mathbf{g}}^+} (\rho_{\mathbf{g}}, \alpha)}. \quad (11)$$

Remark. $V_{w(\mu + \rho_\eta) - \rho_{\mathbf{g}}}$ is, up to a sign, equal to the G -equivariant index of the Kostant's Dirac operator on G/H [11], [12], [13], [14].

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