**FROM THE SQUARE OF DIRAC OPERATOR TO
THE SPIN LANDAU HAMILTONIAN ON
HOMOGENEOUS SPECIAL ORTHOGONAL SPACES**

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Abstract: The solutions of the eigenvalue problems of the square of Dirac operator and the spin Landau Hamiltonian with the normal metric induced by the Killing form on homogeneous special orthogonal space $SO(2k+1)/SO(2k)$ are given respectively.

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1. Introduction

In [1], we gave the solution of the eigenvalue problem of the square of Dirac operator with the normal metric induced by the Killing form on a homogeneous space G/H , where G is a compact, semi-simple Lie group, H is a closed subgroup of G , and the rank of H is equal to the rank of G . In [2], the solution of the eigenvalue problem of the Laplacian on even dimensional spheres $SO(2k+1)/SO(2k) = S^{2k}$ was given.

In this paper, we apply our result in [1] to the case of $SO(2k+1)/SO(2k) = S^{2k}$ and give the solutions of the eigenvalue problems of the square of Dirac operator and the spin Landau Hamiltonian on even dimensional spheres respectively.

2. The Solution of Eigenvalue Problem of the Square of Dirac Operator on $SO(2k+1)/SO(2k)$

Let D be the Dirac operator on G/H . We first review the square of Dirac operator D^2 on a homogeneous space G/H [1], [3]. Here G is a compact, semi-simple Lie group, H is a closed subgroup of G , and the rank of H is equal to the rank of G .

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H , respectively.

We suppose that G/H is reductive, i.e. $\mathfrak{g}$ has an orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$.

We can choose a common Cartan subalgebra

$$\mathfrak{h} \subset \mathfrak{h} \subset \mathfrak{g}.$$

The Killing form of Lie algebra $\mathfrak{g}$ ($\mathfrak{h}$) of G (H) induces a canonical metric $(\cdot, \cdot)$ on $G(H)$ and the dual space of $\mathfrak{g}$ ($\mathfrak{h}$). The metric $(\cdot, \cdot)$ of G induces a natural reductive metric $(\cdot, \cdot)_{\mathfrak{m}}$ on G/H .

Definition 1. The D^2 on G/H is

$$D^2 = C_2(\mathfrak{g}, \cdot) - C_2(\mathfrak{h}, U) + (\rho_{\mathfrak{g}}, \rho_{\mathfrak{g}}) - (\rho_{\mathfrak{h}}, \rho_{\mathfrak{h}}). \quad (1)$$

Here $C_2(\mathfrak{g}, \cdot)$ is the quadratic Casimir element of $\mathfrak{g}$ calculated in an irreducible representation of $\mathfrak{g}$. $C_2(\mathfrak{h}, U)$ is the quadratic Casimir element of $\mathfrak{h}$ calculated in a given irreducible representation U . Let $\rho_{\mathfrak{g}}$ and $\rho_{\mathfrak{h}}$ are the Weyl vectors of $\mathfrak{g}$ and $\mathfrak{h}$ respectively.

We take $so(2k)$ and $so(2k+1)$ be the Lie algebras of $SO(2k)$ and $SO(2k+1)$, respectively. We shall work in the space $\mathbf{R}^k$, where the inner product is the usual one. Let $\epsilon_1, \dots, \epsilon_k$ denote the usual orthonormal unit vectors which form a basis of $\mathbf{R}^k$. We make the usual choice of Cartan subalgebra and positive roots, $\epsilon_i \pm \epsilon_j$ ($1 \leq i < j \leq k$) for $so(2k)$ and these together with ϵ_i , ($1 \leq i \leq k$) for $so(2k+1)$. The Weyl vectors of $so(2k+1)$ and $so(2k)$ are

$$\rho_{so(2k+1)} = \left(\frac{2k-1}{2}, \frac{2k-3}{2}, \dots, \frac{1}{2} \right),$$

and

$$\rho_{so(2k)} = (k-1, k-2, \dots, 1, 0).$$

One has

$$\begin{aligned} \rho_{so(2k+1)} - \rho_{so(2k)} &= \left(\frac{1}{2}, \dots, \frac{1}{2}\right), \\ (\rho_{so(2k+1)}, \rho_{so(2k+1)}) - (\rho_{so(2k)}, \rho_{so(2k)}) &= \frac{1}{2}k^2 - \frac{1}{4}k. \end{aligned} \quad (2)$$

Corollary 2. *The square of Dirac operator D^2 with the normal metric on $SO(2k+1)/SO(2k)$ is*

$$D^2 = C_2(so(2k+1), \cdot) - C_2(so(2k), U) + \frac{1}{2}k^2 - \frac{1}{4}k. \quad (3)$$

Here $C_2(so(2k+1), \cdot)$ is the quadratic Casimir element of $so(2k+1)$ calculated in an irreducible representation of $so(2k+1)$. $C_2(so(2k), U)$ is the quadratic Casimir element of $so(2k)$ calculated in a given irreducible representation U .

The spectrum of $C_2(SO(2k+1), V_\lambda)$ in an irreducible representation V_λ of $so(2k+1)$ with highest weight $\lambda = (\lambda_1, \dots, \lambda_k)$, $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0)$, is

$$\begin{aligned} C_2(\lambda) &= (\lambda + \rho_{so(2k+1)}, \lambda + \rho_{so(2k+1)}) - (\rho_{so(2k+1)}, \rho_{so(2k+1)}) \\ &= \sum_{i=1}^k [\lambda_i^2 + (2k - 2i + 1)\lambda_i]. \end{aligned} \quad (4)$$

The spectrum of $C_2(SO(2k), U_\mu)$ in a given irreducible representation U_μ of $so(2k)$ with highest weight $\mu = (\mu_1, \dots, \mu_k)$, $(\mu_1 \geq \mu_2 \geq \dots \geq \mu_{k-1} \geq \mu_k \geq 0)$, is

$$\begin{aligned} C_2(\mu) &= (\mu + \rho_{so(2k)}, \mu + \rho_{so(2k)}) - (\rho_{so(2k)}, \rho_{so(2k)}) \\ &= \sum_{i=1}^k [\mu_i^2 + 2(k - i)\mu_i]. \end{aligned} \quad (5)$$

We have the following result:

Theorem 3. *Given an irreducible representation U_μ of $so(2k)$ with highest weight $\mu = (\mu_1, \dots, \mu_k)$, $(\mu_1 \geq \mu_2 \geq \dots \geq \mu_{k-1} \geq \mu_k \geq 0)$, the eigenvalue of D^2 labelled by highest weight $\lambda = (\lambda_1, \dots, \lambda_k)$, $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0)$ reads*

$$\begin{aligned} E_\lambda &= (\lambda + \rho_{so(2k+1)}, \lambda + \rho_{so(2k+1)}) - (\mu + \rho_{so(2k)}, \mu + \rho_{so(2k)}) \\ &= \sum_{i=1}^k [\lambda_i^2 - \mu_i^2 + (2k - 2i + 1)\lambda_i - 2(k - i)\mu_i] \end{aligned}$$

$$+\frac{1}{2}k^2 - \frac{1}{4}k. \quad (6)$$

The multiplicity of the eigenvalue E_λ is given by the Weyl dimension formula:

$$\begin{aligned} \dim V_\lambda &= \frac{\prod_{\alpha \in \Phi_{so(2k+1)}^+} (\lambda + \rho_{so(2k+1)}, \alpha)}{\prod_{\alpha \in \Phi_{so(2k+1)}^+} (\rho_{so(2k+1)}, \alpha)} \\ &= \prod_{i=1}^k \frac{2\lambda_i + 2k + 1 - 2i}{2k + 1 - 2i} \\ &\quad \times \prod_{1 \leq i < j \leq k} \frac{(\lambda_i - \lambda_j + j - i)(\lambda_i + \lambda_j + 2k + 1 - i - j)}{(j - i)(2k + 1 - i - j)}. \end{aligned} \quad (7)$$

The eigenspace of the lowest eigenvalue of D^2 on $SO(2k+1)/SO(2k)$ is V_λ with highest weight

$$\begin{aligned} \lambda &= \mu + \rho_{so(2k)} - \rho_{so(2k+1)} \\ &= (\mu_1 - \frac{1}{2}, \dots, \mu_k - \frac{1}{2}). \end{aligned} \quad (8)$$

It follows that

$$(\lambda + \rho_{so(2k+1)}, \lambda + \rho_{so(2k+1)}) = (\mu + \rho_{so(2k)}, \mu + \rho_{so(2k)}).$$

The lowest eigenvalue of D^2 is

$$E_{(\mu_1 - \frac{1}{2}, \dots, \mu_k - \frac{1}{2})} = 0. \quad (9)$$

The multiplicity of the lowest eigenvalue of D^2 which is the index of D on $SO(2k+1)/SO(2k)$ reads

$$\begin{aligned} \text{Ind} D &= \dim V_{(\mu_1 - \frac{1}{2}, \dots, \mu_k - \frac{1}{2})} \\ &= \prod_{i=1}^k \frac{2\mu_i + 2k - 2i}{2k + 1 - 2i} \\ &\quad \times \prod_{1 \leq i < j \leq k} \frac{(\mu_i - \mu_j + j - i)(\mu_i + \mu_j + 2k - i - j)}{(j - i)(2k + 1 - i - j)}. \end{aligned} \quad (10)$$

The above highest weights λ 's can be determined by the following inverse branching rule from $SO(2k)$ to $SO(2k+1)$.

Theorem 4. *Given an irreducible representation U_μ of $so(2k)$ with highest weight $\mu = (\mu_1, \dots, \mu_k)$, $(\mu_1 \geq \mu_2 \geq \dots \geq \mu_{k-1} \geq \mu_k \geq 0)$, the highest weights*

appeared in Theorem 3, $\lambda = (\lambda_1, \dots, \lambda_k)$ with $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0)$ satisfy the following inequalities:

$$(\lambda + \rho_{so(2k+1)}, \lambda + \rho_{so(2k+1)}) \geq (\mu + \rho_{so(2k)}, \mu + \rho_{so(2k)}). \quad (11)$$

$$\lambda_1 + \frac{1}{2} \geq \mu_1 \geq \lambda_2 + \frac{1}{2} \geq \mu_2 \geq \dots \geq \lambda_{k-1} + \frac{1}{2} \geq \mu_{k-1} \geq \lambda_k + \frac{1}{2} \geq \mu_k \geq 0. \quad (12)$$

The Hilbert space of square integrable sections of the associated vector bundle $SO(2k+1) \times_{SO(2k)} U_\mu$ of the principal bundle $P(SO(2k+1)/SO(2k), SO(2k))$ decomposes into the direct sum of the eigenspaces of D^2 on $SO(2k+1)/SO(2k)$, which are irreducible representations V_λ of $so(2k+1)$ with the above highest weights λ 's.

3. The Solution of Eigenvalue Problem of the Spin Landau Hamiltonian H on $SO(2k+1)/SO(2k)$

Now we apply Theorem 3 to the case of the spin Landau Hamiltonian on $SO(2k+1)/SO(2k)$.

Corollary 5. *The spin Landau Hamiltonian H on $SO(2k+1)/SO(2k)$ is*

$$\begin{aligned} H &= \frac{1}{2M} D^2 \\ &= \frac{1}{2M} [C_2(so(2k+1), \cdot) - C_2(so(2k), U) + \frac{1}{2}k^2 - \frac{1}{4}k]. \end{aligned} \quad (13)$$

Here M is the inertia mass of the particle.

From (4), the spectrum of $C_2(SO(2k+1), V_\lambda)$ in an irreducible representation V_λ of $so(2k+1)$ with highest weight $\lambda = (\overbrace{(n+J, J, \dots, J)}^k)$, is

$$\begin{aligned} C_2(\lambda) &= (\lambda + \rho_{so(2k+1)}, \lambda + \rho_{so(2k+1)}) - (\rho_{so(2k+1)}, \rho_{so(2k+1)}) \\ &= n^2 + n(2J + 2k - 1) + kJ(J + k). \end{aligned} \quad (14)$$

Here n is the Landau energy level index.

From (5), The spectrum of $C_2(SO(2k), U_\mu)$ in a given irreducible representation

U_μ of $so(2k)$ with highest weight $\mu = (\overbrace{(\frac{I}{2}, \dots, \frac{I}{2})}^k)$, is

$$C_2(\mu) = (\mu + \rho_{so(2k)}, \mu + \rho_{so(2k)}) - (\rho_{so(2k)}, \rho_{so(2k)})$$

$$= k \frac{I}{2} \left(\frac{I}{2} + k - 1 \right). \quad (15)$$

We have the following result:

Corollary 6. *Given an irreducible representation U_μ of $so(2k)$ with highest weight $\mu = (\overbrace{\frac{I}{2}, \dots, \frac{I}{2}}^k)$, the n th Landau energy level labelled by highest weight $\lambda = (\overbrace{n+J, J, \dots, J}^k)$ reads*

$$\begin{aligned} E_\lambda &= (\lambda + \rho_{so(2k+1)}, \lambda + \rho_{so(2k+1)}) - (\mu + \rho_{so(2k)}, \mu + \rho_{so(2k)}) \\ &= \frac{1}{2M} [n^2 + n(2J + 2k - 1) + kJ(J + k) - k \frac{I}{2} \left(\frac{I}{2} + k - 1 \right)] \\ &\quad + \frac{1}{4M} k^2 - \frac{1}{8M} k. \end{aligned} \quad (16)$$

The degeneracy of E_λ is given by the Weyl dimension formula:

$$\begin{aligned} \dim V_\lambda &= \frac{\prod_{\alpha \in \Phi_{so(2k+1)}^+} (\lambda + \rho_{so(2k+1)}, \alpha)}{\prod_{\alpha \in \Phi_{so(2k+1)}^+} (\rho_{so(2k+1)}, \alpha)} \\ &= \frac{2n + 2J + 2k - 1}{(2k - 2)!(2k - 1)!!} \prod_{i=2}^k (2J + 2k + 1 - 2i) \\ &\quad \times \prod_{i=2}^k (n + i - 1)(n + 2J + 2k - i) \\ &\quad \times \prod_{2 \leq i < j \leq k} \frac{2J + 2k + 1 - i - j}{2k + 1 - i - j}. \end{aligned} \quad (17)$$

The eigenspace of the ground Landau energy level ($n = 0$) is V_λ with highest

weight $\lambda = (\overbrace{\frac{I}{2} - \frac{1}{2}, \dots, \frac{I}{2} - \frac{1}{2}}^k)$. The ground energy level is

$$E_{(\frac{I}{2}-\frac{1}{2}, \dots, \frac{I}{2}-\frac{1}{2})} = 0. \quad (18)$$

The degeneracy of the ground energy level which is the index of the Dirac operator D on $SO(2k + 1)/SO(2k)$ reads

$$\begin{aligned} \text{Ind} D &= \dim V_{(\frac{I}{2}-\frac{1}{2}, \dots, \frac{I}{2}-\frac{1}{2})} \\ &= \prod_{i=1}^k \frac{I + 2k - 2i}{2k + 1 - 2i} \prod_{1 \leq i < j \leq k} \frac{I + 2k - i - j}{2k + 1 - i - j}. \end{aligned} \quad (19)$$

Remark. The Landau energy levels in Corollary 6 coincide with the result in [4].

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