STATISTICAL CONVERGENCE IN FUZZY n -NORMED SPACES

B. Surender Reddy¹, M. Srinivas^{2 §}

^{1,2}Department of Mathematics

University College of Science

Saifabad, Osmania University

Hyderabad, Telangana 500004, INDIA

Abstract: In this paper we introduce the concept of statistical convergence of a sequence in fuzzy n -normed space and defined limit point, statistical limit point and statistical cluster point of a sequence in fuzzy n -normed space. Also we establish the relationship between limit point, statistical limit point and statistical cluster point of a sequence in fuzzy n -normed space.

AMS Subject Classification: 46S40, 41A50

Key Words: fuzzy n -normed space, statistical convergence and statistical Cauchy sequence in fuzzy n -normed Space, statistical limit point and statistical cluster point in fuzzy n -normed space

1. Introduction and Preliminaries

The idea of fuzzy norm on a vector space was initiated by Katsaras [15] in [1984]. Later Felbin [17] introduced the concept of fuzzy normed space in [1992] which is based on the concept of fuzzy metric given by Kaleva and Siekkala [16]. They

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Received: March 24, 2015

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[§]Correspondence author

also defined fuzzy bounded linear operator and norm of which is a fuzzy number. In [2003] Bag and Samanta [1] discussed some of the properties of finite dimensional fuzzy normed linear spaces. In [1979] Mizumoto and Tanaka [5] defined fuzzy number and discussed some of the properties of fuzzy numbers. The theory of fuzzy n -normed spaces was studied by Narayanan and Vijayabalaji [18].

The idea of statistical convergence was given by Steinhaus [7] in [1951]. Fridy [14] developed the idea of statistical limit point, statistical cluster point of a number sequence. In [2010] B.S.Reddy [10] studied statistical convergence in n -normed space and some results are established.

Statistical convergence applied in many branches of mathematics like trigonometric series, number theory and summability theory [8]. The concept of statistical convergence in fuzzy normed space was given by Sencimen, Pehlivan [4] later Mohiuddine, Sevli and Cancan [12] extended the concept of statistical convergence in fuzzy normed space to statistical convergence in fuzzy 2-normed space.

In this paper we introduce the concept of statistical convergence of a sequence in fuzzy n -normed space and defined limit point, statistical limit point and statistical cluster point of a sequence in fuzzy n -normed space. Also we establish the relationship between limit point, statistical limit point and statistical cluster point of a sequence in fuzzy n -normed space.

According to Mizumoto and Tanaka [5], a fuzzy number is a mapping $x : R \rightarrow [0, 1]$ over the set R of real numbers. A fuzzy number x is said to be convex if for any real numbers $t, s, r \in R$ with $s \leq t \leq r$, $x(t) \geq \min\{x(s), x(r)\}$. A fuzzy number x is said to be normal if $\exists t_0 \in R$ such that $x(t_0) = 1$. For $0 < \alpha \leq 1$, α -Level set of an upper semi continuous convex normal fuzzy number η (denoted by $[\eta]_\alpha$) is a closed interval $[a_\alpha, b_\alpha]$, where $a_\alpha = -\infty$ and $b_\alpha = +\infty$ admissible. When $a_\alpha = -\infty$, for instance, then $[a_\alpha, b_\alpha]$ means the interval $(-\infty, b_\alpha]$. Similarly when $b_\alpha = +\infty$ then $[a_\alpha, b_\alpha]$ means the interval $[a_\alpha, +\infty)$. A fuzzy number x is said to be non-negative if $x(t) = 0, \forall t < 0$. The set of all convex, normal, upper semi continuous fuzzy real numbers is denoted by $L(R)$ and the set of all non-negative, convex, normal, upper semi continuous fuzzy real numbers is denoted by $L^*(R)$.

For any real number $r \in R$, we define a corresponding fuzzy number $\hat{r}$ by

$$\begin{aligned}\hat{r} &= 1 \text{ if } t = r \\ &= 0 \text{ if } t \neq r\end{aligned}$$

A partial ordering $\preceq$ on $L(R)$ is defined by $u \preceq v$ if and only if $u_\alpha^- \leq v_\alpha^-$ and $u_\alpha^+ \leq v_\alpha^+ \forall \alpha \in [0, 1]$, where $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ and $[v]_\alpha = [v_\alpha^-, v_\alpha^+]$. The strict

inequality in $L(R)$ is defined by $u \prec v$ if and only if $u_\alpha^- < v_\alpha^-$ and $u_\alpha^+ < v_\alpha^+$ $\forall \alpha \in [0, 1]$. For $k > 0$, (ku) is defined as $(ku)(t) = u(t/k)$ and $(0u)(t) = u(t/k)$ is defined to be $\hat{0}(t)$.

According to Mizumoto and Tanaka [5], the arithmetic operations $\oplus, \ominus, \otimes$ on $L(R) \times L(R)$ are defined by

$$(x \oplus y)(t) = \sup_{s \in R} \min\{x(s), y(t-s)\}$$

$$(x \ominus y)(t) = \sup_{s \in R} \min\{x(s), y(s-t)\}$$

and

$$(x \otimes y)(t) = \sup_{s \in R, s \neq 0} \min\{x(s), y(t/s)\}, \quad \forall t \in R$$

Let $u, v \in L(R)$. Define $D(u, v) = \sup_{\alpha \in [0, 1]} \max\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\}$. Then D is called supremum metric on $L(R)$. Let $(u_n) \subset L(R)$ and $u \in L(R)$. A sequence (u_n) is said to be converges to u in the metric D if $\lim_{n \rightarrow \infty} D(u_n, u) = 0$ (for short, D -converges to u), It can be written as $u_n \xrightarrow{D} u$ or $(D) - \lim_{n \rightarrow \infty} u_n = u$.

2. Fuzzy n -Normed Spaces

Definition 1. Let $n \in N$ and let X be a real linear space of dimension $d \geq n$. A real valued function $\|\bullet, \bullet, \dots, \bullet\|$ on $\underbrace{X \times X \times \dots \times X}_n = X^n \rightarrow R^n$

satisfying the following conditions

nN_1 : $\|x_1, x_2, \dots, x_n\| = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent

nN_2 : $\|x_1, x_2, \dots, x_n\|$ is invariant under any permutation of $x_1, x_2, \dots, x_n$

nN_3 : $\|x_1, x_2, \dots, x_{n-1}, \alpha x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$ for all $\alpha \in R$

nN_4 : $\|x_1, x_2, \dots, x_{n-1}, y+z\| \leq \|x_1, x_2, \dots, x_{n-1}, y\| + \|x_1, x_2, \dots, x_{n-1}, z\|$ for all $y, z, x_1, x_2, \dots, x_{n-1} \in X$, then the function $\|\bullet, \bullet, \dots, \bullet\|$ is called an n -norm on X and the pair $(X, \|\bullet, \bullet, \dots, \bullet\|)$ is called n -normed linear space.

Definition 2. Let X be a real linear space of dimension d , where $2 \leq d < \infty$. Let $\|\bullet, \bullet, \dots, \bullet\| : \underbrace{X \times X \times \dots \times X}_n \rightarrow L^*(R)$ and the mappings $L, R :$

$[0, 1] \times [0, 1] \rightarrow [0, 1]$ be symmetric, non-decreasing in both arguments such that $L(0, 0) = 0$, $R(1, 1) = 1$ then the quadruple $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$ is called fuzzy n -normed space (in short f- n -N) if it satisfies the following conditions for every $x_1, x_2, \dots, x_n, y \in X$ and $s, t \in R$

$(f - nN_1)$: $\|x_1, x_2, \dots, x_n\| = \bar{0}$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent vectors,

$(f - nN_2)$: $\|x_1, x_2, \dots, x_n\|$ is invariant under any permutation of $x_1, x_2, \dots, x_n$,

$(f - nN_3)$: $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$, $\alpha \in R$

$(f - nN_4)$: $\|x_1 + y, x_2, \dots, x_n\|(s + t) \geq L(\|x_1, x_2, \dots, x_n\|(s), \|y, x_2, \dots, x_n\|(t))$, whenever $s \leq \|x_1, x_2, \dots, x_n\|_1^-$, $t \leq \|y, x_2, \dots, x_n\|_1^-$ and $s + t \leq \|x_1 + y, x_2, \dots, x_n\|_1^-$,

$(f - nN_5)$: $\|x_1 + y, x_2, \dots, x_n\|(s + t) \leq R(\|x_1, x_2, \dots, x_n\|(s), \|y, x_2, \dots, x_n\|(t))$, whenever $s \geq \|x_1, x_2, \dots, x_n\|_1^-$, $t \geq \|y, x_2, \dots, x_n\|_1^-$ and $s + t \geq \|x_1 + y, x_2, \dots, x_n\|_1^-$,

where $[\|x_1, x_2, \dots, x_n\|]_\alpha = [\|x_1, x_2, \dots, x_n\|_\alpha^-, \|x_1, x_2, \dots, x_n\|_\alpha^+]$ for $x_1, x_2, \dots, x_n \in X$, $0 \leq \alpha \leq 1$ and $\inf_{\alpha \in [0,1]} \|x_1, x_2, \dots, x_n\|_\alpha^- > 0$. Hence the norm $\|\bullet, \bullet, \dots, \bullet\|$ is called fuzzy n -norm.

Let us consider the topological structure of fuzzy n -normed space $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$. For any $\epsilon > 0$, $\alpha \in [0,1]$ and $x \in X$, the (ϵ, α) -neighborhood of x is the set $N_x(\epsilon, \alpha) = \{y \in X : \|x - y, z_2, z_3, \dots, z_n\|_\alpha^+ < \epsilon\}$ for each $z_2, z_3, \dots, z_n \neq 0$, $z_2, z_3, \dots, z_n \in X$.

Definition 3. A sequence $\{x_k\}$ in fuzzy n -normed space $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$ is said to be convergent to an element $x \in X$ with respect to the fuzzy n -norm on X if for every $\epsilon > 0$ and for every $z_2, z_3, \dots, z_n \neq 0$, $z_2, z_3, \dots, z_n \in X$, $\exists$ a number $N = N(\epsilon, z_2, z_3, \dots, z_n)$ such that $D(\|x_k - x, z_2, z_3, \dots, z_n\|, \bar{0}) < \epsilon$, $\forall k \geq N$ or equivalently $(D) - \lim_{k \rightarrow \infty} \|x_k - x, z_2, z_3, \dots, z_n\| = \bar{0}$.

It can be represented by $(x_k, z_2, z_3, \dots, z_n) \xrightarrow{f-n-N} (x, z_2, z_3, \dots, z_n)$ for every $z_2, z_3, \dots, z_n \neq 0$, $z_2, z_3, \dots, z_n \in X$

This means that for every $\epsilon > 0$, $\exists$ a number $N = N(\epsilon, z_2, z_3, \dots, z_n)$ such that $\sup_{\alpha \in [0,1]} \|x_k - x, z_2, z_3, \dots, z_n\|_\alpha^+ = \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ < \epsilon \quad \forall k \geq N$.

In terms of neighborhood, we have

$(x_k, z_2, z_3, \dots, z_n) \xrightarrow{f-n-N} (x, z_2, z_3, \dots, z_n)$ if for any $\epsilon > 0$, $\exists$ a number $N = N(\epsilon, z_2, z_3, \dots, z_n)$ such that $x_k \in N_x(\epsilon, 0)$, whenever $k \geq N$.

3. Statistical Convergence and Statistically Cauchy Sequences in Fuzzy n -Normed Spaces

In this section, we define the notion of statistically convergence and statistically Cauchy sequences in fuzzy n -normed space.

Let K be a subset of natural numbers N . Then the natural density of K denoted by $\delta(K)$, is defined as $\delta(K) = \lim_n \frac{1}{n} |\{k \leq n : k \in K\}|$, where the vertical bars denote the cardinality of the enclosed set.

A number sequence $\{x_k\}$ is said to be statistically convergent to the number ℓ if for each $\epsilon > 0$, the set $K(\epsilon) = \{k \leq n : \|x_k - \ell\| \geq \epsilon\}$ has natural density zero. i.e., $\lim_n \frac{1}{n} |\{k \leq n : \|x_k - \ell\| \geq \epsilon\}| = 0$. In this case we write $st\text{-}\lim x_k = \ell$. Every convergent sequence is statistically convergent to the same limit, but the converse need not be true.

Statistical convergence in n -normed space has been studied by B.S. Reddy in [10].

Let $\{x_k\}$ be a sequence in n -normed space $(X, \|\bullet, \bullet, \dots, \bullet\|)$. The sequence $\{x_k\}$ is said to be statistically convergent to ℓ , if for every $\epsilon > 0$, the set $\{k \in N : \|x_k - \ell, z_2, z_3, \dots, z_n\| \geq \epsilon\}$ has natural density zero for each $z_2, z_3, \dots, z_n \neq 0, z_2, z_3, \dots, z_n \in X$. In other words $\{x_k\}$ statistically converges to ℓ in n -normed space $(X, \|\bullet, \bullet, \dots, \bullet\|)$ if $\lim_{m \rightarrow \infty} \frac{1}{m} |\{k \leq m : \|x_k - \ell, z_2, z_3, \dots, z_n\| \geq \epsilon\}| = 0$ for each $z_2, z_3, \dots, z_n \neq 0, z_2, z_3, \dots, z_n \in X$. It means that for every $z_2, z_3, \dots, z_n \in X$ we have $\|x_k - \ell, z_2, z_3, \dots, z_n\| < \epsilon$ for almost all k (for short a.a.k). In this case we write $st - \lim_{k \rightarrow \infty} \|x_k, z_2, z_3, \dots, z_n\| = \|\ell, z_2, z_3, \dots, z_n\|$.

Definition 4. Let $\{x_k\}$ be a sequence in fuzzy n -normed space $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$. Then the sequence $\{x_k\}$ is said to be statistically convergent to an element $x \in X$ with respect to the fuzzy n -norm on X if for every $\epsilon > 0$ and every $z_2, z_3, \dots, z_n \neq 0, z_2, z_3, \dots, z_n \in X$ we have $\delta(\{k \in N : D(\|x_k - x, z_2, z_3, \dots, z_n\|, \bar{0}) \geq \epsilon\}) = 0$ or equivalently $\delta(\{k \in N : D(\|x_k - x, z_2, z_3, \dots, z_n\|, \bar{0}) < \epsilon\}) = 1$. Which gives that for each $\epsilon > 0$ and $z_2, z_3, \dots, z_n \in X$, the set $K(\epsilon) = \{k \in N : \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ \geq \epsilon\}$ has natural density zero; namely for each $\epsilon > 0, \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ < \epsilon$ for almost all k . In this case we write $st(f - n - N) - \lim_{k \rightarrow \infty} \|x_k - x, z_2, z_3, \dots, z_n\| = 0$ either $st - \lim_{k \rightarrow \infty} \|x_k - x, z_2, z_3, \dots, z_n\| = 0$ or $(x_k, z_2, z_3, \dots, z_n) \xrightarrow{st(f-n-N)} (x, z_2, z_3, \dots, z_n)$.

In terms of neighborhoods, we have

$$(x_k, z_2, z_3, \dots, z_n) \xrightarrow{st(f-n-N)} (x, z_2, z_3, \dots, z_n)$$

if for every $\epsilon > 0, \delta(\{k \in N : x_k \notin N_x(\epsilon, 0)\}) = 0$, i.e., for each $\epsilon > 0, \{x_k\} \in N_x(\epsilon, 0)$ for almost all k .

A useful interpretation of the above definition is the following:

$$(x_k, z_2, z_3, \dots, z_n) \xrightarrow{st(f-n-N)} (x, z_2, z_3, \dots, z_n)$$

iff $st\text{-}\lim_{k \rightarrow \infty} \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ = 0$. Note that $st\text{-}\lim_{k \rightarrow \infty} \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ = 0$ implies that $st\text{-}\lim_{k \rightarrow \infty} \|x_k - x, z_2, z_3, \dots, z_n\|_\alpha^- = st\text{-}\lim_{k \rightarrow \infty} \|x_k - x, z_2, z_3, \dots, z_n\|_\alpha^+ = 0$ for each $\alpha \in [0, 1]$ since $0 \leq \|x_k - x, z_2, z_3, \dots, z_n\|_\alpha^- \leq \|x_k - x, z_2, z_3, \dots, z_n\|_\alpha^+ \leq \|x_k - x, z_2, z_3, \dots, z_n\|_0^+$ holds for every $k \in N$ and for each $\alpha \in [0, 1]$. Hence the result.

Definition 5. Let $\{x_k\}$ be a sequence in fuzzy n -normed space $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$. Then the sequence $\{x_k\}$ is said to be statistically Cauchy sequence with respect to the fuzzy n -norm on X if for every $\epsilon > 0$, $\exists$ a number $N = N(\epsilon, z_2, z_3, \dots, z_n)$ such that $\delta(\{k \in N : \|x_k - x_{N(\epsilon, z_2, z_3, \dots, z_n)}, z_2, z_3, \dots, z_n\|_0^+ \geq \epsilon\}) = 0$.

Theorem 6. Let $\{x_k\}, \{y_k\}$ be two sequences in fuzzy n -normed space $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$ such that

$$(x_k, z_2, z_3, \dots, z_n) \longrightarrow^{st(f-n-N)} (x, z_2, z_3, \dots, z_n)$$

and

$$(y_k, z_2, z_3, \dots, z_n) \longrightarrow^{st(f-n-N)} (y, z_2, z_3, \dots, z_n)$$

for all $x, y \in X$ and $z_2, z_3, \dots, z_n \neq 0, z_2, z_3, \dots, z_n \in X$ then:

- (i) $(x_k + y_k, z_2, z_3, \dots, z_n) \longrightarrow^{st(f-n-N)} (x + y, z_2, z_3, \dots, z_n)$.
- (ii) $(\alpha x_k, z_2, z_3, \dots, z_n) \longrightarrow^{st(f-n-N)} (\alpha x, z_2, z_3, \dots, z_n)$.
- (iii) $st(f - n - N) - \|(x_k, z_2, z_3, \dots, z_n) = \|(x, z_2, z_3, \dots, z_n)\|$.

Proof. (i) Let us assume that

$$(x_k, z_2, z_3, \dots, z_n) \longrightarrow^{st(f-n-N)} (x, z_2, z_3, \dots, z_n)$$

and

$$(y_k, z_2, z_3, \dots, z_n) \longrightarrow^{st(f-n-N)} (y, z_2, z_3, \dots, z_n).$$

Since $\|\bullet, \bullet, \dots, \bullet\|_0^+$ is a n -norm in the usual sense, we get

$$\begin{aligned} \|(x_k + y_k) - (x + y), z_2, z_3, \dots, z_n\|_0^+ &\leq \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ \\ &\quad + \|y_k - y, z_2, z_3, \dots, z_n\|_0^+, \quad (1) \end{aligned}$$

for all $k \in N$ and every $z_2, z_3, \dots, z_n \neq 0, z_2, z_3, \dots, z_n \in X$. Setting

$$\begin{aligned} K(\epsilon) &= \{k \in N : \|(x_k + y_k) - (x + y), z_2, z_3, \dots, z_n\|_0^+ \geq \epsilon\}, \\ K_1(\epsilon) &= \{k \in N : \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ \geq \frac{\epsilon}{2}\}, \end{aligned}$$

$$K_2(\epsilon) = \{k \in N : \|y_k - y, z_2, z_3, \dots, z_n\|_0^+ \geq \frac{\epsilon}{2}\}.$$

From (1) we have $K(\epsilon) \subseteq K_1(\epsilon) \cup K_2(\epsilon)$. Now by our assumption $\delta(K_1(\epsilon)) = \delta(K_2(\epsilon)) = 0$. Which gives $\delta(K(\epsilon)) = 0$, i.e.

$$(x_k + y_k, z_2, z_3, \dots, z_n) \xrightarrow{st(f-n-N)} (x + y, z_2, z_3, \dots, z_n).$$

(ii) This proof is very simple.

(iii) Since $\|\bullet, \bullet, \dots, \bullet\|_\alpha^-$ and $\|\bullet, \bullet, \dots, \bullet\|_\alpha^+$ are n -norms in the usual sense, we have

$$0 \leq | \|x_k, z_2, z_3, \dots, z_n\|_\alpha^- - \|x, z_2, z_3, \dots, z_n\|_\alpha^- | \leq \|x_k - x, z_2, z_3, \dots, z_n\|_\alpha^-$$

and

$$0 \leq | \|x_k, z_2, z_3, \dots, z_n\|_\alpha^+ - \|x, z_2, z_3, \dots, z_n\|_\alpha^+ | \leq \|x_k - x, z_2, z_3, \dots, z_n\|_\alpha^+,$$

for all $\alpha \in [0, 1]$. Therefore

$$0 \leq \max\{ | \|x_k, z_2, z_3, \dots, z_n\|_\alpha^- - \|x, z_2, z_3, \dots, z_n\|_\alpha^- |, | \|x_k, z_2, z_3, \dots, z_n\|_\alpha^+ - \|x, z_2, z_3, \dots, z_n\|_\alpha^+ | \} \leq \|x_k - x, z_2, z_3, \dots, z_n\|_\alpha^+,$$

for all $\alpha \in [0, 1]$. Taking supremum over $\alpha \in [0, 1]$, we get

$$0 \leq D(\|x_k, z_2, z_3, \dots, z_n\|, \|x, z_2, z_3, \dots, z_n\|) \leq \|x_k - x, z_2, z_3, \dots, z_n\|_0^+. \text{ Hence } st(f - n - N) - \|(x_k, z_2, z_3, \dots, z_n)\| = \|(x, z_2, z_3, \dots, z_n)\|. \quad \square$$

Lemma 7. Let $\{A_i : i \in I\}$ be a countable collection of subset of N such that $\delta(A_i) = 1$ for each $i \in I$. then there is a set $A \subset N$ such that $\delta(A) = 1$ and $|A \setminus A_i| < \infty$ for all $i \in I$.

Theorem 8. Let $\{x_k\}$ be a sequence in fuzzy n -normed space $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$. Then $\{x_k\}$ is statistically convergent to an element $x \in X$ with respect to the fuzzy n -norm on X if and only if $\{x_k\}$ is a sequence for which there is a sequence $\{y_k\}$ that is convergent such that $x_k = y_k$ for a.a.k.

Proof. Let us assume that $(x_k, z_2, z_3, \dots, z_n) \xrightarrow{st(f-n-N)} (x, z_2, z_3, \dots, z_n)$. For each $i \in N$, let $A_i = \{k \in N : \|(x_k - x, z_2, z_3, \dots, z_n)\|_0^+ \leq \frac{1}{i}\}$ and $\delta(A_i) = 1$ for each i , since $(x_k, z_2, z_3, \dots, z_n) \xrightarrow{st(f-n-N)} (x, z_2, z_3, \dots, z_n)$. Let A be as given in Lemma 3.4. For every $\epsilon > 0$, $\exists$ a number $N = N(\epsilon, z_2, z_3, \dots, z_n)$ such that $k \geq N$ and $k \in A$ imply $\|(x_k - x, z_2, z_3, \dots, z_n)\|_0^+ < \epsilon$. Define a sequence $\{y_k\}$ as $y_k = x_k$ for each $k \in A$ and $y_k = x$ for $k \notin A$.

This shows that the sequence $\{y_k\}$ is convergent to x with respect to the fuzzy n -norm on X such that $y_k = x_k$ for almost all k . conversely, assume that $y_k = x_k$ for almost all k and $(y_k, z_2, z_3, \dots, z_n) \xrightarrow{st(f-n-N)} (x, z_2, z_3, \dots, z_n)$. Let $\epsilon > 0$ be given. Then for each m , define the set $\{k \leq m : \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ \geq \epsilon\} \subseteq \{k \leq m : x_k \neq y_k\} \cup \{k \leq m : \|y_k - x, z_2, z_3, \dots, z_n\|_0^+ > \epsilon\}$. Since $(y_k, z_2, z_3, \dots, z_n) \xrightarrow{st(f-n-N)} (x, z_2, z_3, \dots, z_n)$, the set $\{k \leq m : \|y_k - x, z_2, z_3, \dots, z_n\|_0^+ > \epsilon\}$ contains a finite number of elements, say

$$p = p(\epsilon, z_2, z_3, \dots, z_n).$$

Therefore

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{1}{m} |\{k \leq m : \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ \geq \epsilon\}| \\ \leq \lim_{m \rightarrow \infty} \frac{1}{m} |\{k \leq m : x_k \neq y_k\}| + \lim_{m \rightarrow \infty} \frac{p}{m} = 0. \end{aligned}$$

Since $x_k = y_k$ for almost all k . Hence $\|x_k - x, z_2, z_3, \dots, z_n\|_0^+ < \epsilon$ for almost all k . Hence the sequence $\{x_k\}$ is statistically convergent with respect to the fuzzy n -norm on X . $\square$

Theorem 9. *Let $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$ be a fuzzy n -normed space. Then every statistically convergent sequence $\{x_k\}$ is statistically Cauchy sequence with respect to the fuzzy n -norm on X .*

Proof. Suppose that $(x_k, z_2, z_3, \dots, z_n) \xrightarrow{st(f-n-N)} (x, z_2, z_3, \dots, z_n)$. Then for given $\epsilon > 0$ we have $\|x_k - x, z_2, z_3, \dots, z_n\|_0^+ < \frac{\epsilon}{2}$ for a.a.k. Choose $N = N(\epsilon, z_2, z_3, \dots, z_n) \in \mathbb{N}$ such that $\|x_{N(\epsilon, z_2, z_3, \dots, z_n)} - x, z_2, z_3, \dots, z_n\|_0^+ < \frac{\epsilon}{2}$. Now $\|\bullet, \bullet, \dots, \bullet\|_0^+$ being a n -norm in the usual sense, we get

$$\begin{aligned} & \|x_k - x_{N(\epsilon, z_2, z_3, \dots, z_n)}, z_2, z_3, \dots, z_n\|_0^+ \\ &= \|(x_k - x) + (x - x_{N(\epsilon, z_2, z_3, \dots, z_n)}), z_2, z_3, \dots, z_n\|_0^+ \\ &\leq \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ + \|x_{N(\epsilon, z_2, z_3, \dots, z_n)} - x, z_2, z_3, \dots, z_n\|_0^+ < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

for almost all k . Hence $\{x_k\}$ is statistically Cauchy sequence with respect to the fuzzy n -norm on X . $\square$

Theorem 10. *Let $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$ be a fuzzy n -normed space. Let $\{x_k\}$ be a sequence in X and denote $E_{N(\epsilon, z_2, z_3, \dots, z_n)} = \{k \in \mathbb{N} : \|x_k - x_{N(\epsilon, z_2, z_3, \dots, z_n)}, z_2, z_3, \dots, z_n\|_0^+ \geq \epsilon\}$. If the sequence $\{x_k\}$ is statistically Cauchy sequence, then for every $\epsilon > 0$, $\exists A \subset \mathbb{N}$ with $\delta(A) = 0$ such that $\|x_m - x_k, z_2, z_3, \dots, z_n\|_0^+ < \epsilon \forall m, k \notin A$.*

Proof. For a given $\epsilon > 0$, write $A = E_{N(\frac{\epsilon}{2}, z_2, z_3, \dots, z_n)}$. Since the sequence $\{x_k\}$ is statistically Cauchy sequence, we can write $\delta(A) = 0$. Then for any $m, k \notin A$, we have

$$\|x_k - x_{N(\epsilon, z_2, z_3, \dots, z_n)}, z_2, z_3, \dots, z_n\|_0^+ < \frac{\epsilon}{2}$$

and

$$\|x_m - x_{N(\epsilon, z_2, z_3, \dots, z_n)}, z_2, z_3, \dots, z_n\|_0^+ < \frac{\epsilon}{2}.$$

Hence $\|x_m - x_k, z_2, z_3, \dots, z_n\|_0^+ < \epsilon, \forall m, k \notin A$. $\square$

Definition 11. A fuzzy n -norm $\|\bullet, \bullet, \dots, \bullet\|$ on a vector space X is said to be fuzzy equivalent to a fuzzy n -norm $\|\bullet, \bullet, \dots, \bullet\|$ written as $\|\bullet, \bullet, \dots, \bullet\| \cong \|\bullet, \bullet, \dots, \bullet\|$ on X if $\exists \mu, \nu \in L(R)$ and $\mu, \nu \succ \bar{0}$ such that $\forall x \in X$ and every $z_2, z_3, \dots, z_n \neq 0, z_2, z_3, \dots, z_n \in X$,

$$\mu \bigotimes \|x, z_2, z_3, \dots, z_n\| \preceq \|\bullet, \bullet, \dots, \bullet\| \preceq \nu \bigotimes \|x, z_2, z_3, \dots, z_n\|.$$

Theorem 12. Let X be a vector space over the real field R and let $\|\bullet, \bullet, \dots, \bullet\|$ and $|\bullet, \bullet, \dots, \bullet|$ be fuzzy equivalent fuzzy n -norms on X . Let $\{x_k\}$ be a sequence in X . Then:

(i) $\{x_k\}$ is statistically convergent to x in $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$ if and only if $\{x_k\}$ is statistically convergent to x in $(X, |\bullet, \bullet, \dots, \bullet|, L, R)$.

(ii) $\{x_k\}$ is statistically Cauchy sequence in $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$ if and only if $\{x_k\}$ is statistically Cauchy sequence in $(X, |\bullet, \bullet, \dots, \bullet|, L, R)$.

Proof. (i) Let the sequence $\{x_k\}$ be statistically convergent to x in $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$. Since $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$ and $(X, |\bullet, \bullet, \dots, \bullet|, L, R)$ are fuzzy equivalent, $\exists \mu, \nu \in L(R)$ and $\mu, \nu \succ \bar{0}$ such that

$$\begin{aligned} \mu \bigotimes \|x_k - x, z_2, z_3, \dots, z_n\| &\preceq \|\bullet, \bullet, \dots, \bullet\| \\ &\preceq \nu \bigotimes \|x_k - x, z_2, z_3, \dots, z_n\|, \end{aligned}$$

$\forall x_k, x \in X$ and $z_2, z_3, \dots, z_n \in X$. Thus

$$\begin{aligned} \mu_0^+ \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ &\leq \|\bullet, \bullet, \dots, \bullet\|_0^+ \\ &\leq \nu_0^+ \|x_k - x, z_2, z_3, \dots, z_n\|_0^+, \end{aligned}$$

$\forall k \in N$. By our supposition, we have

$$st(f - n - N) - \lim_{k \rightarrow \infty} \|(x_k - x, z_2, z_3, \dots, z_n)\|_0^+ = 0.$$

Hence

$$st(f - n - N) - \lim_{k \rightarrow \infty} \| (x_k - x, z_2, z_3, \dots, z_n) \|_0^+ = 0,$$

i.e.

$$(x_k, z_2, z_3, \dots, z_n) \xrightarrow{st(f-n-N)} (x, z_2, z_3, \dots, z_n)$$

in $(X, \| \bullet, \bullet, \dots, \bullet \|, L, R)$.

In the same manner if $(x_k, z_2, z_3, \dots, z_n) \xrightarrow{st(f-n-N)} (x, z_2, z_3, \dots, z_n)$ in $(X, \| \bullet, \bullet, \dots, \bullet \|, L, R)$ then $(x_k, z_2, z_3, \dots, z_n) \xrightarrow{st(f-n-N)} (x, z_2, z_3, \dots, z_n)$ in $(X, \| \bullet, \bullet, \dots, \bullet \|, L, R)$.

(ii) Let the sequence $\{x_k\}$ be statistically Cauchy sequence in $(X, \| \bullet, \bullet, \dots, \bullet \|, L, R)$. Since $(X, \| \bullet, \bullet, \dots, \bullet \|, L, R)$ and $(X, \| \bullet, \bullet, \dots, \bullet \|, L, R)$ are fuzzy equivalent, $\exists \mu, \nu \in L(R)$ and $\mu, \nu \succ \bar{0}$ such that

$$\mu_0^+ \|x_k, z_2, z_3, \dots, z_n\|_0^+ \leq \|x_k, z_2, z_3, \dots, z_n\|_0^+ \leq \nu_0^+ \|x_k, z_2, z_3, \dots, z_n\|_0^+,$$

$\forall x \in X$ and $z_2, z_3, \dots, z_n \in X$. For any $\epsilon > 0$, $\exists N = N(\epsilon, z_2, z_3, \dots, z_n) \in N$ such that $\|x_k - x_{N(\epsilon, z_2, z_3, \dots, z_n)}, z_2, z_3, \dots, z_n\|_0^+ < \frac{\epsilon}{\nu_0^+}$ for almost all k . Hence

$$\begin{aligned} & \|x_k - x_{N(\epsilon, z_2, z_3, \dots, z_n)}, z_2, z_3, \dots, z_n\|_0^+ \\ & \leq \nu_0^+ \|x_k - x_{N(\epsilon, z_2, z_3, \dots, z_n)}, z_2, z_3, \dots, z_n\|_0^+ < \epsilon, \end{aligned}$$

for a.a.k. Therefore the sequence $\{x_k\}$ is statistically Cauchy sequence in $(X, \| \bullet, \bullet, \dots, \bullet \|, L, R)$. In the same manner if $\{x_k\}$ is statistically Cauchy sequence in $(X, \| \bullet, \bullet, \dots, \bullet \|, L, R)$ then it is statistically Cauchy sequence in $(X, \| \bullet, \bullet, \dots, \bullet \|, L, R)$. $\square$

4. Statistical Limit Point and Statistical Cluster Point in Fuzzy n -Normed Spaces

Statistical limit point of sequence $\{x_k\}$ has been define and studied by Fridy[14] and for fuzzy number by Aytar[9]. In this section, we define the notions of thin subsequence, non-thin subsequence, statistical limit point and statistical cluster point in fuzzy n -normed space.

Definition 13. Let $(X, \| \bullet, \bullet, \dots, \bullet \|, L, R)$ be fuzzy n -normed space and (x_k) be a sequence in $(X, \| \bullet, \bullet, \dots, \bullet \|, L, R)$. An element $x \in X$ is said to be a limit point of the sequence (x_k) if there is a subsequence of (x_k) which converges to x with respect to the fuzzy n -norm on X . The set of all limit points of the sequence (x_k) is denoted by $L_{(f-n-N)}(x_k)$.

Definition 14. Let (x_k) be a sequence in fuzzy n -normed space $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$ and (x_{k_j}) be a subsequence of (x_k) . Write $K = \{k_j : j \in N\}$. If $\delta(K) = 0$ then we say that (x_{k_j}) is a thin subsequence of (x_k) . A subsequence (x_{k_j}) is said to be a non-thin subsequence of (x_k) if $\delta(K) > 0$ or $\delta(K)$ does not exists, namely, $\bar{\delta}(K) > 0$

Definition 15. Let $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$ be fuzzy n -normed space and (x_k) be a sequence in $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$. An element $x \in X$ is said to be statistical limit point of the sequence (x_k) if $\exists$ a non-thin subsequence of (x_k) which converges to x with respect to the fuzzy n -norm on X . The set of all statistical limit points of the sequence (x_k) is denoted by $SL_{(f-n-N)}(x_k)$.

Definition 16. Let $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$ be fuzzy n -normed space and (x_k) be a sequence in $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$. An element $x \in X$ is said to be statistical cluster point of the sequence (x_k) with respect to the fuzzy n -norm on X if for every $\epsilon > 0$ and $z_2, z_3, \dots, z_n \in X$, $\bar{\delta}(\{k \in N : \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ < \epsilon\}) > 0$. The set of all statistical cluster points of the sequence (x_k) is denoted by $SC_{(f-n-N)}(x_k)$.

Remark 17. An element $x \in SC_{(f-n-N)}(x_k)$ implies that $\bar{\delta}(\{k \in N : \|x_k - x, z_2, z_3, \dots, z_n\|_\alpha^+ < \epsilon\}) > 0$ and $\bar{\delta}(\{k \in N : \|x_k - x, z_2, z_3, \dots, z_n\|_\alpha^- < \epsilon\}) > 0 \forall \epsilon > 0, \alpha \in [0, 1]$ and $z_2, z_3, \dots, z_n \in X$.

Theorem 18. Let $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$ be fuzzy n -normed space. Then for every sequence (x_k) in X , we have $SL_{(f-n-N)}(x_k) \subseteq SC_{(f-n-N)}(x_k) \subseteq L_{(f-n-N)}(x_k)$.

Proof. Let $x \in SL_{(f-n-N)}(x_k)$. Then $\exists$ a non-thin sub sequence (x_{k_j}) of the sequence (x_k) which converges to x , namely, $\bar{\delta}(\{k_j : j \in N\}) = d > 0$. Since $\{k \in N : \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ < \epsilon\} \supseteq \{k_j \in N : \|x_{k_j} - x, z_2, z_3, \dots, z_n\|_0^+ < \epsilon\}$ for every $\epsilon > 0$ and so $\{k \in N : \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ < \epsilon\} \supseteq \{k_j : j \in N\} \setminus \{k_j \in N : \|x_{k_j} - x, z_2, z_3, \dots, z_n\|_0^+ \geq \epsilon\}$. Since

$$(x_{k_j}, z_2, z_3, \dots, z_n) \xrightarrow{st(f-n-N)} (x, z_2, z_3, \dots, z_n),$$

the set $\{k_j \in N : \|x_{k_j} - x, z_2, z_3, \dots, z_n\|_0^+ \geq \epsilon\}$ is finite for any $\epsilon > 0$, hence we have $\bar{\delta}(\{k \in N : \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ < \epsilon\}) \geq \bar{\delta}(\{k_j : j \in N\}) - \bar{\delta}(\{k_j \in N : \|x_{k_j} - x, z_2, z_3, \dots, z_n\|_0^+ \geq \epsilon\}) = d > 0$. Thus, for any $\epsilon > 0$ and $z_2, z_3, \dots, z_n \in X$, $\bar{\delta}(\{k \in N : \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ < \epsilon\}) > 0$, i.e., $x \in SC_{(f-n-N)}(x_k)$.

Let $x \in SC_{(f-n-N)}(x_k)$. For every $\epsilon > 0$ and $z_2, z_3, \dots, z_n \in X$, write $\bar{\delta}(\{k \in N : \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ < \epsilon\}) > 0$ which gives every $(\epsilon, 0)$ -

neighborhood of x contains infinitely many terms of the sequence (x_k) . i.e., $x \in L_{(f-n-N)}(x_k)$. Therefore $SL_{(f-n-N)}(x_k) \subseteq SC_{(f-n-N)}(x_k) \subseteq L_{(f-n-N)}(x_k)$. $\square$

Theorem 19. Let $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$ be fuzzy n -normed space and (x_k) be a sequence in $(X, \|\bullet, \bullet, \dots, \bullet\|, L, R)$. Then

$$SL_{(f-n-N)}(x_k) = SC_{(f-n-N)}(x_k) = \{x\},$$

if $(x_k, z_2, z_3, \dots, z_n) \longrightarrow^{st(f-n-N)} (x, z_2, z_3, \dots, z_n)$.

Proof. Let $(x_k, z_2, z_3, \dots, z_n) \longrightarrow^{st(f-n-N)} (x, z_2, z_3, \dots, z_n)$. Therefore $x \in SC_{(f-n-N)}(x_k)$. Let us assume that $\exists$ at least one $y \in SC_{(f-n-N)}(x_k)$ such that $y \neq x$. For every $\epsilon > 0$ and every $z_2, z_3, \dots, z_n \neq 0$, $z_2, z_3, \dots, z_n \in X$ such that $\{k \in N : \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ \geq \epsilon\} \supseteq \{k \in N : \|x_k - y, z_2, z_3, \dots, z_n\|_0^+ < \epsilon\}$ is true. Therefore $\bar{\delta}(\{k \in N : \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ \geq \epsilon\}) \geq \bar{\delta}(\{k \in N : \|x_k - y, z_2, z_3, \dots, z_n\|_0^+ < \epsilon\})$. Since

$$(x_k, z_2, z_3, \dots, z_n) \longrightarrow^{st(f-n-N)} (x, z_2, z_3, \dots, z_n),$$

we have

$$\delta(\{k \in N : \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ \geq \epsilon\}) = 0.$$

This gives $\bar{\delta}(\{k \in N : \|x_k - x, z_2, z_3, \dots, z_n\|_0^+ \geq \epsilon\}) = 0$. Thus

$$\bar{\delta}(\{k \in N : \|x_k - y, z_2, z_3, \dots, z_n\|_0^+ < \epsilon\}) = 0.$$

This is a contradiction to our assumption $y \in SC_{(f-n-N)}(x_k)$. Therefore $SC_{(f-n-N)}(x_k) = \{x\}$. But from hypothesis $(x_k, z_2, z_3, \dots, z_n) \longrightarrow^{st(f-n-N)} (x, z_2, z_3, \dots, z_n)$. By Theorem 8 and definition 15, we get $x \in SL_{(f-n-N)}(x_k)$.

Now Theorem 18 yields $SL_{(f-n-N)}(x_k) = SC_{(f-n-N)}(x_k) = \{x\}$. $\square$

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