ON ADOMIAN'S DECOMPOSITION METHOD FOR SOLVING A FRACTIONAL ADVECTION-DISPERSION EQUATION

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Abstract: In this paper, the Adomian's decomposition method (ADM) is considered to solve a fractional advection-dispersion model. This model can be represented if the first order derivative in time is replaced by the Caputo fractional derivative of order α ($0 < \alpha \leq 1$). In addition, the space derivative orders are replaced by the alternative orders $0 < \beta \leq 1$ and $1 < \gamma \leq 2$. The obtained solutions are formulated in a convergent infinite series in terms of Mittag-Leffler functions. Finally, two illustrative examples are introduced to ensure the effectiveness of the used method.

AMS Subject Classification: 34A08, 34K37

Key Words: fractional calculus, Advection-dispersion equation, Adomian's decomposition method, Mittag-Leffler function

1. Introduction

It is well known that, the advection-dispersion equation(ADE) is the mathemat-

Received: March 24, 2015

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ical model that has commonly been applied to describe the Brownian motion of particles[1]. (ADE) is used to describe the transport and spreading of substances (e.g. solute) or conserved property existing in a fluid due to the fluid's bulk motion. The classical (ADE) for the motion of a concentration field $C(\xi, \tau)$ of one space variable ξ at time τ has the form (see [2])

$$\frac{\partial C(\xi, \tau)}{\partial \tau} = \left[D \frac{\partial^2}{\partial \xi^2} - V \frac{\partial}{\partial \xi} \right] C(\xi, \tau), \quad (1)$$

where D and V are positive constants representing diffusion and drift coefficients respectively. The diffusion and drift coefficient may also be functions depending on the space variable ξ and the time τ to be in the form of Fokker-Planck equation(FPE)[3]. For some applications of this equation, see [4, 5, 6]. Fractional derivatives play key role in modeling of particle transport in diffusion including the fractional Advection-dispersion equation(FADE)[3]. Thus, (FADE) may be introduced to provide a good simulation for diffusion. Some special forms of the fractional (ADE) were solved analytically in [7]. Rocca *et al.* [8] considered the fractional diffusion-advection equation for solar cosmic-ray transport and gave its general solution. The authors in [9] presented a level IV fugacity model coupled to a dispersion-advection equation to simulate the environmental concentration of a pesticide in rice fields. Zhuang and Liu [10] proposed an implicit difference approximation for the time fractional diffusion equation and gave error. Liu *et al.* [11] considered the space fractional Fokker-Planck equation with instantaneous source and presented a fractional method of lines. The authors in [12] developed practical numerical methods to solve one dimensional fractional advection-dispersion equations with variable coefficients on a finite domain. A. El-Sayed *et al.* [13] concerned with a model that describes the intermediate process between advection and dispersion via fractional derivative, an analytic approximate solution for that model is obtained by the powerful method called Adomian decomposition method(ADM).

Adomian decomposition method (ADM) [14, 15] is a mathematical technique for solving large classes of ordinary and partial differential equations that gives an analytic approximate solution. George Adomian(1923-1996) is the first one who introduced this method. The basic idea of the (ADM) is based on representing the solution as infinite series. In fact, our actual problem that we can face is the convergence of that approximate solution to the exact solution. The convergence analysis of the method was studied by many researchers, for example see [16, 17]. Sufficient conditions for convergence are given in [18, 19]. Applying the (ADM) for solving Fractional differential equations (FDEs) comes from the method applied to the linear and nonlinear differential equations with

integer derivatives. The application of the method to (FDEs) has been established in [20]. For solving a system of fractional differential equations by (ADM) has been investigated in [21, 22].

In this work, the fractional derivative in the Caputo sense is considered. Fractional advection-dispersion model is discussed for general case of the fractional derivative order. This paper is organized to have some important preliminaries in section 2. Section 3 deals with reviewing the basic idea of (ADM) briefly. Section 4 is concerned with applying the (ADM) to find the general solution of the (FADE) for different three cases of arbitrary fractional orders of derivatives. Finally, some illustrative examples are given in section 5.

2. Preliminaries

In this section we review some basic definitions in fractional calculus, and some of its properties [23].

Definition 1. The function $f(x)$ is said to be in the space $L_1[a, b]$ if $f(x)$ is Lebesgue integrable on $[a, b]$.

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f(x) \in L_1$ is defined by

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt$$

where Γ denotes the gamma function. When $a = 0$, we can write $I_a^\alpha \rightarrow I^\alpha$. Some properties of the operator I_a^α can be found in [24].

For $f(x) \in L_1, \alpha, \beta \geq 0$ and $\gamma > -1$:

1. $I_a^\alpha I_a^\beta f(x) = I_a^\beta I_a^\alpha f(x)$;
2. $I_a^\alpha I_a^\beta f(x) = I_a^{\alpha+\beta} f(x)$;
3. $I_a^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} x^{\alpha+\gamma}$.

Definition 3. The fractional derivative of order $\alpha (m-1 < \alpha \leq m)$ in the Caputo sense for a function $f(x) \in L_1$ is defined as

$$D_a^\alpha f(x) = I^{m-\alpha} \frac{d^m}{dx^m} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} \frac{d^m}{dt^m} f(t) dt.$$

Lemma 1. For $\alpha \in (m-1, m)$, the Caputo fractional derivative of the power function x^p is given by

$$D^\alpha x^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha} & \text{if } p > m-1, \\ 0 & \text{if } p \leq m-1. \end{cases}$$

Lemma 2. For $\alpha \in (m-1, m)$, $m \in N$ and $f(x) \in L_1$, then:

- $D_a^\alpha I_a^\alpha f(x) = f(x)$,
- $I_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{k=0}^m \frac{f^{(k)}(a)}{k!} (x-a)^k$.

Definition 4. A two-parameter function of the Mittag-Leffler type is defined by the series expansion (see [24]):

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

It follows from Definition 2.4 that the one parameter Mittag-Leffler function is

$$E_\alpha(z) = E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

3. The Basic Idea of the ADM

In reviewing the basic idea of the used method [14, 15]. Consider a nonlinear differential equation

$$Lu + Ru + Nu = g(x), \quad (2)$$

where $L = D^n$, $D = \frac{d}{dx}$, and n represents the highest order of the linear operator, R is the remaining linear operator grouping the lower derivatives which can be written as $R = \sum_{k=0}^{n-1} a_k(x) \frac{d^k}{dx^k}$ and Nu represents the nonlinear term in equation (2). For the initial value problems, integrating (2) by the operator L^{-1} from 0 to x yields

$$u(x) = \sum_{k=0}^{n-1} \frac{\partial^k u(0)}{\partial x^k} - L^{-1}Ru - L^{-1}Nu + L^{-1}g(x). \quad (3)$$

The method is based on setting the solution in an infinite series to be in the form

$$u(x) = \sum_{n=0}^{\infty} u_n, \quad (4)$$

and the nonlinear operator can be decomposed as

$$Nu = \sum_{n=0}^{\infty} A_n,$$

where A_n depends upon $u_0, u_1, \dots, u_n$, are called Adomian polynomials which are defined by

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} [N(\sum_{n=0}^{\infty} \lambda^n u_n)]_{\lambda=0}. \quad (5)$$

The u_n 's are determined recursively as

$$u_0 = \sum_{k=0}^{n-1} \frac{\partial^k u(0)}{\partial x^k} + L^{-1}g(x)$$

$$u_n = -L^{-1}Ru_{n-1} - L^{-1}A_{n-1}. \quad (6)$$

Making use equation (5) and equation (6), the required solution (4) of the nonlinear differential equation (2) is obtained.

4. Space-Time Fractional Advection-Dispersion Equation

Numerous models for fractional advection-dispersion equation(FADE) has been introduced by many researchers (see [2, 3, 8, 10, 11] and references therein).

In this work we introduce the fractional advection-dispersion equation(FADE) in the general fractional form

$$\frac{\partial^\alpha u}{\partial \tau^\alpha} = D \frac{\partial^\gamma u}{\partial \xi^\gamma} - V \frac{\partial^\beta u}{\partial \xi^\beta}, \quad 0 < \xi < L, \quad (7)$$

with $0 < \alpha \leq 1$, $\beta > 0$, and $\gamma > 0$. For simplicity of the equation, we set $t = \tau V$, $x = \xi$ and $C(\xi, \tau)$ is replaced by $u(x, t)$ to become in the form

$$D_t^\alpha u = \mu D_x^\gamma u - D_x^\beta u, \quad x, t > 0, \quad (8)$$

where $\mu = \frac{D}{V}$, $D_t = \frac{\partial}{\partial t}$, and $D_x = \frac{\partial}{\partial x}$. with the initial condition

$$u(x, 0) = f(x) \quad (9)$$

Our main aim is to get solution for the (FADE) modeled in (8) with the initial condition given in (9) for different three cases. Case I is concerned with

an exponential function as initial condition and for arbitrary fractional order $0 < \alpha \leq 1$ and $\beta = 1$, $\gamma = 2$. Considering $\gamma = 2\beta$ and any function $f(x)$ for the initial condition (9), we get the solution for equation (8) in case II. Finally, case III is dealt with the space-time fractional (ADE) (8) for any arbitrary fractional order $0 < \alpha, \beta \leq 1$ and $1 < \gamma \leq 2$ under the general initial condition $u(x, 0) = f(x)$.

4.1. Case I: Time fractional ADE ($\beta = 1$, $\gamma = 2$)

Consider the time fractional ADE which has been discussed by [25].

$$D_t^\alpha u = \mu D_x^2 u - D_x u, \quad x, t > 0, \quad (10)$$

with the initial condition

$$u(x, 0) = e^{-x} \quad (11)$$

equation (10) can be written in the operator form as

$$u(x, t) = u(x, 0) + \mu I^\alpha (D_x^2 u) - I^\alpha (D_x u) \quad (12)$$

Applying the Adomian decomposition method, the solution $u(x, t)$ can be given by the infinite series

$$u(x, t) = \sum_{m=0}^{\infty} u_m, \quad (13)$$

where each term of the series (13) will be determined recursively. So, the recursive relation can be written as

$$\begin{aligned} u_0 &= u(x, 0) = e^{-x} \\ u_{n+1} &= \mu I^\alpha (D_x^2 u_n) - I^\alpha (D_x u_n) \end{aligned} \quad (14)$$

So, the m th term takes the form

$$u_m = \frac{t^{m\alpha}}{\Gamma(m\alpha + 1)} (1 + \mu)^m e^{-x} \quad (15)$$

Substituting (15) into (13), we have the required solution as

$$\begin{aligned} u(x, t) &= e^{-x} \sum_{m=0}^{\infty} \frac{t^{m\alpha}}{\Gamma(m\alpha + 1)} (1 + \mu)^m \\ &= e^{-x} E_\alpha((1 + \mu)t^\alpha), \end{aligned} \quad (16)$$

where $E_\alpha(x)$ denotes the Mittag-Leffler Function in one parameter, see [24].

Figure 1: The concentration $u(x, t)$ of equation (16).

Remark 1. If $\alpha = 1$, the solution (16) converges to the exact solution $e^{(1+\mu)t-x}$ which is the same result obtained by El-Sayed in [13].

In Figure 1, the concentration $u(x, t)$ is drawn for $\mu = 0.01$ and $\alpha = 1, 0.5, 0.25$ respectively.

4.2. Case II: Space-time fractional ADE ($\gamma = 2\beta$)

Consider the space-time fractional ADE

$$D_t^\alpha u = \mu D_x^{2\beta} u - D_x^\beta u, \quad x, t > 0, \quad (17)$$

with the initial condition (9). So, the equivalent Adomian's recurrence relation of equation (17) is

$$u_{n+1} = \mu I^\alpha (D_x^{2\beta} u_n) - I^\alpha (D_x^\beta u_n). \quad (18)$$

Applying this recurrence relation, we can get the first four terms of u_n as follows

$$\begin{aligned} u_1 &= \frac{t^\alpha}{\Gamma(\alpha + 1)} [\mu D_x^{2\beta} f(x) - D_x^\beta f(x)], \\ u_2 &= \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} [\mu^2 D_x^{4\beta}(f) - 2\mu D_x^{3\beta}(f) + D_x^{2\beta}(f)], \\ u_3 &= \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} [\mu^3 D_x^{6\beta}(f) - 3\mu^2 D_x^{5\beta}(f) + 3\mu D_x^{4\beta}(f) - D_x^{3\beta}(f)], \\ u_4 &= \frac{t^{4\alpha}}{\Gamma(4\alpha + 1)} [\mu^4 D_x^{8\beta}(f) - 4\mu^3 D_x^{7\beta}(f) + 6\mu^2 D_x^{6\beta}(f) - 4\mu D_x^{5\beta}(f) + D_x^{4\beta}(f)]. \end{aligned}$$

Consequently, the m -th term can be calculated as

$$u_m = \frac{t^{m\alpha}}{\Gamma(m\alpha + 1)} \sum_{p=0}^m (-1)^p \binom{m}{p} \mu^{m-p} D_x^{(2m-p)\beta}(f(x)). \quad (19)$$

Combining (19) with (13) we obtain the solution of equation (17) and (18a) as follows

$$u(x, t) = \sum_{m=0}^{\infty} \frac{t^{m\alpha}}{\Gamma(m\alpha + 1)} \sum_{p=0}^m (-1)^p \binom{m}{p} \mu^{m-p} D_x^{(2m-p)\beta} (f(x)). \quad (20)$$

Remark 2. As a particular situation, let $f(x) = e^{-x}$ then we have

$$u(x, t) = \sum_{m=0}^{\infty} \frac{t^{m\alpha}}{\Gamma(m\alpha + 1)} \sum_{p=0}^m (-1)^p \binom{m}{p} \mu^{m-p} \times \sum_{k=\lceil (2m-p)\beta \rceil}^{\infty} \frac{(-1)^k x^{k-(2m-p)\beta}}{\Gamma(k - (2m-p)\beta + 1)} \quad (21)$$

where $\lceil x \rceil$ denotes the ceiling function.

Remark 3. Setting $\beta = 1$, the solution (20) with condition $u(x, 0) = e^{-x}$ reduces to the solution (16) of case I.

Proof. Substituting $\beta = 1$ in (21) yields

$$u(x, t) = \sum_{m=0}^{\infty} \frac{t^{m\alpha}}{\Gamma(m\alpha + 1)} \sum_{p=0}^m (-1)^p \binom{m}{p} \mu^{m-p} \times \sum_{k=\lceil (2m-p) \rceil}^{\infty} \frac{(-1)^k x^{k-(2m-p)}}{\Gamma(k - (2m-p) + 1)}. \quad (22)$$

Let $k - (2m - p) = s$, then the solution (22) is reduced to

$$\begin{aligned} u(x, t) &= \sum_{m=0}^{\infty} \left[\frac{t^{m\alpha}}{\Gamma(m\alpha + 1)} \sum_{p=0}^m \binom{m}{p} \mu^{m-p} \sum_{s=0}^{\infty} \frac{(-1)^s x^s}{\Gamma(s + 1)} \right] \\ &= \sum_{m=0}^{\infty} \left[\frac{t^{m\alpha}}{\Gamma(m\alpha + 1)} \sum_{p=0}^m \binom{m}{p} \mu^{m-p} e^{-x} \right] \\ &= e^{-x} E_{\alpha}((1 + \mu)t^{\alpha}). \end{aligned}$$

4.3. Case III: Space-time fractional ADE ($0 < \beta \leq 1$ and $1 < \gamma \leq 2$)

In this case we solve equation (8) for any arbitrary fractional order $0 < \alpha, \beta \leq 1$ and $1 < \gamma \leq 2$ under the initial condition $u(x, 0) = f(x)$.

So, the first four terms of the Adomian's recursive relation can be written as

$$\begin{aligned}
 u_1 &= \frac{t^\alpha}{\Gamma(\alpha+1)} [\mu D_x^\gamma f(x) - D_x^\beta f(x)] = \frac{t^\alpha}{\Gamma(\alpha+1)} [\mu D_x^\gamma - D_x^\beta] f(x), \\
 u_2 &= \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} [\mu^2 D_x^{2\gamma}(f) - 2\mu D_x^{\gamma+\beta}(f) + D_x^{2\beta}(f)] \\
 &= \frac{t^\alpha}{\Gamma(\alpha+1)} [\mu D_x^\gamma - D_x^\beta]^2 f(x), \\
 u_3 &= \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} [\mu^3 D_x^{3\gamma}(f) - 3\mu^2 D_x^{2\gamma+\beta}(f) + 3\mu D_x^{\gamma+2\beta}(f) - D_x^{3\beta}(f)] \\
 &= \frac{t^\alpha}{\Gamma(\alpha+1)} [\mu D_x^\gamma - D_x^\beta]^3 f(x), \\
 u_4 &= \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} [\mu^4 D_x^{4\gamma}(f) - 4\mu^3 D_x^{3\gamma+\beta}(f) \\
 &\quad + 6\mu^2 D_x^{2\gamma+2\beta}(f) - 4\mu D_x^{\gamma+3\beta}(f) + D_x^{4\beta}(f)] \\
 &= \frac{t^\alpha}{\Gamma(\alpha+1)} [\mu D_x^\gamma - D_x^\beta]^4 f(x).
 \end{aligned}$$

$$\begin{aligned}
 u_1 &= \frac{t^\alpha}{\Gamma(\alpha+1)} [\mu D_x^\gamma f(x) - D_x^\beta f(x)] \\
 &= \frac{t^\alpha}{\Gamma(\alpha+1)} [\mu D_x^\gamma - D_x^\beta] f(x), \\
 u_2 &= \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} [\mu^2 D_x^{2\gamma}(f) - 2\mu D_x^{\gamma+\beta}(f) + D_x^{2\beta}(f)] \\
 &= \frac{t^{2\alpha}}{\Gamma(\alpha+1)} [\mu D_x^\gamma - D_x^\beta]^2 f(x), \\
 u_3 &= \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} [\mu^3 D_x^{3\gamma}(f) - 3\mu^2 D_x^{2\gamma+\beta}(f) + 3\mu D_x^{\gamma+2\beta}(f) - D_x^{3\beta}(f)] \\
 &= \frac{t^{3\alpha}}{\Gamma(\alpha+1)} [\mu D_x^\gamma - D_x^\beta]^3 f(x), \\
 u_4 &= \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} [\mu^4 D_x^{4\gamma}(f) - 4\mu^3 D_x^{3\gamma+\beta}(f) + 6\mu^2 D_x^{2\gamma+2\beta}(f) \\
 &\quad - 4\mu D_x^{\gamma+3\beta}(f) + D_x^{4\beta}(f)] \\
 &= \frac{t^{4\alpha}}{\Gamma(\alpha+1)} [\mu D_x^\gamma - D_x^\beta]^4 f(x).
 \end{aligned}$$

In view of equation (13), we have

$$u(x, t) = \sum_{m=0}^{\infty} \frac{t^{m\alpha}}{\Gamma(m\alpha + 1)} \sum_{p=0}^m (-1)^p \binom{m}{p} \mu^{m-p} D_x^{(m-p)\gamma+p\beta} (f(x)), \quad (23)$$

which also can be obtained in the equivalent form

$$u(x, t) = [E_{\alpha}(t^{\alpha}(\mu D_x^{\gamma} - D_x^{\beta}))]f(x). \quad (24)$$

Remark 4. Setting $\gamma = 2\beta$, the solution (24) reduces to equation (20).

Proof. $\gamma = 2\beta$, the solution (24) is equivalent to

$$u(x, t) = [E_{\alpha}(t^{\alpha}(\mu D_x^{2\beta} - D_x^{\beta}))]f(x) = [E_{\alpha}(t^{\alpha} D_x^{\beta}(\mu D_x^{\beta} - 1))]f(x).$$

By Definition 2.4 and the binomial theorem, it easy to see

$$u(x, t) = \sum_{m=0}^{\infty} \frac{t^{m\alpha}}{\Gamma(m\alpha + 1)} \sum_{p=0}^m (-1)^p \binom{m}{p} \mu^{m-p} D_x^{(2m-p)\beta} (f(x)).$$

5. Test Examples

In this section, we give some examples for the different cases that we give their analytic solution in the previous sections.

Example 1. According to the relation (24), if $\alpha = 1$, $\beta = 1$, and $\gamma = 2$ with the initial condition $u(x, 0) = e^{-x}$. We get the solution to be in the form

$$\begin{aligned} u(x, t) &= \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(m+1)} \sum_{p=0}^m (-1)^p \binom{m}{p} \mu^{m-p} D_x^{2m-p} (e^{-x}) \\ &= \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(m+1)} \sum_{p=0}^m (-1)^p \binom{m}{p} \mu^{m-p} [(-1)^{2m-p} e^{-x}] \\ &= \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(m+1)} \sum_{p=0}^m (-1)^{2m} \binom{m}{p} \mu^{m-p} [e^{-x}] \\ &= e^{-x} \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(m+1)} \sum_{p=0}^m \binom{m}{p} \mu^{m-p} \\ &= e^{-x} \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(m+1)} (1 + \mu)^m \end{aligned}$$

Figure 2: The Solution $u(x, t)$ of example 2 for selected values of α, β and $n = 1$.

$$= e^{-x+(1+\mu)t},$$

which is the exact solution of the classical Advection-dispersion equation (1) under the initial condition $u(x, 0) = e^{-x}$ which is obtained in [13], see Figure 1a.

Example 2. Consider the equation (8) with the initial condition:

$$u(x, 0) = x^n, \quad n \in R.$$

Using the obtained formula (24), we can get the m -th term in the series of the solution as

$$\begin{aligned} u_m &= (-1)^m \frac{t^{m\alpha}}{\Gamma(m\alpha + 1)} D_x^{m\beta} x^n \\ &= (-1)^m \frac{t^{m\alpha}}{\Gamma(m\alpha + 1)} \frac{n!}{\Gamma(n - m\beta + 1)} x^{n-m\beta}. \end{aligned}$$

So, the solution is given by

$$u(x, t) = \sum_{m=0}^{\infty} (-1)^m \frac{t^{m\alpha}}{\Gamma(m\alpha + 1)} \frac{n!}{\Gamma(n - m\beta + 1)} x^{n-m\beta}.$$

If $\alpha = \beta = 1, \gamma = 2$ and $n = 1$, whence the solution is $u(x, t) = x - t$, which is the exact solution of the problem, see [26]) (see Figure 2a). Also the solution of example2 when $\alpha = 0.5, \beta = 1$ and $\alpha = 0.75, \beta = .25$ are shown in Figures 2b, 2c.

6. Conclusion

The main aim of this work is to use the Adomian's decomposition method (ADM) to solve the space-time fractional advection-dispersion equation (FADE). Different three cases of (FADE) based on the order of the derivatives were discussed. General formulas of the solutions for these three cases are analyzed under any initial conditions. These solutions are always given in terms of Mittag-Leffler functions. (ADM) is direct and powerful to get the exact solution. Finally, some examples are given to illustrate that our results generalize some important considerable works. Moreover, a graphical solution for an example worked out using Matlab program is given.

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