LIOUVILLE-TYPE RESULT FOR QUASILINEAR ELLIPTIC PROBLEMS WITH VARIABLE EXPONENT

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Abstract: In this paper we discuss several aspects of relations between the Keller-Osserman property and the validity of the Liouville-type theorem for quasilinear elliptic equation with variable exponent that include non-homogeneous $p(\cdot)$ -Laplace equations for p measurable function.

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1. Introduction

The study of singular quasilinear elliptic equation, and specifically boundary blow-up problems, has attracted considerable attention starting with the original work of Bieberbach (1916) [6]. This approach was introduced in the last decades of the need to provide answers to important questions of the nonlinear problems with variable exponent for example see [18], [5].

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Set in order to obtain and even formulate Liouville's theorem, for example, for $p(\cdot)$ -superharmonic functions on $\mathbb{R}^d$, $d \geq 3$, one needs to compare an arbitrary $p(\cdot)$ -superharmonic function with a constant which is a trivial $p(\cdot)$ -subharmonic function. In this paper, we study the quasilinear elliptic equation

$$-\operatorname{div} \mathcal{A}(x, \nabla u) + \mathcal{B}(x, u) = 0, \quad (1)$$

where $\mathcal{A} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\mathcal{B} : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$, are Carathéodory functions satisfying the structure conditions given in assumptions **(H0)**, **(H1)**, **(H2)**, **(H3)**, and **(H4)** below.

A typical example for the operator $\mathcal{A}$ and $\mathcal{B}$ are

$$\mathcal{A}(x, \nabla u) = |\nabla u|^{p(x)-2} \nabla u$$

and $\mathcal{B}(x, u) = |u|^{p(x)-2} u$ respectively, for all $x \in \mathbb{R}^d$ thus

$$\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u).$$

This problem appears in the study of quantum mechanics, *classical statistical* and *Hamiltonian mechanics* we refer to [13]. There are also related mathematical results in ergodic theory. In [2] Baalal and Boukricha study the same problem when the function $p(x) = p$ is constant. Here we provide a more elementary proof, which in particular we are interested in the nonlinear potential theory associated with *variable exponent* of quasilinear elliptic equations (1).

For the existence and uniqueness of $p(\cdot)$ -solutions $u \in \mathcal{W}^{1,p(x)}(\Omega)$ where $1 < p(x) < d$ for all $x \in \Omega$, of the variational Dirichlet problem associated with the quasilinear elliptic equation (1) see [3], these $p(\cdot)$ -solutions are obtained by the $p(\cdot)$ -obstacle problem. *The Harnack inequality* is valid as it is a fundamental tool for the crucial question for the regularity of weak $p(\cdot)$ -solutions for the equation (1) with nonstandard $p(x)$ -growth conditions, see [4].

In [15], [19] Keller and Osserman prove that a necessary and sufficient condition for the considered problem to have an *entire solution* is that $\mathcal{B}$. In the case where $p(x) = p$ is constant the validity of *the Keller-Osserman property* with variable exponent see [5].

The attention is focused on those problems whose $p(\cdot)$ -solution of (1) depends on some Liouville-type theorem. Let us mention, in this respect, some examples of Liouville's theorem: *Cauchy and Liouville*, *Hadamard and Liouville*, *Poisson and Liouville*, *De Giorgi and Liouville*, *Harnack and Liouville* then *Moser and Liouville*. The potential theory focuses more on the Poisson-Liouville theorem for more details see [12].

The contribution of this paper is to extend some of the results in [2] to the equation (1) under general assumptions below. We discuss several aspects of relations between *the Keller-Osserman property* and the validity of *the Liouville-type theorem* for quasilinear elliptic equation with variable exponent that include $p(x)$ -Laplace equations for p measurable function.

This paper deal with a crucial question concerning the Non-homogeneity of $p(x)$ -Laplacian in the presence of the nonlinear term given as $\mathcal{B}$. Because of *the Non-homogeneity of $p(x)$ -Laplacian*, $p(x)$ -Laplacian problems are more complicated than those of p -Laplacian ones; and another difficulty of this paper is that $\mathcal{B}$ cannot be represented as $b(x)f(u)$.

In the first section, we introduce some generalization and position of the problem. In second section we give some basic facts about variable exponent spaces and a rough overview of properties of $p(\cdot)$ -solutions of the prototype equality. In Section 3, we generalize, with detailed proofs, we prove that the Keller-Osserman property in $(\mathbb{R}^d, \mathcal{H})$ is valid; i.e., *every open ball admits a regular Evans function*, we discuss several aspects of relations between the Keller-Osserman property and the validity of the Liouville-type theorem for quasilinear elliptic equation with variable exponent.

2. Preliminaries

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $p : \Omega \rightarrow]1, +\infty[$ called a *variable exponent* and we denote

$$p^- := \operatorname{ess\,inf}_{x \in \bar{\Omega}} p(x) \quad \text{and} \quad p^+ := \operatorname{ess\,sup}_{x \in \bar{\Omega}} p(x).$$

For each open bounded subset Ω of $\mathbb{R}^d$ ($d \geq 2$), we denote

$$\mathcal{C}_+(\bar{\Omega}) = \{p \in \mathcal{C}(\bar{\Omega}) : p(x) > 1 \text{ for any } x \in \bar{\Omega}\}.$$

We introduce also *the convex modular*

$$\varrho_{p(\cdot)}(u) = \int_{\Omega} |u|^{p(x)} dx.$$

If the exponent is bounded, i.e., if $p^+ < \infty$, then the expression

$$\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called *the Luxemburg norm*.

Proposition 2.1 (*Generalized Hölder inequality [23]*). For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have

$$|\int_{\Omega} uv dx| \leq (\frac{1}{p^-} + \frac{1}{p'^-}) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}.$$

We define the variable exponent Sobolev space (see [16], [9], [23]) by

$$\mathcal{W}^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : \|\nabla u\| \in L^{p(\cdot)}(\Omega)\}.$$

with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)} \quad \forall u \in \mathcal{W}^{1,p(\cdot)}(\Omega). \quad (2)$$

The local Sobolev space $\mathcal{W}_{\text{loc}}^{1,p(\cdot)}(\Omega)$ consists of functions u that belong to $\mathcal{W}_{\text{loc}}^{1,p(\cdot)}(U)$ for all open sets U compactly contained in Ω . The Sobolev space with zero boundary values, $\mathcal{W}_0^{1,p(\cdot)}(\Omega)$, is defined as the completion of $C_0^\infty(\Omega)$ in the norm of $\mathcal{W}^{1,p(\cdot)}(\Omega)$.

Let $p^*(x)$ be the critical Sobolev exponent of $p(x)$ defined by

$$p^*(x) = \begin{cases} \frac{dp(x)}{d-p(x)} & \text{for } p(x) < d \\ +\infty & \text{for } p(x) \geq d \end{cases}$$

We assume further on that, there exist positive constant C such that the function p satisfies *logarithmic Hölder continuity condition* if :

$$(\star) \begin{cases} \exists C > 0 : |p(x) - p(y)| \leq \frac{C}{-\log|x-y|} & \text{for } |x - y| < \frac{1}{2}, \\ 1 < p^- \leq p^+ < d. \end{cases}$$

Proposition 2.2 (*The $p(\cdot)$ -Poincaré inequality*). Let Ω be a bounded open set and let $p : \Omega \rightarrow [1, \infty[$ satisfy $(\star)$, there exists a constant C , depending only on $p(\cdot)$ and Ω , such that the inequality

$$\|u\|_{p(\cdot)} \leq C \|\nabla u\|_{p(\cdot)} \quad \text{for all } u \in \mathcal{W}_0^{1,p(\cdot)}(\Omega).$$

Definition 1. We say that a $u \in \mathcal{W}_{\text{loc}}^{1,p(\cdot)}(\Omega)$ is a $p(\cdot)$ -solution of (1) in Ω provided that for all $\varphi \in \mathcal{W}_0^{1,p(\cdot)}(\Omega)$ if ,

$$\int_{\Omega} \mathcal{A}(x, \nabla u) \cdot \nabla \varphi dx + \int_{\Omega} \mathcal{B}(x, u) \varphi dx = 0. \quad (3)$$

Definition 2. A function $u \in \mathcal{W}_{\text{loc}}^{1,p(\cdot)}(\Omega)$ is termed $p(\cdot)$ -supersolution of (1), if and only if, for all non-negative functions $\varphi \in \mathcal{W}_0^{1,p(\cdot)}(\Omega)$ we have,

$$\int_{\Omega} \mathcal{A}(x, \nabla u) \cdot \nabla \varphi dx + \int_{\Omega} \mathcal{B}(x, u) \varphi dx \geq 0.$$

A function u is a $p(\cdot)$ -subsolution in Ω if $-u$ is a $p(\cdot)$ -supersolution in Ω , and a $p(\cdot)$ -solution in Ω if it is both a $p(\cdot)$ -supersolution and a $p(\cdot)$ -subsolution in Ω .

The following comparison principle is useful for the potential theory associated with equation (1):

Lemma 3. Suppose that u is a $p(\cdot)$ -supersolution and v is a $p(\cdot)$ -subsolution on Ω such that

$$\limsup_{x \rightarrow y} v(x) \leq \liminf_{x \rightarrow y} u(x) \quad (4)$$

for all $y \in \partial\Omega$ and if both sides of the inequality are not simultaneously $+\infty$ or $-\infty$, then $v \leq u$ in Ω .

Throughout the paper we suppose that the functions $\mathcal{A} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a Carathéodory function satisfying the following assumptions:

(H0) For all $\xi, \xi' \in \mathbb{R}^d$ with $\xi \neq \xi'$, $[\mathcal{A}(x, \xi) - \mathcal{A}(x, \xi')] \cdot (\xi - \xi') > 0$.

(H1) $|\mathcal{A}(x, \xi)| \leq \beta[k(x) + |\xi|^{p(x)-1}]$;

(H2) $\mathcal{A}(x, \xi)\xi \geq \nu|\xi|^{p(x)}$;

for a.e. $x \in \Omega$, all $\xi \in \mathbb{R}^d$, where k is a positive bounded function lying in $L^{p'(x)}(\Omega)$ and $\beta, \nu > 0$.

In this paper we suppose that the function $\mathcal{B} : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is given Carathéodory function and the following condition is satisfied:

(H3) $\zeta \rightarrow \mathcal{B}(x, \zeta)$ is increasing and $\mathcal{B}(x, 0) = 0$ for every $x \in \mathbb{R}^d$.

(H4) $|\mathcal{B}(x, \zeta)| \leq g(x) + |\zeta|^{\delta(x)}$; for a.e. $x \in \Omega$, all $\zeta \in \mathbb{R}^d$, where g is a positive bounded function lying in $L^{p'(x)}(\Omega)$ and

$$p(x) \leq \delta(x) + 1 < p^*(x)$$

3. Keller-Osserman Property

For every open set U we shall denote by $\mathcal{U}(U)$ the set of all relatively compact open regular subset V in U with $\overline{V} \subset U$.

By previous section and in order to obtain an axiomatic nonlinear potential theory, we shall investigate the harmonic sheaf associated with (1) and defined as follows:

For every open subset U of $\mathbb{R}^d$ ($d \geq 1$), we set

$$\begin{aligned}\mathcal{H}(U) &= \{u \in \mathcal{C}(U) \cap \mathcal{W}_{\text{loc}}^{1,p(\cdot)}(U) : u \text{ is a } p(\cdot)\text{-solution of (1)}\} \\ &= \{u \in \mathcal{C}(U) : H_V u = u \text{ for every } V \in \mathcal{U}(U)\}.\end{aligned}$$

Element in the set $\mathcal{H}(U)$ are called *harmonic* on U .

In this section, we prove that functions for which the comparison principle holds are monotone limits of $p(\cdot)$ -supersolutions see [17], [1].

Definition 4. We say that a function $u : \Omega \rightarrow (-\infty, +\infty]$ is $p(\cdot)$ -superharmonic in Ω if:

1. u is lower semicontinuous;
2. u is finite almost everywhere and;
3. If h is a $p(\cdot)$ -solution in $D \subset \subset \Omega$, continuous in $\overline{D}$ and $u \geq h$ on $\partial\Omega$, then $u \geq h$ in D .

Theorem 5. Let u be a $p(\cdot)$ -supersolution in with the property

$$u(x) = \text{ess} \liminf_{x \rightarrow y} u(y) \quad \text{for all } x \in \Omega$$

Then u is $p(\cdot)$ -superharmonic.

Let $\mathcal{H}$ be the sheaf of continuous $p(\cdot)$ -solutions related to the equation (1) and we consider ball $B=B(x_0, R)$ with center x_0 and radius R and for every $u \in \mathcal{H}(B)$,

Definition 6. A function $u \in \mathcal{H}^+(B)$ is called $p(\cdot)$ -regular Evans function for $\mathcal{H}$ and B if $\lim_{B \ni x \rightarrow z} u(x) = +\infty$ for every regular point z in the boundary of B .

We also introduce the *Keller-Osserman property*, the $p(\cdot)$ -regular Evans functions apparently for the first time to proof Liouville theorem for the equation (1) with variable exponent.

For an investigation of regular Evans functions see [7], [10], [20], [21].

Definition 7. We shall say that $\mathcal{H}$ satisfies the Keller-Osserman property, denoted (KO), if every ball admits a $p(\cdot)$ -regular Evans function for $\mathcal{H}$.

As in [7, Proposition 1.3], we have the following proposition.

Proposition 3.1. $\mathcal{H}$ satisfies the (KO) condition if and only if $\mathcal{H}^+$ is locally uniformly bounded (i.e. for every non empty open set U in $\mathbb{R}^d$ and for every compact $K \subset U$, there exists a constant $C > 0$ such that $\sup_K u \leq C$ for every $u \in \mathcal{H}^+(U)$).

Corollary 8. If $\mathcal{H}$ fulfills the (KO) property, then $\mathcal{H}$ satisfies the BreLOT convergence property.

Theorem 9. Assume that $\mathcal{A}$ and $\mathcal{B}$ satisfies the following supplementary conditions

- (A0) For every $x \in \mathbb{R}^d$, the function p is differentiable and $|\nabla p(x)|$ bounded.
- (A1) For every $x_0 \in \mathbb{R}^d$, the function F from $\mathbb{R}^d$ to $\mathbb{R}^d$ defined by $F(x) = \mathcal{A}(x, x - x_0)$ is differentiable and $\operatorname{div} F$ is locally bounded.
- (A2) $|\mathcal{B}(x, \zeta)| \leq b(x) |\zeta|^{\alpha(x)}$, $\alpha(x) > p(x) - 1$ where $b \in L_{\operatorname{loc}}^{\frac{d}{d-\epsilon}}(\mathbb{R}^d)$, $0 < \epsilon < 1$, with $\operatorname{ess\,inf}_B b(x) > 0$ for every B in $\mathbb{R}^d$.

Then the (KO) property is valid by $\mathcal{H}$.

Proof. Let B be the ball with center $x_0 \in \mathbb{R}^d$ and radius R . Put $f(x) = R^2 - \|x - x_0\|^2$ and $\psi(x) = cf^{-\beta}(x)$ where $\beta > 0$, we obtain the desired property if we find a constant $c > 0$ such that ψ is a $p(\cdot)$ -supersolution of the equation (1). We have $\nabla f(x) = -2(x - x_0)$ and $\nabla \psi(x) = 2c\beta f(x)^{-(\beta+1)}(x - x_0)$ and then

$$\mathcal{A}(x, \nabla \psi(x)) = (2c\beta f(x)^{-(\beta+1)})^{p(x)-1} \mathcal{A}(x, x - x_0).$$

Therefore

$$\begin{aligned} \operatorname{div} \mathcal{A}(x, \nabla \psi(x)) &\leq \left[\left(\nabla p(x) (2c\beta f(x)^{-(\beta+1)}) + 2(p(x) - 1)(\beta + 1)f(x)^{-1}(x - x_0) \right) \right. \\ &\quad \times \left. \mathcal{A}(x, x - x_0) + \operatorname{div} \mathcal{A}(x, x - x_0) \right] (2c\beta f(x)^{-(\beta+1)})^{p(x)-1} \end{aligned}$$

On the other hand and since of hypothesis (A2), we have

$$\mathcal{B}(x, \psi(x)) \leq b(x)(cf(x)^{-\beta})^{\alpha(x)}$$

Let $\varphi \in \mathcal{C}_c^\infty(B)$, $\varphi \geq 0$ and we set

$$I_\varphi = \int \mathcal{A}(x, \nabla \psi) \nabla \varphi dx + \int \mathcal{B}(x, \psi) \varphi dx$$

then

$$\begin{aligned}
I_\varphi &= - \int \left(\operatorname{div} \mathcal{A}(x, \nabla \psi) dx - \mathcal{B}(x, \psi) \right) \varphi dx \\
&\geq - \int \left(\left[\left(2\beta \nabla p(x) f(x)^{-(\beta+1)} + 2(p(x)-1)(\beta+1)f(x)^{-1}(x-x_0) \right) \mathcal{A}(x, x-x_0) \right. \right. \\
&\quad \left. \left. + \operatorname{div} \mathcal{A}(x, x-x_0) \right] c^{p(x)-\alpha(x)-1} (2\beta)^{p(x)-1} f(x) - b(x) f(x)^{(\beta+1)(p(x)-1)+1-\beta\alpha(x)} \right) \\
&\quad c^{\alpha(x)} \varphi f(x)^{-(\beta+1)(p(x)-1)-1} dx
\end{aligned}$$

Putting $\beta = \operatorname{ess\,inf}_{x \in \overline{B}} \left(\frac{p(x)}{\alpha(x)-p(x)+1} \right)$ for $x \in B$, we obtain

$$\begin{aligned}
I_\varphi &\geq - \int \left(\left[\left(\frac{p(x) \nabla p(x) f(x)^{-\frac{p(x)}{\alpha(x)-p(x)+1}}}{\alpha(x)-p(x)+1} + \frac{(p(x)-1)(\alpha(x)+1)(x-x_0)}{\alpha(x)-p(x)+1} \right) \right. \right. \\
&\quad \times \left. \left. 2\mathcal{A}(x, x-x_0) + f(x) \operatorname{div} \mathcal{A}(x, x-x_0) \right] \left(\frac{2p(x)}{\alpha(x)-p(x)+1} \right)^{p(x)-1} - c^{\alpha(x)-p(x)+1} b(x) \right) \\
&\quad \times c^{p(x)-1} \varphi f(x)^{-\left(\frac{(\alpha(x)+1)(p(x)-1)}{\alpha(x)-p(x)+1} + 1 \right)} dx
\end{aligned}$$

It follows from **(A2)** that $\mathcal{A}(x, x-x_0) \cdot (x-x_0)$ is locally bounded, we obtain

$$\begin{aligned}
c^{\alpha(x)-p(x)+1} &\geq \left[\left(\frac{p(x) |\nabla p(x)| f(x)^{-\frac{p(x)}{\alpha(x)-p(x)+1}}}{\alpha(x)-p(x)+1} + \frac{(p(x)-1)(\alpha(x)+1)|x-x_0|}{\alpha(x)-p(x)+1} \right) \right. \\
&\quad \times \left. 2 \frac{|\mathcal{A}(x, x-x_0)|}{b(x)} + f(x) \frac{|\operatorname{div} \mathcal{A}(x, x-x_0)|}{b(x)} \right] \left(\frac{2p(x)}{\alpha(x)-p(x)+1} \right)^{p(x)-1}
\end{aligned}$$

Then

$$\begin{aligned}
c &\geq \sup_{x \in B} \left\{ \left[\left(\frac{p(x) |\nabla p(x)|}{\alpha(x)-p(x)+1} + \frac{(p(x)-1)(\alpha(x)+1)|x-x_0|}{\alpha(x)-p(x)+1} \right) 2 \frac{|\mathcal{A}(x, x-x_0)|}{b(x)} \right. \right. \\
&\quad \left. \left. + R^2 \frac{|\operatorname{div} \mathcal{A}(x, x-x_0)|}{b(x)} \right] \left(\frac{2p(x)}{\alpha(x)-p(x)+1} \right)^{p(x)-1} \right\}^{\frac{1}{\alpha(x)-p(x)+1}}
\end{aligned}$$

therefore $I_\varphi \geq 0$ holds for every $\varphi \in \mathcal{C}_c^\infty(B)$ with $\varphi \geq 0$.

Thus the function $\psi(x) = c(R^2 - \|x - x_0\|^2)^{\frac{p(x)}{\alpha(x)-p(x)+1}}$ is a $p(\cdot)$ -supersolution satisfying $\lim_{x \rightarrow z} \psi(x) = +\infty$ for every $z \in \partial B$.

By the comparison principle we have $H_B n \leq \psi$ for every $n \in \mathbb{N}$ and therefore, the increasing sequence $(H_B n)_n$ of harmonic functions is locally uniformly bounded on B .

The Bauer convergence property implies that $u = \sup_n H_B n \in \mathcal{H}(B)$ see [2]. Therefore we have $\liminf_{x \rightarrow z} u(x) \geq n$ for every z in ∂B , thus $\lim_{x \rightarrow z} u(x) = +\infty$ for every z in ∂B and u is a $p(\cdot)$ -regular Evans function. Since B is an arbitrary ball. $\square$

4. Liouville Theorem with Variable Exponent

In this section we prove the Liouville theorem, since that the original proof of Baalal and Boukricha is based on the explicit knowledge of positive supersolutions of the equation (1) with $p > 1$ positive constant on arbitrary open balls, with the further property that Evans functions explode at the boundary of the considered balls. This fact is crucially related to the shape of the non-linear function $\mathcal{B}(t) = |t|^{p-1}t$ and it does not easily extend to more general functions $\mathcal{B}$ see [8].

For this reason, we give here a different proof based on Theorem (9). This approach has the advantage to apply to a larger class of problems. We may now state our main results. First we have to consider a lemma:

Lemma 10. *Under the assumptions in Theorem 9, for every ball $B=B(x_0, R)$ with center x_0 and radius R and for every $u \in \mathcal{H}(U)$,*

$$|u(x_0)| \leq cR^{\frac{2p(x)}{p(x)-1-\alpha(x)}} \quad \text{for all} \quad x \in B$$

where

$$\begin{aligned} c &= \sup_{x \in B} \left\{ \left[\left(\frac{p(x)|\nabla p(x)|}{\alpha(x) - p(x) + 1} + \frac{(p(x) - 1)(\alpha(x) + 1)|x - x_0|}{\alpha(x) - p(x) + 1} \right) 2 \frac{|\mathcal{A}(x, x - x_0)|}{b(x)} \right. \right. \\ &\quad \left. \left. + R^2 \frac{|\operatorname{div} \mathcal{A}(x, x - x_0)|}{b(x)} \right]^{\frac{1}{\alpha(x) - p(x) + 1}} \left(\frac{2p(x)}{\alpha(x) - p(x) + 1} \right)^{\frac{p(x)-1}{\alpha(x) - p(x) + 1}} \right\} \end{aligned}$$

Proof. From the proof of the previous theorem, if $B_n = B(x_0, R(1 - n^{-1}))$, $n \geq 2$, we have

$$u(x_0) \leq c_n \left(\frac{R(n-1)}{n} \right)^{\frac{2p(x)}{p(x)-1-\alpha(x)}}$$

for every $n \geq 2$ and

$$\begin{aligned} c_n &= \sup_{x \in B_n} \left\{ \left[\left(\frac{p(x)|\nabla p(x)|}{\alpha(x) - p(x) + 1} + \frac{(p(x) - 1)(\alpha(x) + 1)|x - x_0|}{\alpha(x) - p(x) + 1} \right) 2 \frac{|\mathcal{A}(x, x - x_0)|}{b(x)} \right. \right. \\ &\quad \left. \left. + \left(\frac{R(n-1)}{n} \right)^2 \frac{|\operatorname{div} \mathcal{A}(x, x - x_0)|}{b(x)} \right]^{\frac{1}{\alpha(x) - p(x) + 1}} \left(\frac{2p(x)}{\alpha(x) - p(x) + 1} \right)^{p(x)-1} \right\} \\ &\leq \sup_{x \in B} \left\{ \left[\left(\frac{p(x)|\nabla p(x)|}{\alpha(x) - p(x) + 1} + \frac{(p(x) - 1)(\alpha(x) + 1)|x - x_0|}{\alpha(x) - p(x) + 1} \right) 2 \frac{|\mathcal{A}(x, x - x_0)|}{b(x)} \right. \right. \\ &\quad \left. \left. + R^2 \frac{|\operatorname{div} \mathcal{A}(x, x - x_0)|}{b(x)} \right]^{\frac{1}{\alpha(x) - p(x) + 1}} \left(\frac{2p(x)}{\alpha(x) - p(x) + 1} \right)^{p(x)-1} \right\} \end{aligned}$$

Then we obtain the inequality

$$u(x_0) \leq cR^{\frac{2p(x)}{p(x)-1-\alpha(x)}}.$$

Since $-u$ is a $p(\cdot)$ -solution of similarly equation, we get

$$-u(x_0) \leq cR^{\frac{2p(x)}{p(x)-1-\alpha(x)}}$$

with the same constant c as before. Then we have the desired inequality. $\square$

The proof of the *Liouville theorem* uses an idea of A. Baalal and A. Boukricha in [2], who proved the corresponding Liouville theorem for quasilinear elliptic equation with *Keller-Osserman condition*. Here we make use of various have a Liouville like theorem associated with variable exponent.

We put

$$\begin{aligned} M(R) &= \sup_{\|x-x_0\| \leq R} \left\{ \left[\left(\frac{p(x)|\nabla p(x)|}{\alpha(x)-p(x)+1} + \frac{(p(x)-1)(\alpha(x)+1)|x-x_0|}{\alpha(x)-p(x)+1} \right) \right. \right. \\ &\times \left. \left. 2 \frac{|\mathcal{A}(x, x-x_0)|}{b(x)} + R^2 \frac{|\operatorname{div} \mathcal{A}(x, x-x_0)|}{b(x)} \right] \left(\frac{2p(x)}{\alpha(x)-p(x)+1} \right)^{p(x)-1} \right\}^{\frac{1}{\alpha(x)-p(x)+1}} \end{aligned}$$

Theorem 11. *Assume that the conditions in Theorem 9 are satisfied and that*

$$\liminf_{R \rightarrow \infty} \left(R^{-2p(x)} M(R) \right) = 0.$$

Then $u \equiv 0$ is the unique $p(\cdot)$ -solution of the equation (1) on $\mathbb{R}^d$.

Proof. Let u be a $p(\cdot)$ -solution of the equation (1) on $\mathbb{R}^d$. By the previous corollary, we have for every $x_0 \in \mathbb{R}^d$ and every $R > 0$

$$\begin{aligned} |u(x_0)| &\leq \sup_{\|x-x_0\| \leq R} \left\{ \left[\left(\frac{p(x)|\nabla p(x)|}{\alpha(x)-p(x)+1} + \frac{(p(x)-1)(\alpha(x)+1)|x-x_0|}{\alpha(x)-p(x)+1} \right) 2 \frac{|\mathcal{A}(x, x-x_0)|}{b(x)} \right. \right. \\ &+ \left. \left. R^2 \frac{|\operatorname{div} \mathcal{A}(x, x-x_0)|}{b(x)} \right] R^{-2p(x)} \left(\frac{2p(x)}{\alpha(x)-p(x)+1} \right)^{p(x)-1} \right\}^{\frac{1}{\alpha(x)-p(x)+1}} \end{aligned}$$

Hence $u(x_0) = 0$ and $u \equiv 0$. $\square$

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