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GRACEFUL LABELING OF GENERALIZED ROMAN RINGS WITH 4M VERTICES

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Abstract: Roman rings can be obtained by introducing n copies of cycle C_m with m vertices, which are merged respectively to n teeth of comb graph $P_n \odot L_1$. In this paper it is proved that generalized Roman rings with cycle C_{4m} are graceful.

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Key Words: graceful labeling, roman rings, cycle graph

1. Introduction

Graphs considered in this paper are simple finite and undirected. In general G(V, E) denotes the graph G with vertex set V(G), edge set E(G), such that$|V(G)|$ = p vertices $|E(G)|$ = q edges. A labeling of the vertices of G with the numbers from 0 to q is an injective map $\phi : V \to \{0, 1, ...q\}$. A graph G is graceful if there exists a labeling of its vertices such that the map $\phi^{\Box}$: E $\to \{1, 2, ...q\}$ given by $\phi^{\Box}$ (uv) = $|\phi(u) - \phi(v)|$, where u, v $\in$ V and uv$\in$ E is a bijection.

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A graph that admits graceful labeling is called graceful graph. The notation graceful labeling was introduced Rosa[4] with the name valuation.

Gallian[2] gives the extensive survey of contributions to graceful labeling of variety of graphs. The notation and terminology used in this paper are taken from [2].

Rosa[4], [3] showed that the n-cycle C is graceful if and only if n = 0 or 3 (mod 4). Bhat-Nayak and Selvam [1] have shown that the n-cone (also called the n-point suspension of C_m) $C_m + \overline{K_n}$ is graceful when m = 0 or 3 (mod 12) . They also proved the gracefulness of $C_4 + \overline{K_n}$, $C_5 + \overline{K_2}$, $C_7 + \overline{K_n}$, $C_9 + \overline{K_2}$, $C_{11} + \overline{K_n}$ and $C_{19} + \overline{K_n}$ and others[5], [6].

2. Main Result

Let $R_1, R_2, \ldots, R_n$ be n copies of cycle C_m(we term here as rings). Let the supporting points on the n rings $R_1, R_2, \ldots, R_n$ be $t_1, t_2, \ldots, t_n$, which are merged respectively to n teeth of comb graph $P_n \odot L_1$. Let $b_1, b_2, b_3, \ldots, b_n$ be base points of the comb graph from which n rings of equal length say m, are hanging, each of which at a tooth of n teeth respectively. The resulting structure is called Roman rings R (m, n) (the term taken from gymnastic exercise). Let the points of ith ring be $c_1^i, c_2^i, \ldots, c_{m-1}^i$ for i = 1, 2,. . ., n.

From the above definition of R(m, n) graph, it is clear that $|V(R(m,n))| =$ n (m + 1). Also, the number of edges of R (m, n) is $|E(R(m,n))|$ = n (m + 2) – 1.

Theorem 1. *The generalized Roman rings R (m, n), m = 4j for j ≥ 1, n ≥ 1 is graceful.*

Proof. The labeling of vertices in the first ring are as follows:

Step 1 $\phi(b_1) = 0$. $\phi(t_1) = q$. $\phi(c_1^1) = 1$. $\phi(c_2^1) = q - 1$.

Step 2 $\phi(c_{2i+1}^1) = \phi(c_{2i-1}^1) + 1$, $1 \le i \le \frac{m}{4} - 1$.

Step 3 $\phi(c_{2i+2}^1) = \phi(c_{2i}^1) - 1$, $1 \le i \le \frac{m}{2} - 2$.

Step 4 $\phi(c_{2i+1}^1) = \phi(c_{2i-1}^1) + 2$, for $i = \frac{m}{4}$.

Step 5 $\phi(c_{2i+1}^1) = \phi(c_{2i-1}^1) + 1$, for $\frac{m}{4} + 1 \le i \le \frac{m-2}{2}$.

The labeling of 2^{nd} ring are as follows:

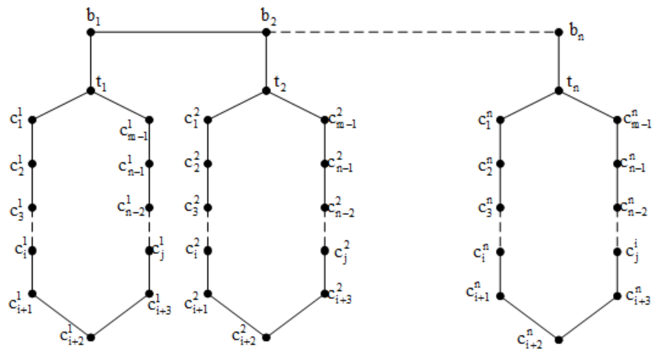

Figure 1: General form of R (4j, n)

Step 6 ϕ (b_2) = q − (m+1), ϕ (t_2) = m + 1. ϕ (c_1^2) = q − $\frac{m}{2}$, ϕ (c_2^2) = $\frac{m}{2}$ + 2.

Step 7 ϕ (c_{2i+1}^2) = ϕ (c_{2i-1}^2) − 1, $1 \leq i \leq \frac{m}{4} - 1$.

Step 8 ϕ (c_{2i+2}^2) = ϕ (c_{2i}^2) + 1, $1 \leq i \leq \frac{m}{2} - 2$.

Step 9 ϕ (c_{2i+1}^2) = ϕ (c_{2i-1}^2) − 2, for $i = \frac{m}{4}$.

Step 10 ϕ (c_{2i+1}^2) = ϕ (c_{2i-1}^2) − 1, for $\frac{m}{4} + 1 \leq i \leq \frac{m-2}{2}$.
(Let $n = 2\lambda + 1$ for odd and $n = 2\lambda$ for even)

Case 1: n **is odd.** Then ODD segment vertices are labeled as follows:

Step 11 ϕ (b_{2d+1}) = ϕ (b_{2d-1}) + m + 2, $1 \leq d \leq \lambda$.
ϕ (t_{2d+1}) = q − ϕ (b_{2d+1}), $1 \leq d \leq \lambda$.

Step 12 ϕ (c_{2i+1}^{2d+1}) = ϕ (c_{2i-1}^{2d-1}) + m + 2, $1 \leq i \leq \frac{m}{4} - 1$, $1 \leq d \leq \lambda$.

Step 13 ϕ (c_{2i+2}^{2d+1}) = ϕ (c_{2i}^{2d-1}) − (m+2), $1 \leq i \leq \frac{m}{2} - 2$, $1 \leq d \leq \lambda$.

Step 14 ϕ (c_{2i+1}^{2d+1}) = ϕ (c_{2i-1}^{2d-1}) + (m + 2), for $i = \frac{m}{4}$, $1 \leq d \leq \lambda$.

Step 15 ϕ (c_{2i+1}^{2d+1}) = ϕ (c_{2i-1}^{2d-1}) + m + 2, for $\frac{m}{4} + 1 \leq i \leq \frac{m-2}{2}$, $1 \leq d \leq \lambda$.
EVEN segment vertices are labeled as follows:

Step 16 ϕ (b_{2d+2}) = q − ϕ (t_{2d+2}), $1 \leq d \leq \frac{2\lambda-1}{2}$. ϕ (t_{2d+2}) = ϕ (t_{2d}) + m + 2, $1 \leq d \leq \lambda - 1$.

Step 17 ϕ (c_{2i+1}^{2d+2}) = ϕ (c_{2i-1}^{2d}) – (m + 2), $1 \leq i \leq \frac{m}{4} - 1$, $1 \leq d \leq \lambda - 1$.

Step 18 ϕ (c_{2i+2}^{2d+2}) = ϕ (c_{2i}^{2d}) + (m + 2), $1 \leq i \leq \frac{m}{2} - 2$, $1 \leq d \leq \lambda - 1$.

Step 19 ϕ (c_{2i+1}^{2d+2}) = ϕ (c_{2i-1}^{2d}) – (m + 2), for $i = \frac{m}{4}$, $1 \leq d \leq \lambda - 1$.

Step 20 ϕ (c_{2i+1}^{2d+2}) = ϕ (c_{2i-1}^{2d}) – (m + 2), for $\frac{m}{4} + 1 \leq i \leq \frac{m-2}{2}$, $1 \leq d \leq \lambda - 1$.

Case 2: n is even. Then ODD segment vertices are labeled as follows:

Step 11 ϕ (b_{2d+1}) = ϕ (b_{2d-1}) + m + 2, $1 \leq d \leq \frac{2\lambda-1}{2}$.
ϕ (t_{2d+1}) = q – ϕ (b_{2d+1}), $1 \leq d \leq \lambda - 1$.

Step 12 ϕ (c_{2i+1}^{2d+1}) = ϕ (c_{2i-1}^{2d-1}) + m + 2, $1 \leq i \leq \frac{m}{4} - 1$, $1 \leq d \leq \lambda - 1$.

Step 13 ϕ (c_{2i+2}^{2d+1}) = ϕ (c_{2i}^{2d-1}) – (m+2), $1 \leq i \leq \frac{m}{2} - 2$, $1 \leq d \leq \lambda - 1$.

Step 14 ϕ (c_{2i+1}^{2d+1}) = ϕ (c_{2i-1}^{2d-1}) + (m + 2), for $i = \frac{m}{4}$, $1 \leq d \leq \lambda - 1$.

Step 15 ϕ (c_{2i+1}^{2d+1}) = ϕ (c_{2i-1}^{2d-1}) + m + 2, for $\frac{m}{4} + 1 \leq i \leq \frac{m-2}{2}$, $1 \leq d \leq \lambda - 1$.

EVEN segment vertices are labeled as follows:

Step 16 ϕ (b_{2d+2}) = q – ϕ (t_{2d+2}), $1 \leq d \leq (\lambda - 1)$.
ϕ (t_{2d+2}) = ϕ (t_{2d}) + m + 2, $1 \leq d \leq (\lambda - 1)$.

Step 17 ϕ (c_{2i+1}^{2d+2}) = ϕ (c_{2i-1}^{2d}) – (m + 2), $1 \leq i \leq \frac{m}{4} - 1$, $1 \leq d \leq (\lambda - 1)$.

Step 18 ϕ (c_{2i+2}^{2d+2}) = ϕ (c_{2i}^{2d}) + (m + 2), $1 \leq i \leq \frac{m}{2} - 2$, $1 \leq d \leq (\lambda - 1)$.

Step 19 ϕ (c_{2i+1}^{2d+2}) = ϕ (c_{2i-1}^{2d}) – (m + 2), for $i = \frac{m}{4}$, $1 \leq d \leq (\lambda - 1)$.

Step 20 ϕ (c_{2i+1}^{2d+2}) = ϕ (c_{2i-1}^{2d}) – (m + 2), for $\frac{m}{4} + 1 \leq i \leq \frac{m-2}{2}$, $1 \leq d \leq (\lambda - 1)$.

Now, the induced edge labeling are as follows:

1 $\phi^{*}(b_1 t_1)$ = q.

2 $\phi^{*}(t_1 c_1^1)$ = q – 1.

3 $\phi^{*}(c_i^1 c_{i+1}^1)$ = q – 1 – i, $1 \leq i < \frac{m}{2}$.

4 $\phi^{*}(t_1 c_{m-1}^1)$ = q – ($\frac{m}{2}$ + 1).

5 $\phi^{*}(c_{i-1}^1 c_i^1)$ = q – 1 – i, $\frac{m}{2} + 1 \leq i \leq m - 1$.

6 $\phi^{\square}(b_1b_2)$ = q – (m + 1).

7 $\phi^{\square}(c^2_{i-1}c^2_i)$ = q – m – i, 2 ≤ i ≤ $\frac{m}{2}$.

8 $\phi^{\square}(t_2c^2_1)$ = q – ($\frac{3m}{2}$ + 1).

9 $\phi^{\square}(c^2_{i-1}c^2_i)$ = q – (m + 1) – i, $\frac{m}{2}$ + 1 ≤ i ≤ m – 1.

10 $\phi^{\square}(t_2c^2_{m-1})$ = q – 2m – 1.

11 $\phi^{\square}(b_2t_2)$ = q – 2m – 2.

12 $\phi^{\square}(b_2b_3)$ = q – 2m – 3.

Case 1: n **is odd.** Then, induced odd segment edge labeling are as follows:

13 $\phi^{\square}(b_{2d+1}t_{2d+1})$ = q – (2m + 4) d, 1 ≤ d ≤ λ.

14 $\phi^{\square}(t_{2d+1}c^{2d+1}_1)$ = q – 1 – (2m + 4) d, 1 ≤ d ≤ λ.

15 $\phi^{\square}(c^{2d+1}_i c^{2d+1}_{i+1})$ = q – 1 – i – (2m + 4) d, 1 ≤ d ≤ λ, 1 ≤ i < $\frac{m}{2}$.

16 $\phi^{\square}(t_{2d+1}c^{2d+1}_{m-1})$ = q – ($\frac{m}{2}$ + 1) – (2m + 4) d, 1 ≤ d ≤ λ.

17 $\phi^{\square}(c^{2d+1}_{i-1}c^{2d+1}_i)$ = q – 1 – i – (2m + 4) d, 1 ≤ d ≤ λ, $\frac{m}{2}$ + 1 ≤ i ≤ m – 1.

18 $\phi^{\square}(b_{2d+1}b_{2d+2})$ = q – (m + 1) – (2m + 4) d, 1 ≤ d ≤ λ – 1.

Induced even segment edge labeling are as follows:

19 $\phi^{\square}(c^{2d+2}_{i-1}c^{2d+2}_i)$ = q – m – i – (2m + 4) d, 1 ≤ d ≤ λ – 1, 2 ≤ i ≤ $\frac{m}{2}$.

20 $\phi^{\square}(t_{2d+2}c^{2d+2}_1)$ = q – ($\frac{3m}{2}$ + 1) – (2m + 4) d, 1 ≤ d ≤ λ – 1.

21 $\phi^{\square}(c^{2d+2}_{i-1}c^{2d+2}_i)$ = q – (m + 1) – i – (2m + 4) d, 1 ≤ d ≤ λ – 1, $\frac{m}{2}$ + 1 ≤ i ≤ m – 1.

22 $\phi^{\square}(t_{2d+2}c^{2d+2}_{m-1})$ = q – 2m – 1 – (2m + 4) d, 1 ≤ d ≤ λ – 1.

23 $\phi^{\square}(b_{2d+2}t_{2d+2})$ = q – 2m – 2 – (2m + 4) d, 1 ≤ d ≤ λ – 1.

24 $\phi^{\square}(b_{2d+2}b_{2d+3})$ = q – 2m – 3 – (2m + 4) d, 1 ≤ d ≤ λ – 1.

Case 2: n **is even.** Then induced odd segment edge labeling are as follows:

13 $\phi^{\square}(b_{2d+1}t_{2d+1})$ = q – (2m + 4) d, 1 ≤ d ≤ λ – 1.

14 $\phi^{\square}(t_{2d+1}c_1^{2d+1})$ = q – 1 – (2m + 4) d, $1 \leq d \leq \lambda - 1$.

15 $\phi^{\square}(c_i^{2d+1}c_{i+1}^{2d+1})$ = q – 1 – i – (2m + 4) d, $1 \leq d \leq \lambda - 1$, $1 \leq i < \frac{m}{2}$.

16 $\phi^{\square}(t_{2d+1}c_{m-1}^{2d+1})$ = q – $(\frac{m}{2} + 1)$ – (2m + 4) d, $1 \leq d \leq \lambda - 1$.

17 $\phi^{\square}(c_{i-1}^{2d+1}c_i^{2d+1})$ = q – 1 – i – (2m + 4) d, $1 \leq d \leq \lambda - 1$, $\frac{m}{2} + 1 \leq i \leq m - 1$.

18 $\phi^{\square}(b_{2d+1}b_{2d+2})$ = q – (m + 1) – (2m + 4) d, $1 \leq d \leq (\lambda - 1)$.

Induced even segment edge labeling are as follows:

19 $\phi^{\square}(c_{i-1}^{2d+2}c_i^{2d+2})$ = q – m – i – (2m + 4) d, $1 \leq d \leq \frac{n-2}{2}(\lambda - 1)$, $2 \leq i \leq \frac{m}{2}$.

20 $\phi^{\square}(t_{2d+2}c_1^{2d+2})$ = q – $(\frac{3m}{2} + 1)$ – (2m + 4) d, $1 \leq d \leq (\lambda - 1)$.

21 $\phi^{\square}(c_{i-1}^{2d+2}c_i^{2d+2})$ = q – (m + 1) – i – (2m + 4) d, $1 \leq d \leq (\lambda - 1)$, $\frac{m}{2} + 1 \leq i \leq m - 1$.

22 $\phi^{\square}(t_{2d+2}c_{m-1}^{2d+2})$ = q – 2m – 1 – (2m + 4) d, $1 \leq d \leq (\lambda - 1)$.

23 $\phi^{\square}(b_{2d+2}t_{2d+2})$ = q – 2m – 2 – (2m + 4) d, $1 \leq d \leq (\lambda - 1)$.

24 $\phi^{\square}(b_{2d+2}b_{2d+3})$ = q – 2m – 3 – (2m + 4) d, $1 \leq d \leq \lambda - 2$.

This results the induced edge values are distinct. □

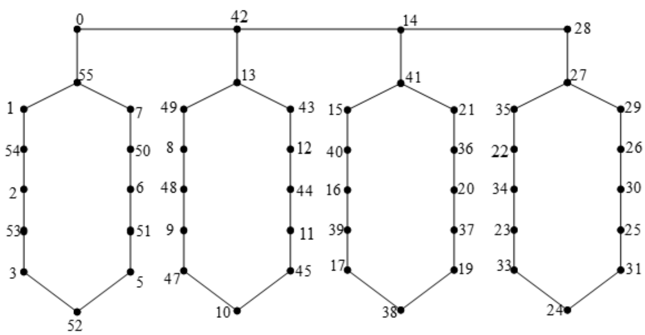

Figure 2: Example for R (12, 4)

3. Conclusion

The generalized Roman rings $R(m,n)$, $m = 4j$ for $j \geq 1$, $n \geq 1$ are graceful.

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