HYERS-ULAM STABILITY OF SOME FREDHOLM INTEGRAL EQUATION

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Abstract: We prove the Hyers-Ulam stability of some kinds of Fredholm integral equation. That is, if $\phi(t)$ is an approximate solution of a Fredholm integral equation, then there exists an exact solution of the differential equation near to $\phi(t)$.

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1. Introduction and Preliminaries

S.M. Ulam [15] gave a wide-ranging talk about a series of important unsolved problems in 1940. The question concerning the stability of group homomorphisms is one of them. D.H. Hyers (see [1]) proved the stability for the case of approximately additive mappings under the assumption that G_1 and G_2 are Banach spaces. After then, the Hyers-Ulam stability of function equation (see [2, 11, 12, 13]) and differential function (see [3, 4, 6, 7, 8, 9, 10, 14]) was investigated by several mathematicians.

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Fredholm integral equation is a kind of important integral equation. In this paper, we will try to investigate the Hyers-Ulam stability to some kinds of Fredholm integral equation:

The first class of convolutional Fredholm integral

$$\phi(t) = f(t) + \int_{-}^{+} k(t-s)\phi(s)ds, \quad (1.1)$$

and the second class of Fredholm integral

$$\int_{-}^{+} k(t-s)\phi(s)ds = f(t), \quad (1.2)$$

and

$$\int_{-}^{+} k(t+s)\phi(s)ds = f(t), \quad (1.3)$$

where the kernel of integral $k(t)$ is continuous and satisfies some properties, and $f(t)$ is continuous.

Firstly, we should introduce these prerequisite knowledge.

Fourier transform:

$$F(f)(s) = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-}^{+} f(t)e^{-ist}dt.$$

Inverse Fourier transform:

$$F^{-1}[F](t) = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-}^{+} F(s)e^{ist}ds,$$

and we know that Fourier transform has some properties:

- (a) $F[\alpha f_1 + \beta f_2] = \alpha F[f_1] + \beta F[f_2] \quad \forall \alpha, \beta \in \mathbb{C};$
- (b) $F[f(t)] = isF[f(t)];$
- (c) $F^{-1}[F(s)](t) = f(t);$
- (d) Suppose that $h(t) = \int_0^t f(x)dx$, then $F[h(t)] = \frac{1}{is}F[f(t)];$
- (e) Suppose that

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_{-}^{+} f(t-x)g(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-}^{+} g(t-x)f(x)dx,$$

we have

$$H(s) = F(h(t)) = F[f] \cdot F[g] = F(s) \cdot G(s);$$

$$(f) \quad F[f(t-a)] = e^{-ias} F[f(t)];$$

$$(g) \quad \int_{-}^{+} |F(f)|^2 ds = \int_{-}^{+} |f(x)|^2 dx.$$

And the following inequality is useful to our proof to the main results.

Theorem 1.1 (Hölder's Inequality). *Assume that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $x \in L^p(E)$, $y \in L^q(E)$, then $xy \in L(E)$ and*

$$\int_E |x(t)y(t)| dt \leq \left(\int_E |x(t)|^p dt \right)^{\frac{1}{p}} \left(\int_E |y(t)|^q dt \right)^{\frac{1}{q}}.$$

2. Main Results

Theorem 2.1. *Suppose that $f(t), k(t)$ are real continuous functions and $f(t), k(t) \in L^2(-\infty, +\infty)$, $K(s) \neq \frac{1}{2\pi}$ and $\frac{1}{1 - \frac{1}{2\pi K(s)}} \in L^2(-\infty, +\infty)$, and if a real function $\phi(t) : \mathbb{R} \rightarrow \mathbb{R}$ and $F(\phi)(s)$ exists, satisfies that*

$$\|\phi(t) - f(t) - \int_{-}^{+} k(t-s)\phi(s)ds\| \leq \varphi(t), \quad (2.1)$$

where $\varphi(t) \in L^2(-\infty, +\infty)$, $\varphi(t) \geq 0$. Then there exists a unique real solution $\phi_1(t)$ ($F(\phi_1)(s)$ exists) satisfies Eq.(1.1) and $\|\phi - \phi_1\| \leq M(\int_{-}^{+} \varphi(x)^2 dx)^{\frac{1}{2}}$ where M is a constant.

Proof. (1) Firstly, we will prove the stability.

Let ϕ be a solution of Ineq.(2.1) and put

$$\phi(t) - f(t) - \int_{-}^{+} k(t-s)\phi(s)ds := g(t) \quad (2.2)$$

where $|g(t)| \leq \varphi(t)$, since $\varphi(t) \in L^2(-\infty, +\infty)$, we know that $g(t) \in L^2(-\infty, +\infty)$.

It follows Fourier transform and the property(e) that

$$\Phi(s) = F(s) + G(s) + \sqrt{2\pi}K(s)\Phi(s). \quad (2.3)$$

Then we have

$$\Phi(s) = \frac{F(s) + G(s)}{1 - \sqrt{2\pi}K(s)}. \quad (2.4)$$

It follows the inverse Fourier transform that

$$p(t) = \frac{1}{\sqrt{2\pi}} \int_{-}^{+} \frac{1}{1 - \sqrt{2\pi}K(s)} e^{ist} ds. \quad (2.5)$$

Then, we can know from the property (a) and (e) that

$$\Phi(s) = (F(s) + G(s)) \cdot P(s) = F\left[\frac{1}{\sqrt{2\pi}} \int_{-}^{+} (f(t-x) + g(t-x))p(x)dx\right](s). \quad (2.6)$$

It follows the inverse Fourier transform and properties (c) that

$$\begin{aligned} \phi(t) &= \frac{1}{\sqrt{2\pi}} \int_{-}^{+} (f(t-x) + g(t-x))p(x)dx \\ &= \frac{1}{2\pi} \int_{-}^{+} \int_{-}^{+} (f(t-x) + g(t-x)) \frac{1}{1 - \sqrt{2\pi}K(s)} e^{isx} ds dx, \end{aligned}$$

and if we let $g(t) = 0$, we get a solution of Eq. (1.1)

$$\phi_1(t) = \frac{1}{2\pi} \int_{-}^{+} \int_{-}^{+} f(t-x) \frac{1}{1 - \sqrt{2\pi}K(s)} e^{isx} ds dx$$

Then, follows the Theorem 1.1

$$\begin{aligned} |\phi(t) - \phi_1(t)| &= \frac{1}{2\pi} \left| \int_{-}^{+} \int_{-}^{+} g(t-x) \frac{1}{1 - \sqrt{2\pi}K(s)} e^{isx} ds dx \right| \\ &\leq \frac{1}{2\pi} \left(\int_{-}^{+} \varphi(t-x)^2 dx \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{-}^{+} \left| \int_{-}^{+} \frac{1}{1 - \sqrt{2\pi}K(s)} e^{isx} ds \right|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

Moreover, following property (g), we know that

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \left(\int_{-}^{+} \left| \frac{1}{\sqrt{2\pi}} \int_{-}^{+} \frac{1}{1 - \sqrt{2\pi}K(s)} e^{isx} ds \right|^2 dx \right)^{\frac{1}{2}} \\ = \frac{1}{\sqrt{2\pi}} \left(\int_{-}^{+} \left| \frac{1}{1 - \sqrt{2\pi}K(x)} \right|^2 dx \right)^{\frac{1}{2}} = M < \infty. \end{aligned}$$

Therefore $|\phi(t) - \phi_1(t)| \leq M \left(\int_{-}^{+} \varphi(x)^2 dx \right)^{\frac{1}{2}}$.

(2) Now, we will prove the existence and uniqueness of Eq.(1.1), and that the solution is real function.

Existence. Let $f \in L^2(-\infty, +\infty)$, follows the property (g), we know that $F \in L^2(-\infty, +\infty)$, and $\frac{1}{1-\sqrt{2\pi}K(s)} \in L^2(-\infty, +\infty)$, such that

$$\int_{-}^{+} \left| \frac{F(s)}{1-\sqrt{2\pi}K(s)} \right| ds \leq \left| \int_{-}^{+} |F(s)|^2 ds \right|^{\frac{1}{2}} \left| \int_{-}^{+} \left| \frac{1}{1-\sqrt{2\pi}K(s)} \right|^2 ds \right|^{\frac{1}{2}}.$$

Thus $\left| \frac{F(s)}{1-\sqrt{2\pi}K(s)} \right| \in L(-\infty, +\infty)$, and the inverse fourier transform of $\frac{F(s)}{1-\sqrt{2\pi}K(s)}$ exists, that means $\phi_1(t)$ is actually a solution of Eq. (1.1).

Uniqueness. If there exists another solution $\tilde{\phi}_1(t)$ satisfies Eq.(1.1) and the fourier transform of it exists. By using fourier transform, we have

$$\tilde{\Phi}_1(s) = \frac{F(s) + G(s)}{1 - \sqrt{2\pi}K(s)} = \Phi_1(s)$$

follows the property (c), we know $\tilde{\phi}_1(t) = \phi_1(t)$.

Real function. Since f, k are real functions, we know it is impossible that the solution of Eq.(1.1) ϕ_1 is a pure imaginary function. If ϕ_1 is a complex function, from the above proof we know that $\phi_1(t) = \Re\phi_1(t) + i\Im(\phi_1(t))$ and fourier transform of $\Re(\phi_1(t))$ and $\Im(\phi_1(t))$ exist, and $\Im(\phi_1(t))$ is a solution of

$$\Im(\phi_1(t)) = \int_{-}^{+} k(t-s)\Im(\phi_1(t))ds$$

by fourier transform and property(e)

$$F(\Im(\phi_1(s))) = \sqrt{2\pi}K(s)F(\Im(\phi_1(s)))$$

and $K(s) \neq \frac{1}{2\pi}$, such that $F(\Im(\phi_1(s))) = 0 = F(\Im(\phi_1(s)))$. Thus, $\phi(t)_1$ is a real function. $\square$

Theorem 2.2. Suppose that $f(t), k(t)$ are real continuous functions and $f(t), k(t) \in L^2(-\infty, +\infty)$, $K(s) \neq 0$ and $\frac{1}{K(s)} \in L^2(-\infty, +\infty)$, and and if a real function $\phi(t) : \mathbb{R} \rightarrow \mathbb{R}$ and $F(\phi)(s)$ exists, satisfies that

$$\|f(t) - \int_{-}^{+} k(t-s)\phi(s)ds\| \leq \varphi(t) \quad (2.7)$$

where $\varphi(t) \in L^2(-\infty, +\infty)$, $\varphi(t) \geq 0$. Then there exists a unique real solution $\phi_1(t)$ ($F(\phi_1)(s)$ exists) satisfies Eq.(1.2) and $\|\phi - \phi_1\| \leq M(\int_{-}^{+} \varphi(x)^2 dx)^{\frac{1}{2}}$ where M is a constant.

Proof. (1) Firstly, we will prove the stability.

Let ϕ be a solution of Ineq.(2.7) and put

$$f(t) - \int_{-}^{+} k(t-s)\phi(s)ds := g(t) \quad (2.8)$$

where $|g(t)| \leq \varphi(t)$, since $\varphi(t) \in L^2(-\infty, +\infty)$, we know that $g(t) \in L^2(-\infty, +\infty)$.

It follows Fourier transform and the property(e) that

$$F(s) - \sqrt{2\pi}K(s)\Phi(s) = G(s). \quad (2.9)$$

Then we have

$$\Phi(s) = \frac{F(s) - G(s)}{\sqrt{2\pi}K(s)}. \quad (2.10)$$

It follows the inverse Fourier transform that

$$p(t) = \frac{1}{\sqrt{2\pi}} \int_{-}^{+} \frac{1}{\sqrt{2\pi}K(s)} e^{ist} ds. \quad (2.11)$$

Then, we can know from the property(a) and (e) that

$$\Phi(s) = (F(s) - G(s)) \cdot P(s) = F\left[\frac{1}{\sqrt{2\pi}} \int_{-}^{+} (f(t-x) - g(t-x))p(x)dx\right](s). \quad (2.12)$$

It follows the inverse Fourier transform and properties(c) that

$$\begin{aligned} \phi(t) &= \frac{1}{\sqrt{2\pi}} \int_{-}^{+} (f(t-x) - g(t-x))p(x)dx \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-}^{+} \int_{-}^{+} (f(t-x) - g(t-x)) \frac{1}{K(s)} e^{isx} ds dx, \end{aligned}$$

and if we let $g(t) = 0$, we get a solution of Eq.(1.2)

$$\phi_1(t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-}^{+} \int_{-}^{+} f(t-x) \frac{1}{K(s)} e^{isx} ds dx.$$

Therefore (using Theorem 1.1):

$$\begin{aligned} |\phi(t) - \phi_1(t)| &= \frac{1}{(2\pi)^{\frac{3}{2}}} \left| \int_{-}^{+} \int_{-}^{+} g(t-x) \frac{1}{K(s)} e^{isx} ds dx \right| \\ &\leq \frac{1}{(2\pi)^{\frac{3}{2}}} \left(\int_{-}^{+} \varphi(t-x)^2 dx \right)^{\frac{1}{2}} \left(\int_{-}^{+} \left| \int_{-}^{+} \frac{1}{K(s)} e^{isx} ds \right|^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

Moreover, following property (g), we have

$$\frac{1}{2\pi} \left(\int_{-}^{+} \left| \frac{1}{\sqrt{2\pi}} \int_{-}^{+} \frac{1}{K(s)} e^{isx} ds \right|^2 dx \right)^{\frac{1}{2}} = \frac{1}{2\pi} \left(\int_{-}^{+} \left| \frac{1}{K(x)} \right|^2 dx \right)^{\frac{1}{2}} = M < \infty.$$

Therefore $|\phi(t) - \phi_1(t)| \leq M \left(\int_{-}^{+} \varphi(x)^2 dx \right)^{\frac{1}{2}}.$

(2) Now, we will prove the existence and uniqueness of Eq.(1.2), and the solution is actually a real function.

Existence. $f \in L^2(-\infty, +\infty)$, follows the property(g), we know that $F \in L^2(-\infty, +\infty)$. And $\frac{1}{K(s)} \in L^2(-\infty, +\infty)$, following the proof of theorem 2.1, we can know that $\phi_1(s)$ exist and satisfy Eq.(1.2).

Uniqueness. If there exists another solution $\tilde{\phi}_1(t)$ satisfies Eq.(1.2) and the fourier transform of it exists, by using fourier transform, by using fourier transform, we have

$$\tilde{\Phi}_1(s) = \frac{F(s) - G(s)}{\sqrt{2\pi}K(s)} = \Phi_1(s)$$

follows the property(c), we know $\tilde{\phi}_1(t) = \phi_1(t)$.

Real function. Since f, k are real functions, we know it is impossible that the solution of Eq.(1.2) ϕ_1 is a pure imaginary function. If ϕ_1 is a complex function, from the above proof we know that $\phi_1(t) = \Re\phi_1(t) + i\Im(\phi_1(t))$ and fourier transform of $\Re(\phi_1(t))$ and $\Im(\phi_1(t))$ exist, and $\Im(\phi_1(t))$ is a solution of

$$0 = \int_{-}^{+} k(t-s)\Im(\phi_1(t))ds$$

by fourier transform and property(e)

$$0 = K(s)\Im(\Phi_1(s))$$

and $K(s) \neq 0$, such that $\Im(\Phi_1(s)) = 0 = \Im(\phi_1(t))$. Thus, $\phi_1(t)$ is a real function. $\square$

Theorem 2.3. Suppose that $f(t), k(t)$ are real continuous functions and $f(t), k(t) \in L^2(-\infty, +\infty)$, $K(s) \neq 0$ and $\frac{1}{K(s)} \in L^2(-\infty, +\infty)$, and and if a real function $\phi(t) : \mathbb{R} \rightarrow \mathbb{R}$ and $F(\phi)(s)$ exists, satisfies that

$$\|f(t) - \int_{-}^{+} k(t+s)\phi(s)ds\| \leq \varphi(t) \quad (2.13)$$

where $\varphi(t) \in L^2(-\infty, +\infty)$, $\varphi(t) \geq 0$. Then there exists a unique real solution $\phi_1(t)$ ($F(\phi_1)(s)$ exists) satisfies Eq.(1.3) and $\|\phi - \phi_1\| \leq M(\int_{-}^{+} \varphi(x)^2 dx)^{\frac{1}{2}}$ where M is a constant.

Proof. (1) We will prove the stability property.

Let ϕ be a solution of Ineq.(2.13) and put

$$f(t) - \int_{-}^{+} k(t+s)\phi(s)ds := g(t), \quad (2.14)$$

where $|g(t)| \leq \varphi(t)$, since $\varphi(t) \in L^2(-\infty, +\infty)$, we know that $g(t) \in L^2(-\infty, +\infty)$.

It follows, using Fourier transform, that

$$\begin{aligned} \int_{-}^{+} f(t)e^{-ist}dt &= \int_{-}^{+} g(t)e^{-ist}dt + \int_{-}^{+} \left[\int_{-}^{+} k(t+x)\phi(x)dx \right] e^{-ist}dt \\ &= \int_{-}^{+} g(t)e^{-ist}dt + \int_{-}^{+} \int_{-}^{+} k(t+x)e^{-is(t+x)}\phi(x) \\ &\quad \times e^{isx}dxdt, \end{aligned}$$

$$F(s) - G(s) = \sqrt{2\pi}K \cdot F^{-1}(\phi(t))(s). \quad (2.15)$$

Then we have

$$\phi(t) = F\left(\frac{F-G}{\sqrt{2\pi}K}\right) \quad (2.16)$$

and if we let $g(t) = 0$, we obtain solution of Eq.(1.3)

$$\phi_1(t) = F\left(\frac{F}{\sqrt{2\pi}K}\right). \quad (2.17)$$

Thus

$$\begin{aligned} |\phi(t) - \phi_1(t)| &= \left| F\left(\frac{G}{\sqrt{2\pi}K}\right) \right| \\ &\leq \frac{1}{2\pi} \left| \int_{-}^{+} G(s)^2 ds \right|^{\frac{1}{2}} \left| \int_{-}^{+} \left(\frac{e^{-ist}}{K(s)} \right)^2 ds \right|^{\frac{1}{2}} \\ &\leq \frac{1}{2\pi} \left| \int_{-}^{+} \varphi(s)^2 ds \right|^{\frac{1}{2}} \left| \int_{-}^{+} \left(\frac{e^{-ist}}{K(s)} \right)^2 ds \right|^{\frac{1}{2}} \\ &\leq \frac{1}{2\pi} \left| \int_{-}^{+} \varphi(s)^2 ds \right|^{\frac{1}{2}} \left| \int_{-}^{+} \left(\frac{1}{K(s)} \right)^2 ds \right|^{\frac{1}{2}} \end{aligned}$$

and let $M = \frac{1}{2\pi} \left| \int_{-}^{+} \left(\frac{1}{K(s)} \right)^2 ds \right|^{\frac{1}{2}}$, we have $\|\phi - \phi_1\| \leq M \left| \int_{-}^{+} \varphi(s)^2 ds \right|^{\frac{1}{2}}$.

(2) Now, we will prove the existence and uniqueness of Eq.(1.3), and the solution is actually a real function.

Existence. $f \in L^2(-\infty, +\infty)$, follows the property(g), we know that $F \in L^2(-\infty, +\infty)$. And $\frac{1}{K(s)} \in L^2(-\infty, +\infty)$, following the proof of theorem 2.1, we can know that $\phi_1(s)$ exist and satisfy Eq.(1.3).

Uniqueness. If there exists another real solution $\tilde{\phi}_1(t)$ satisfies Eq.(1.3) and the fourier transform of it exists, by using fourier transform, we have

$$F^{-1}(\tilde{\phi}_1(s)) = \frac{F}{\sqrt{2\pi}K} = F^{-1}(\phi_1(s))$$

follows the property(c), we know $\tilde{\phi}_1(t) = \phi_1(t)$.

Real function. Since f, k are real functions, we know it is impossible that the solution of Eq.(1.3) ϕ_1 is a pure imaginary function. If ϕ_1 is a complex function, from the above proof we know that $\phi_1(t) = \Re\phi_1(t) + i\Im(\phi_1(t))$ and Fourier transform of $\Re(\phi_1(t))$ and $\Im(\phi_1(t))$ exist, and $\Im(\phi_1(t))$ is a solution of

$$0 = \int_{-}^{+} k(t+s)\Im(\phi_1(t))ds,$$

by Fourier transform:

$$0 = \sqrt{2\pi}K(s) \cdot F^{-1}(\Im(\phi_1(t)))(s)$$

and $K(s) \neq 0$, such that $\Im(\Phi_1(s)) = 0 = \Im(\phi_1(t))$. Thus, $\phi_1(t)$ is a real function. $\square$

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