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SOLITON SOLUTIONS IN A MODIFIED COMBINED SINE-COSINE-GORDON MODEL

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Abstract: We present a combined sine-cosine-Gordon model in (1+1) dimensions modified by the addition of an extra kinetic term similar to the Skyrme term in higher dimensions and a potential term. We obtain soliton solutions of the considered model and show that it possesses the kink solutions of the unmodified model, with the same velocity, as well as a double-kink solutions. We study some properties of the modified model, and end with conclusions and some features and comments.

Key Words: solitons, double-kink solutions, sine-Gordon equation, combined sine-cosine-Gordon equation

1. Introduction

The sine-Gordon model is a very important and a well-studied problem [1], because of its relations to many physical systems, such as, the propagation of fluxons in Josephson junctions between two superconductors [2-4], the motion of a rigid pendula attached to a stretched wire [4], solid state physics, non-linear optics, and stability of fluid motions. It has also been of interest for

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the string theory, since it seems to be related to the classical string on specific manifolds [5]. This model is integrable and can be solved by the inverse scattering method. Due to the wide applications of sine-Gordon type equations, various other methods in (1+1) space-time dimensions, had been developed to investigate the solutions under different circumstances, such as, Bäcklund transformation, Hirota bilinear forms, pseudo-spectral method, the tanh-sech method [6, 7], and the sine-cosine method [8]. Also, a new variable separated ODE method was developed by Sirendaoreji and Jiong [3] and adopted by Fu *et al.* [9] and Wazwaz [10] to study the related problems.

The most important property of the sine-Gordon type equations is that they possess solitonic structures. Solitons widely appear in nonlinear classical field theories as stable, particle-like objects, with finite mass and smooth structures. They are localized waves that propagate without change of its shape and velocity properties and stable against mutual collisions [11]. They range from kinks and antikinks in some simple dynamical systems to monopoles of non-Abelian gauge theories. Constructing exact solitary or soliton like solutions has long been a major concern for both mathematicians and physicists. In (1+1) dimensions, well-known equations possess soliton solutions, such as the sine-Gordon equation, the KdV equation, the nonlinear Schrödinger equation, ... etc. Various models in higher dimensions possess also soliton-like structures. Among them in (2+1) dimensions is the $O(3)$ σ model and its generalizations, the CP^N σ model [12]. Such models in two Euclidean dimensions do have solitonic structures but they are unstable under small fluctuations, however, apart from this instability, these structures do behave very much like solitons.

Peyrard *et al.* [13] have studied a sine-Gordon model in (1+1) dimensions modified by the addition of a further kinetic term similar to the Skyrme term in higher dimensions and an extra potential term to its Lagrangian density. However, as there are no Skyrme-like terms in (1+1) dimensions, Peyrard *et al.* [13] have pointed out that to consider the effect of extra terms, one has to sacrifice one of the conditions imposed on the form of the terms in (2+1) dimensions. Thus as the analog of the Skyrme term, they have taken a term involving two-time and two-space derivatives, and they broken the condition of the Lorentz covariance of the model. Also, one of us [14] have adopted the procedures of ref. [13] and studied soliton solutions in a modified ϕ^4 model in (1+1) dimensions. In both studied models [13,14], it is found that these models have interesting properties. In particular, in addition to their familiar kink solutions they possess a double-kink solutions. Recently, Wazwaz [10] has studied a combined sine-cosine-Gordon (CSCG) model and obtained soliton solutions. A further study of these solutions is presented by Kuo and Hu [15], in particular, they derived the relation between the phase of the CSCG equation and the parameters in these solutions and discussed its applications in physical systems.

In this paper, we shall adopt the procedures of refs. [13,14] for the addition of a further kinetic term and an extra potential term and present a modified CSCG model. We show that this modified model is also possess both kink and double-kink solutions. In the next section, we present the basic formulation of the modified CSCG model. In section 3, we study soliton solutions of the modified CSCG model and show that it possesses the same solutions of the usual model with some conditions on the parameters of the extra terms. In section 4, we discuss some properties of these solutions. In particular, we show that this modified model possesses double-kink solutions. Finally, in section 5, we conclude the paper and give some features and comments.

2. Formulation of the Modified CSCG Model

Let us consider the CSCG model in (1+1) dimensions defined by the Lagrangian density

$$L = \frac{1}{2} u_t^2 - \frac{1}{2} u_x^2 - 1 + \alpha \cos u - \beta \sin u, \tag{1}$$

where α and β are arbitrary constants and u is a single real scalar field. The Euler-Lagrange equation corresponding to this Lagrangian is given by

$$u_{tt} - u_{xx} + \alpha \sin u + \beta \cos u = 0, \tag{2}$$

where subscripts denote differentiation. For $\beta = 0$, the above Lagrangian with the corresponding equation of motion are those of the familiar sine-Gordon model.

In this work, we aim to modify the CSCG model in (1+1) dimensions defined by the Lagrangian density (1). We adopt the procedures of refs. [13,14], and add to the Lagrangian density (1) a further kinetic term similar to the Skyrme term in higher dimensions and an extra potential term. Therefore, we shall consider the modified CSCG model defined by a Lagrangian density in the form

$$\begin{aligned} L = {} & \tfrac{1}{2} u_t^2 - \tfrac{1}{2} u_x^2 + \tfrac{1}{2} \theta\, u_x^2 u_t^2 \left(a \sin\left(\tfrac{u}{2}\right) + b \cos\left(\tfrac{u}{2}\right) \right) \\ & - 1 + \alpha \cos u - \beta \sin u + k \left(a \sin\left(\tfrac{u}{2}\right) + b \cos\left(\tfrac{u}{2}\right) \right)^5, \end{aligned} \tag{3}$$

where a and b are arbitrary constants and θ and k are some parameters. The θ-term represents the extra kinetic term and the k-term represents the extra potential term. For vanishing θ and k, this Lagrangian reduces to the usual CSCG Lagrangian (1).

It is easy to derive the Euler-Lagrange equation corresponding to the Lagrangian density (3). This equation is given by

$$
\begin{aligned}
&u_{tt} - u_{xx} + \alpha \sin u + \beta \cos u \\
&\quad + \theta \left[4 u_x u_t u_{xt} + u_x^2 u_{tt} + u_t^2 u_{xx}\right] \left(a \sin\left(\tfrac{u}{2}\right) + b \cos\left(\tfrac{u}{2}\right) \right) \\
&\quad + \tfrac{3}{4} \theta u_x^2 u_t^2 \left(a \cos\left(\tfrac{u}{2}\right) - b \sin\left(\tfrac{u}{2}\right) \right) \\
&\quad - \tfrac{5}{2} k \left(a \sin\left(\tfrac{u}{2}\right) + b \cos\left(\tfrac{u}{2}\right) \right)^4 \left(a \cos\left(\tfrac{u}{2}\right) - b \sin\left(\tfrac{u}{2}\right) \right) = 0.
\end{aligned}
\tag{4}
$$

If the two parameters θ and k vanish, the equation of motion of the modified CSCG model (4) reduces to the equation of motion of the usual CSCG equation (2). For non-vanishing θ and k, this modified CSCG model exhibits interesting properties which we shall discuss in the next sections.

3. Soliton Solutions

Let us first obtain soliton solutions of the usual CSCG equation (2). As we are looking for localized solutions, we first unite the independent variables x and t into one wave variable $\xi = x - \omega t$, where ω is a positive real number defines the velocity of the imposed solitary wave. This will carry out the partial differential equations into ordinary differential equations. Therefore, we consider

$$
u(x,t) = u(\xi); \qquad \xi = x - \omega t. \tag{5}
$$

Using equation (5) into the equation of motion of the usual CSCG model (2), we obtain

$$
-(1-\omega^2)\, u'' + \alpha \sin u + \beta \cos u = 0; \qquad ' \equiv \frac{d}{d\xi}. \tag{6}
$$

To obtain soliton solutions of equation (6), we follow ref. [10], and assume that $u(\xi)$ satisfies the variable separated ordinary differential equation given by

$$u' = \nu \left(a \sin\left(\frac{u}{2}\right) + b \cos\left(\frac{u}{2}\right) \right), \tag{7}$$

where ν is a parameter that will be determined. Differentiating the ansatz (7) with respect to ξ gives

$$\begin{aligned} u'' &= \tfrac{\nu^2}{2} \left(a \sin\left(\tfrac{u}{2}\right) + b \cos\left(\tfrac{u}{2}\right) \right) \left(a \cos\left(\tfrac{u}{2}\right) - b \sin\left(\tfrac{u}{2}\right) \right) \\ &= \nu^2 \left[\tfrac{(a^2 - b^2)}{4} \sin u + \tfrac{ab}{2} \cos u \right]. \end{aligned} \tag{8}$$

Substituting (8) into (6), we find that the ansatz (7) satisfies the usual CSCG equation (6) when

$$\alpha = \frac{1}{4}(a^2 - b^2), \qquad \beta = \frac{1}{2}ab, \tag{9}$$

and

$$\nu^2 = \frac{1}{1 - \omega^2}. \tag{10}$$

Therefore, the solutions of the variable separated ordinary differential equation (7) are solutions of the usual CSCG model with α and β given by equation (9) and ν given by equation (10).

Equation (7) is separable, hence we set

$$\frac{du}{a \sin\left(\frac{u}{2}\right) + b \cos\left(\frac{u}{2}\right)} = \nu\, d\xi, \tag{11}$$

where by integrating both sides we find the following solutions

$$u = 4 \tan^{-1} \left[\frac{\sqrt{a^2 + b^2}}{b} \tanh \left(\frac{\sqrt{a^2 + b^2}}{4} \nu \xi \right) + \frac{a}{b} \right], \tag{12}$$

$$u = 4 \cot^{-1} \left[\frac{\sqrt{a^2 + b^2}}{b} \tanh \left(\frac{\sqrt{a^2 + b^2}}{4} \nu \xi \right) - \frac{a}{b} \right], \tag{13}$$

$$u = 4 \tan^{-1} \left[\frac{\sqrt{a^2 + b^2}}{b} \coth \left(\frac{\sqrt{a^2 + b^2}}{4} \nu \xi \right) + \frac{a}{b} \right], \tag{14}$$

$$u = 4\cot^{-1}\left[\frac{\sqrt{a^2+b^2}}{b}\coth\left(\frac{\sqrt{a^2+b^2}}{4}\nu\xi\right) - \frac{a}{b}\right]. \tag{15}$$

In fact there are relations between these four solutions. Equation (13) can be transformed into equation (12) and equation (15) can be transformed into equation (14) by the transformations $u \to u - 2\pi$ and $x \to -x$, $t \to -t$. Hence, for the CSCG model, we have the two kinds of solutions given by equations (12) and (14).

Next, we turn our attention to the modified CSCG equation of motion (4). Inserting (5) into (4), we obtain the following ordinary differential equation of the modified CSCG model

$$\begin{aligned}
&-(1-\omega^2)\,u'' + \alpha\sin u + \beta\cos u \\
&+ 6\theta\omega^2 u'^2 u''\left(a\sin\left(\tfrac{u}{2}\right) + b\cos\left(\tfrac{u}{2}\right)\right) + \tfrac{3}{4}\theta\omega^2 u'^4\left(a\cos\left(\tfrac{u}{2}\right) - b\sin\left(\tfrac{u}{2}\right)\right) \\
&- \tfrac{5}{2}k\left(a\sin\left(\tfrac{u}{2}\right) + b\cos\left(\tfrac{u}{2}\right)\right)^4\left(a\cos\left(\tfrac{u}{2}\right) - b\sin\left(\tfrac{u}{2}\right)\right) = 0.
\end{aligned} \tag{16}$$

If we now consider that $u(\xi)$ satisfies the same variable separated ordinary differential equation given by (7), we find that equation (16) is satisfied with α and β are given by equation (9) and ν is given by equation (10) and such that

$$\nu^4\omega^2 = \frac{2k}{3\theta} = c, \tag{17}$$

where c is a constant fixed by the parameters of the model. This means that the ansatz (7) also satisfies the modified CSCG equation (16) when an extra condition given by (17) imposed on ν given by (10). In other words, the two kinds of solutions of the usual CSCG model, given by equations (12) and (14), are also solutions for the modified CSCG model with the extra condition (17). These solutions are kink-like solutions.

Figure 1 shows the kink solution given by equation (12) for $\theta = 0.05$ and $k = 0.75$. Figure 1(a) shows the kink solution for different values of a and b, namely, $a = 0.83$ and $b = 0.23$. Figure 1(b) shows the kink solution for $a = 0.85$ and $b = 10^{-5}$, whereas Figure 1(c) for $a = 10^{-5}$ and $b = 0.85$. For $a = b = 0.6$ the kink solution can be shown in Figure 1(d). Clearly, in all cases, this kink is a 2π-kink.

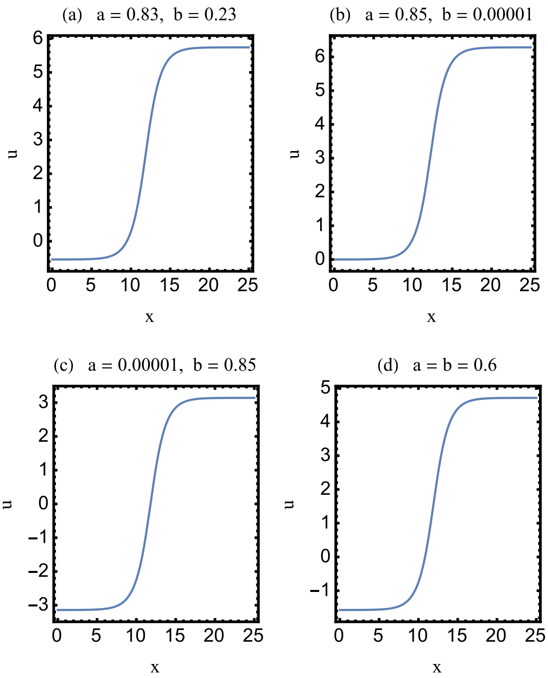

Figure 1: The Kink solution given by equation (12), for $\theta = 0.05$ and $k = 0.75$; and four different cases of a and b. In (a), $a = 0.83$ and $b = 0.23$, in (b), $a = 0.85$ and $b = 10^{-5}$, in (c), $a = 10^{-5}$ and $b = 0.85$, and in (d), $a = b = 0.6$.

4. Properties of the Modified CSCG Model

In this section we study some properties of the considered modified CSCG model. First, from equations (10) and (17), the velocity of the modified CSCG kink is given by

$$\omega = \sqrt{\frac{1 + 2c - \sqrt{1 + 4c}}{2c}}. \tag{18}$$

This means that the velocity of the kink is determined in terms of the ratio of θ and k. In this, the modified model resembles the $O(3)\,\sigma$ model in (2+1) dimensions in which the addition of the Skyrme-like and potential terms has fixed the size of the soliton-like structures.

The dependence of the properties of the model upon θ and k separately is also reflected in the shape of the total potential energy of the modified CSCG model, which is given by

$$V(u) = 1 - \alpha \cos u + \beta \sin u - k \left(a \sin\left(\frac{u}{2}\right) + b \cos\left(\frac{u}{2}\right) \right)^5 . \tag{19}$$

This potential energy can be shown in Figure 2 for the same different cases of a and b as in Figure 1. Full line is for $k = 0.0$, dash-dotted line is for $k = 0.5$, dash line is for $k = 1$, and dotted line is for $k = 1.5$. Clearly, the potential is periodic with its period being equal to 4π and the CSCG-kink interpolates between the two minima situated at the ends of a half of the period. The shape of the potential suggests the existence of other solutions having the shape of a double-kink in the interval of this half period when the potential exhibits a local minimum at the middle of this half period. For example, in Figure 2(b), where we have taken $a = 0.85$ and $b = 10^{-5}$, the CSCG-kink interpolates between 0 and 2π and the double-kink exists in the domain $0 < u < 2\pi$ when the potential has a local minimum at $u = \pi$. Similar situations can be seen in the other cases of Figure 2.

We observe that permanent profile solutions cannot be obtained by simply boosting a static solution because the Lagrangian (3) is not Lorentz invariant. Let us seek localized solutions of equation (4) which depend only on the variable ξ. In this case, equation (4) is transformed into equation (16). Next, we use equations (7)-(9) into equation (16), then multiply it by u' and integrating to get

$$\begin{aligned} &\left(\frac{du}{d\xi}\right)^4 \left[3\theta\omega^2 \left(a \sin\left(\tfrac{u}{2}\right) + b \cos\left(\tfrac{u}{2}\right) \right) \right] - (1-\omega^2)\left(\frac{du}{d\xi}\right)^2 \\ &\quad + \left(a \sin\left(\tfrac{u}{2}\right) + b \cos\left(\tfrac{u}{2}\right) \right)^2 - 2k \left(a \sin\left(\tfrac{u}{2}\right) + b \cos\left(\tfrac{u}{2}\right) \right)^5 = 0. \end{aligned} \tag{20}$$

This equation has special solutions if we require that u satisfies the variable separated ordinary differential equation (7) with the condition (10), as then terms two and three cancel each other. If we imposed the extra condition (17), we find that the first and last terms cancel too. As we showed before, these special solutions are the CSCG-kink (12) and (14) found before.

General solutions of (20) can be found by solving equation (20) for $(du/d\xi)^2$.

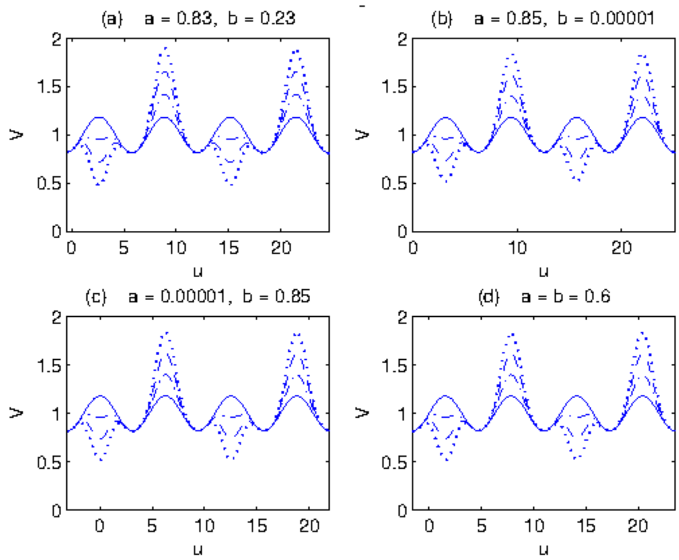

Figure 2: Potential energy of the modified CSCG model for various values of k and the same four different cases of a and b as in Figure 1. Full line is for $k = 0.0$, dash-dotted line is for $k = 0.5$, dash line is for $k = 1$, and dotted line is for $k = 1.5$.

Clearly, equation (20) has real solutions if

$$\begin{aligned}\Delta &= (1-\omega^2)^2 \\ &\quad - 12\theta\omega^2\left(a\,\sin\left(\tfrac{u}{2}\right) + b\,\cos\left(\tfrac{u}{2}\right)\right)^3\left[1 - 2k\left(a\,\sin\left(\tfrac{u}{2}\right) + b\,\cos\left(\tfrac{u}{2}\right)\right)^3\right] \\ &\geq 0.\end{aligned} \tag{21}$$

Therefore, these general solutions which vanish at the ends of the half period are given by

$$\frac{du}{d\xi} = \pm\left[\frac{1 - \omega^2 - \sqrt{\Delta}}{6\theta\omega^2(a\,\sin(u/2) + b\,\cos(u/2))}\right]^{\frac{1}{2}}. \tag{22}$$

This equation cannot be solved analytically, but as it is a one variable first order differential equation, its solutions can be computed numerically, using, *e.g.*, MATHEMATICA.

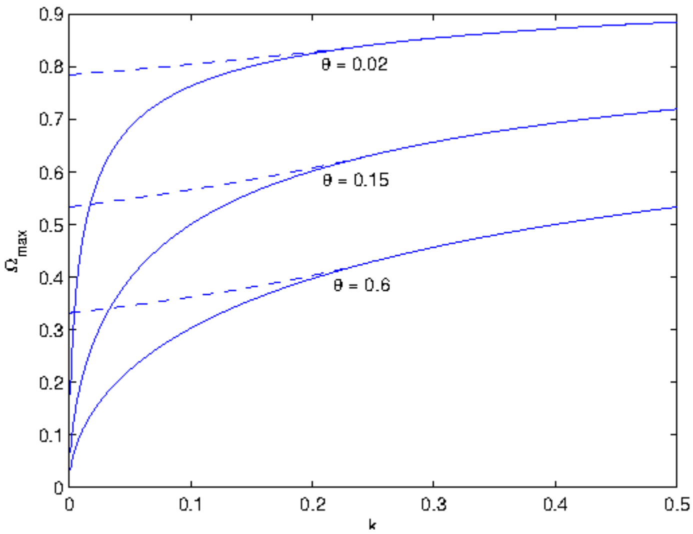

Figure 3: Variation of the maximum velocity of the kink (dash line) and of the velocity of the CSCG kink (full line) versus k for three different values of θ.

The condition $\Delta \geq 0$ for all u in the range of the half period, *i.e.*, $0 \geq (a \sin(\frac{u}{2}) + b \cos(\frac{u}{2}))^3 \leq 1$, imposes a maximum velocity for the solitary wave, given by

$$\nu^4 \omega^2 \leq \frac{1}{12\theta(1-2k)} \qquad \text{for} \quad k \leq 0.25, \tag{23}$$

$$\nu^4 \omega^2 \leq \frac{2k}{3\theta} \qquad \text{for} \quad k \geq 0.25. \tag{24}$$

This means that the maximum velocity $\omega \equiv \Omega_{\max}$ is given by equation (18), where

$$c = \frac{1}{12\theta(1-2k)} \qquad \text{for} \quad k \leq 0.25, \tag{25}$$

and

$$c = \frac{2k}{3\theta} \qquad \text{for} \quad k \geq 0.25. \tag{26}$$

The variation of the maximum velocity and of the velocity of the CSCG-kink versus k can be shown in Figure 3 for different three values of θ, namely, $\theta = 0.6$, 0.15 and 0.02. We notice that for $k \geq 0.25$, the maximum velocity defined by (24) coincides with the velocity of the CSCG-kink, whereas for $k < 0.25$ it is larger than that of the CSCG-kink.

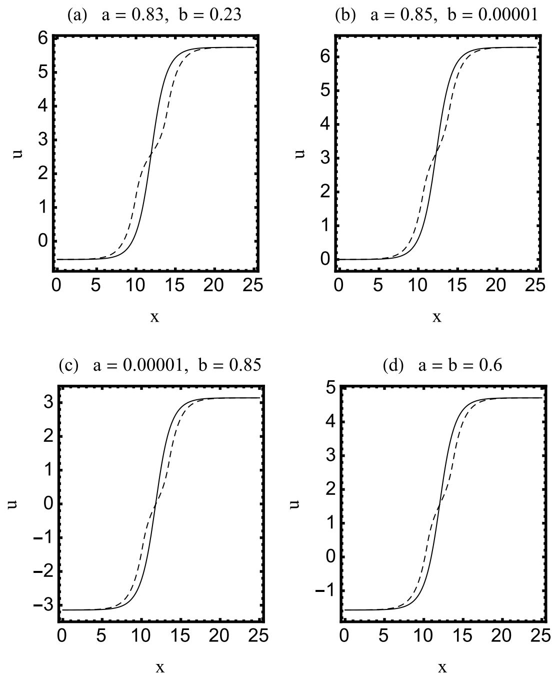

Figure 4: The CSCG-kink (full line) and the general solution (dash line) at the same speed $\omega = \Omega$ for $\theta = 0.05$, $k = 0.75$ and the same four different cases of a and b as in Figure 1.

Using equations (10) and (17) into (21), we see that Δ reduces to Δ_0 given by

$$\Delta_0 = \frac{1}{\nu^4}\left[1 - 4k\left(a\sin\left(\frac{u}{2}\right) + b\cos\left(\frac{u}{2}\right)\right)^3\right]^2. \tag{27}$$

Therefore, for $k \leq 0.25$, the expression of $du/d\xi$ given by equation (22) is identical to the particular variable separated ordinary differential equation (7). Thus, equation (22) gives the general expression of the CSCG-kink with its velocity is in the interior of the allowed velocity range. On the other hand, for $k > 0.25$, the CSCG-kink solution is different from the general expression (22). This implies that the equation of motion (4) of the modified CSCG model has two different solutions in the same domain of the half period mentioned before. This is a remarkable property of this model, which has also been observed for the modified sine-Gordon model [13] and for the modified ϕ^4 model [14]. We have used MATHEMATICA and solved equation (22) numerically. Figure 4 shows these two solutions (dash line) and the special kink (full line) for $k = 0.75$ and $\theta = 0.05$ ($\omega = 0.854309$), and for the four cases of a and b considered in Figure 1.

We observe from Figure 4 that, as expected before, since the potential energy of the model, shown in Figure 2, has a local minimum at the middle of the half period, the general solution (22) can be regarded as if it were as formed out of two subkinks in the first and second quarter of the 4π-period bound to each other. Because of this particular shape, the general solution given by (22) is called the double-kink.

5. Discussion and Conclusion

In conclusion, we have presented some soliton solutions in a CSCG model in (1+1) dimensions modified by the addition of an extra kinetic term similar to the Skyrme term in higher dimensions and a potential term. We obtained the equations of motion of the usual and modified models. We used the method of variable separated ordinary differential equation given in ref. [10] and constructed kink solutions. We showed that the constructed kink solutions are solutions of the usual CSCG model with the parameter ν given by equation (10) as well as are solutions of the modified CSCG model with an extra condition given by equation (17) imposed on the parameter ν.

We studied some properties of the modified CSCG model. In particular, plotting the potential energy of the model, we observed that the potential is periodic with its period equal to 4π and the kink of the model interpolates between the two minima situated at the ends of a half of the period. Moreover, the shape of the potential suggests the existence of other solutions having the shape of a double-kink in the interval of this half period when the potential exhibits a local minimum at the middle of this half period.

This encouraged us to make some more analysis and obtained a general expression of the solution of the model, which include, as special case, the kink solutions obtained before. Using MATHEMATICA, we computed and plotted this general solution and found that it looks as if it were as formed out of two subkinks bound to each other and therefore, is called the double-kink. This property was also hold in both the modified sine-Gordon model [13] and the modified ϕ^4 model [14].

It is important to report here that the additional kinetic and potential terms to the usual Lagrangian of the model, represented by the θ- and k-terms, respectively, are not unique. However, they are related to the form of $du/d\xi$ in the variable separated ordinary differential equation, *e.g.*, in our case, the ansatz (7). To illustrate this in some more details, if the θ- and k-terms are considered as

$$\theta u_x^2 u_t^2 f(u) \qquad \text{and} \qquad kg(u), \tag{28}$$

for some functions f and g, then the functions $f(u)$ and $g(u)$ should be given, respectively, by

$$\left(\frac{du}{d\xi}\right)^{\frac{1}{n}} \qquad \text{and} \qquad \left(\frac{du}{d\xi}\right)^{(4+\frac{1}{n})}, \tag{29}$$

for some integer n.

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