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WEAKLY WO-CONTINUOUS FUNCTIONS ON ASSOCIATED w-SPACES

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Abstract: We introduce the notions of weakly WO-continuity and strongly W-closed graph.

We study some characterizations and properties of such functions.

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1. Introduction

In [17], Siwiec introduced the notions of weak neighborhoods and weak base in a topological space. We introduced the weak neighborhood systems defined by using the notion of weak neighborhoods in [12]. And we also introduced a

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weak neighborhood space (briefly WNS) which is independent of neighborhood spaces [4] and general topological spaces [2]. We introduced the notion of w-spaces in [13] and investigated some basic properties. In [14], we introduced and studied the notions WK-continuity and WO-continuity on associated w-spaces. In this paper, we introduce the notions of weakly WO-continuity and strongly W-closed graph on associated w-spaces. And we study some characterizations and properties of such functions.

2. Preliminaries

Let S be a subset of a topological space X. The closure (resp., interior) of S will be denoted by clS (resp., $intS$). A subset S of X is called a *pre-open* set [10] (resp., α-set [14], *semi-open* [6]) if $S \subset int(cl(S))$ (resp., $S \subset int(cl(int(S))), S \subset cl(int(S))$). The complement of a preopen set (resp., α-set, *semi-open*) is called a *preclosed* set (resp., α-closed set, *semi-closed*). The family of all preopen sets (resp., α-sets, semi-open sets) in X will be denoted by $PO(X)$ (resp., $\alpha(X), SO(X)$). We know the family $\alpha(X)$ is a topology finer than the given topology on X.

A subset A of a topological space (X, τ) is said to be:

(a) g-closed [5] if $Cl(A) \subset U$ whenever $A \subset U$ and U is open in X;

(b) gp-closed [7] if $pCl(A) \subset U$ whenever $A \subset U$ and U is open in X;

(c) gs-closed [1, 3] if $sCl(A) \subset U$ whenever $A \subset U$ and U is open in X;

(d) $g\alpha$-closed [9] if $\tau^\alpha Cl(A) \subset U$ whenever $A \subset U$ and U is α-open in X, where $\tau^\alpha = \alpha(X)$;

(e) $g\alpha^{\square}$-closed [8] if $\tau^\alpha Cl(A) \subset Int(U)$ whenever $A \subset U$ and U is α-open in X;

(f) $g\alpha^{\square\square}$-closed [8] if $\tau^\alpha Cl(A) \subset Int(Cl(U))$ whenever $A \subset U$ and U is α-open in X;

(g) αg-closed [9] if $\tau^\alpha Cl(A) \subset U$ whenever $A \subset U$ and U is open in X;

(h) $\alpha^{\square\square} g$-closed [9] if $\tau^\alpha Cl(A) \subset Int(Cl(U))$ whenever $A \subset U$ and U is open in X;

(i) g-open (resp., gp-open, gs-open, $g\alpha$-open, $g\alpha^{\square}$-open, $g\alpha^{\square\square}$-open, αg-open;

$\alpha^{\square\square} g$-open) if the complement A is g-closed (resp., gp-closed, gs-closed, $g\alpha$-closed, $g\alpha^{\square}$-closed, $g\alpha^{\square\square}$-closed, αg-closed, $\alpha^{\square\square} g$-closed).

The family of all g-open (resp., gp-open, gs-open, $g\alpha$-open, $g\alpha^{*}$-open, $g\alpha^{**}$-open, αg-open, $\alpha^{**}g$-open) sets in X will be denoted by $gO(X)$ (resp., $gpO(X), gsO(X), g\alpha O(X), g\alpha^{*}O(X), g\alpha^{**}O(X), \alpha gO(X), \alpha^{**}gO(X)$.

Definition 2.1 ([13]). Let X be a nonempty set. A subfamily w_X of the power set $P(X)$ is called a *weak structure* on X if it satisfies the following:

(1) $\emptyset \in w_X$ and $X \in w_X$.

(2) For $U_1, U_2 \in w_X$, $U_1 \cap U_2 \in w_X$.

Then the pair (X, w_X) is called a *w-space* on X. Then $V \in w_X$ is called a *w-open* set and the complement of a w-open set is a *w-closed* set.

Let (X, τ) be a topological space. Then the family τ, $GO(X)$, $g\alpha O(X)$, $g\alpha^{*}O(X)$, $g\alpha^{**}O(X)$, $\alpha gO(X)$ and $\alpha^{**}gO(X)$ on X are all weak structures on X. But $PO(X)$, $GPO(X)$ and $SO(X)$ are not weak structures on X.

Definition 2.2 ([13]). Let (X, w_X) be a w-space. For a subset A of X, the *w-closure* of A and the *w-interior* of A are defined as the following:

(1) $wCl(A) = \cap\{F : A \subset F, X - F \in w_X\}$.

(2) $wInt(A) = \cup\{U : U \subset A, U \in w_X\}$.

Theorem 2.3 ([13]). *Let (X, w_X) be a w-space and $A \subset X$.*
(1) $x \in wInt(A)$ if and only if there exists a w-open subset U containing x such that $U \subset A$.

(2) $x \in wCl(A)$ if and only if $A \cap V \neq \emptyset$ for every w-open subset V containing x.

Let X be a nonempty set and let (X, τ) be a topological space. A subfamily w_τ of the power set $P(X)$ is called an *associated weak structure* [14] on X if $\tau \subset w_\tau$. Then the pair (X, w_τ) is called an *associated w-space* with τ [14].

Definition 2.4 ([14]). Let $f : (X, w_\tau) \to (Y, \mu)$ be a function on an associated w-space X with τ and a topological space (Y, μ). Then f is said to be:

(1) *WO-continuous* if for $x \in X$ and for each open subset V containing $f(x)$, there is a w-open subset U of X containing x such that $f(U) \subset V$,

(2) *WK-continuous* if for every open set V in Y, $f^{-1}(V)$ is a w-open set in X.

Remark 2.5. Every WK-continuous function is a WO-continuous function. But the converse may not be true as shown in Example 4.6 [14].

Definition 2.6 ([15]). Let (X_i, w_i) be a w-space for $i \in J$. Set $\mathbf{S} = \{{\pi_i}^{-1}(U_i) : U_i \in w_i$ for $i \in J\}$ where $\pi_i : \prod X_i \to X_i$ is an i-th projection map. We call $\mathbf{W} = \{\cap \mathbf{B} : \mathbf{B} \subseteq \mathbf{S}$ and $\mathbf{B}$ is finite $\}$ the *product weak structure* on $X = \prod X_i$, and $(\prod X_i, \mathbf{W})$ is called the *product weak space* (briefly, *product w-space*).

3. Weakly WO-Continuous Functions

Definition 3.1. Let (X, w_τ) be an associated w-space and (Y, μ) be a topological space. Then $f : X \to Y$ is said to be *weakly WO-continuous* if for $x \in X$ and for each open subset V containing $f(x)$, there is a w-open subset U containing x such that $f(U) \subset Cl(V)$.

From Remark 2.5 and definition of weakly WO-continuity, we get the following implications but the converses are not true:

continuous $\Rightarrow$ WK-continuous $\Rightarrow$ WO-continuous $\Rightarrow$ weakly WO-continuous.

Example 3.2. Let $X = Y = \{a, b, c\}$. Consider an associated weak structures $w_\sigma = \{\emptyset, \{a\}, \{c\}, X\}$ and a topological space $\mu = \{\emptyset, \{a, b\}, Y\}$.

Let $f : (X, w_\sigma) \to (Y, \mu)$ be a function defined by $f(x) = x$, for $x \in X$. Then since $Cl(\{a, b\}) = Y$, f is weakly WO-continuous. But f is not WO-continuous because X is the only w_σ-open set containing b in X.

Theorem 3.3. *Let $f : (X, w_\sigma) \to (Y, \mu)$ be a function on an associated w-space (X, w_σ) and a topological space (Y, μ). Then the following statements are equivalent:*

(1) f is weakly WO-continuous.

(2) $f^{-1}(V) \subseteq wInt(f^{-1}(Cl(V)))$ for every open subset V of Y.

(3) $wCl(f^{-1}(Int(A))) \subseteq f^{-1}(A)$ for every closed set A of Y.

(4) $wCl(f^{-1}(Int(Cl(B)))) \subseteq f^{-1}(Cl(B))$ for every set B of Y.

(5) $f^{-1}(Int(B)) \subseteq wInt(f^{-1}(Cl(Int(B)))$ for every set B of Y.

(6) $wCl(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$ for every open subset V of Y.

Proof. (1) $\Rightarrow$ (2) Let V be an open subset in Y and $x \in f^{-1}(V)$. There exists a w-open subset U of X containing x such that $f(U) \subseteq Cl(V)$. Since $x \in U \subseteq f^{-1}(Cl(V))$, $x \in wInt(f^{-1}(Cl(V)))$. Hence

$$x \in f^{-1}(V) \subseteq wInt(f^{-1}(wCl(V))).$$

(2) $\Rightarrow$ (3) Let A be a closed subset in Y. Then $Y - A$ in open in Y and, by (2)

$$f^{-1}(Y - A) \subseteq wInt(f^{-1}(Cl(Y - A))) = wInt(f^{-1}(Y - Int(A))) \\ \subseteq X - wCl(f^{-1}(Int(A))).$$

Thus $wCl(f^{-1}(Int(A))) \subseteq f^{-1}(A)$.

(3) $\Rightarrow$ (4) Let B be a subset of Y. Since $Cl(B)$ is closed in Y, from (3), it follows $wCl(f^{-1}(Int(Cl(B)))) \subseteq f^{-1}(Cl(B))$.

(4) $\Rightarrow$ (5) Let B be a subset of Y. Then

$$f^{-1}(Int(B)) = X - f^{-1}(Cl(Y - B)) \subseteq X - wCl(f^{-1}Int(Cl(Y - B))) \\ = wInt(f^{-1}Cl(Int(B))).$$

Thus we get the result.

(5) $\Rightarrow$ (6) Let V be an open subset of Y. Suppose $x \notin f^{-1}(Cl(V))$. Then $f(x) \notin Cl(V)$ and so there exists an open set U containing $f(x)$ such that $U \cap V = \emptyset$ and so $Cl(U) \cap V = \emptyset$. By (5), $x \in f^{-1}(U) \subseteq wInt(f^{-1}(Cl(U)))$. Hence there exists an open set G containing x such that $x \in G \subseteq f^{-1}(Cl(U))$. Since

$$Cl(U) \cap V = \emptyset$$

and

$$f(G) \subseteq Cl(U), \qquad G \cap f^{-1}(V) = \emptyset$$

and so $x \notin wCl(f^{-1}(V))$. Hence $wCl(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$.

(6) $\Rightarrow$ (1) Let $x \in X$ and V an open set in Y containing $f(x)$. Since $V = Int(V) \subseteq Int(Cl(V))$, by (6),

$$x \in f^{-1}(V) \subseteq f^{-1}(Int(Cl(V))) = X - f^{-1}(Cl(Y - Cl(V))) \\ \subseteq X - wCl(f^{-1}(Y - Cl(V))) = mInt(f^{-1}(Cl(V))).$$

Hence there exists a w-open subset U in X such that $x \in U \subseteq f^{-1}(Cl(V))$. □

We recall that a point x of a topological space X is said to be θ-adherent of A if $A \cap Cl(V) \neq \emptyset$ for every open set V containing x. The set of all θ-adherent points of A is called θ-closure of A [18] and is denoted by $Cl_\theta(A)$. If $A = Cl_\theta(A)$, then A is called θ-closed. The complement of a θ-closed set is said to be θ-open. It is shown in [18] that $Cl(A) = Cl_\theta(A)$ for every open set A and $Cl_\theta(B)$ is closed for every subset B of X.

Theorem 3.4. *Let $f : (X, w_\sigma) \to (Y, \mu)$ be a function on an associated w-space (X, w_σ) and a topological space (Y, μ). Then the following statements are equivalent:*

(1) f is weakly WO-continuous.

(2) $wCl(f^{-1}(Int(Cl_\theta(B)))) \subseteq f^{-1}(Cl_\theta(B))$ for every set B of Y.

(3) $wCl(f^{-1}(Int(Cl(B)))) \subseteq f^{-1}(Cl_\theta(B))$ for every set B of Y.

(4) $wCl(f^{-1}(Int(Cl(G)))) \subseteq f^{-1}(Cl(G))$ for every open subset G of Y.

(5) $f(wCl(A)) \subseteq Cl_\theta(f(A))$ for every set A of X.

(6) $wCl(f^{-1}(B)) \subseteq f^{-1}(Cl_\theta(B))$ for every set B of Y.

Proof. (1) $\Rightarrow$ (2) Let B be any subset in Y; then $Cl_\theta(B)$ is closed, by Theorem 3.3 (3), we get the result.

(2) $\Rightarrow$ (3) It is obvious since $Cl(B) \subseteq Cl_\theta(B)$ for every subset B of Y.

(3) $\Rightarrow$ (4) It is obvious since $Cl(G) = Cl_\theta(G)$ for every open subset G of Y.

(4) $\Rightarrow$ (1) Since $G \subseteq Int(Cl(G))$ for every open set G of Y, from Theorem 3.3 (6), it follows f is weakly WO-continuous.

(1) $\Rightarrow$ (5) Let A be a subset of X. Let $x \in wCl(A)$ and G be an open subset of Y containing $f(x)$. Since f is weakly WO-continuous, there exists a w-open set U containing x in X such that $f(U) \subseteq Cl(G)$. Since $x \in wCl(A)$, we have $U \cap A \neq \emptyset$ and so $\emptyset \neq f(U) \cap f(A) \subseteq Cl(G) \cap f(A)$. Thus we have $f(x) \in Cl_\theta(f(A))$.

(5) $\Rightarrow$ (6) Let B be a subset of Y; then by (5), we have

$$f(wCl(f^{-1}(B))) \subseteq Cl_\theta(f(f^{-1}(B))) \subseteq Cl_\theta(B)$$

and so we get the result.

(6) $\Rightarrow$ (1) Let B be a subset of Y; then by (6),

$$\begin{aligned} wCl(f^{-1}(Int(Cl(B)))) &\subseteq f^{-1}(Cl_\theta(Int(Cl(B)))) \\ &= f^{-1}(Cl(Int(Cl(B)))) \subseteq f^{-1}(Cl(B)). \end{aligned}$$

Hence f is weakly WO-continuous by Theorem 3.3 (4). □

Let X be a w-space. Then X is said to be w-T_2 [15] if for every two distinct points x and y in X, there exist two disjoint weak open sets U and V such that $x \in U$ and $y \in V$.

Let X be a topological space. Then X is said to be *Urysohn* if for every two distinct points x and y in X, there exist two open sets U and V such that $Cl(U) \cap Cl(V) = \emptyset$.

Theorem 3.5. *Let $f : (X, w_\sigma) \to (Y, \mu)$ be a function on an associated w-space (X, w_σ) and a topological space (Y, μ). If f is a weakly WO-continuous injection and Y is Urysohn, then X is w-T_2.*

Proof. Let x_1 and x_2 be two distinct elements in X, then $f(x_1) \neq f(x_2)$. There exist two open sets U and V in Y containing $f(x_1)$, $f(x_2)$, respectively, such that $Cl(U) \cap Cl(V) = \emptyset$. Since f is weakly WO-continuous, there exist w-open sets U_1, V_2 containing x_1, x_2, respectively, such that $f(U_1) \subseteq Cl(U)$, $f(V_2) \subseteq Cl(V)$. It follows $U_1 \cap V_2 = \emptyset$. Hence X is w-T_2. □

Definition 3.6. Let $f : (X, w_\sigma) \to (Y, \mu)$ be a function on an associated w-space (X, w_σ) and a topological space (Y, μ). We call f has a *strongly W-closed graph* if for each $(x, y) \notin G(f)$, there exist a w-open set U and an open set V containing x and y, respectively, such that $(U \times Cl(V)) \cap G(f) = \emptyset$.

Lemma 3.7. *Let $f : (X, w_\sigma) \to (Y, \mu)$ be a function on an associated w-space (X, w_σ) and a topological space (Y, μ). Then f has a strongly W-closed graph if for each $(x, y) \notin G(f)$, there exist a w-open set U containing and an open set V containing x and y, respectively, such that $f(U) \cap Cl(V) = \emptyset$.*

Proof. Obvious. □

Theorem 3.8. *Let $f : (X, w_\sigma) \to (Y, \mu)$ be a function on an associated w-space (X, w_σ) and a topological space (Y, μ). If f is weakly WO-continuous and Y is Urysohn, then f has a strongly W-closed graph.*

Proof. Let $(x, z) \notin G(f)$. Then $z \neq f(x)$ and since Y is Urysohn, there exist two open sets U and V containing z and $f(x)$, respectively, such that $Cl(U) \cap Cl(V) = \emptyset$. Since f is weakly WO-continuous, there exists a w-open set H containing x such that $f(H) \subseteq Cl(V)$. It implies $f(H) \cap Cl(U) = \emptyset$. Hence f has a strongly W-closed graph. □

Theorem 3.9. *Let $f : (X, w_\sigma) \to (Y, \mu)$ be a function on an associated w-space (X, w_σ) and a topological space (Y, μ). If f is a weakly WO-continuous injection with a strongly W-closed graph, then X is w-T_2.*

Proof. Let x_1 and x_2 be two distinct elements in X, then $f(x_1) \neq f(x_2)$. This implies that $(x_1, f(x_2)) \in (X \times Y) - G(f)$. Since f has a strongly W-closed graph, there exist a w-open set U and an open set V containing x_1 and $f(x_2)$, respectively, such that $f(U) \cap Cl(V) = \emptyset$. Since f is weakly WO-continuous, there exist a w-open set W containing x_2 such that $f(W) \subset Cl(V)$. It implies $f(W) \cap f(U) = \emptyset$. Therefore $W \cap U = \emptyset$ and so X is a w-T_2 space. □

A subset A of a w-space (X, w_X) is called *W-compact* [15] relative to A if every collection $\{U_i : i \in J\}$ of w-open subsets of X such that $A \subset \cup\{U_i : i \in J\}$, there exists a finite subset J_0 of J such that $A \subset \cup\{U_i : i \in J_0\}$. A subset A of a w-space (X, w_X) is said to be *W-compact* if A is W-compact as a subspace of X.

A subset A of a w-space X is said to be *quasi H-closed* relative to (Y, μ) [18] if every collection $\{U_i : i \in J\}$ of open subsets of X such that $A \subset \cup\{U_i : i \in J\}$, there exists a finite subset J_0 of J such that $A \subset \cup\{Cl(U_i) : i \in J_0\}$.

Theorem 3.10. *Let $f : (X, w_\sigma) \to (Y, \mu)$ be a function on an associated w-space (X, w_σ) and a topological space (Y, μ). If f is weakly WO-continuous and A is a W-compact subset of X, then $f(A)$ is quasi H-closed relative to (Y, μ).*

Proof. Let $\{V_i : i \in J\}$ be a cover of $f(A)$ by open subsets of Y. For each $x \in A$, there exists $i(x) \in J$ such that $f(x) = y \in V_{i(x)}$. Since f is weakly WO-continuous, there exists a w-open set $U(x)$ containing x such that $f(U(x)) \subseteq Cl(V_{i(x)})$. The family $\{U(x) : x \in A\}$ is a cover of A by w-open sets in X. Since A is w-compact, there is a finite subcover $\{U(x_1), U(x_2), \cdots, U(x_n) : x_j \in A, j = 1, 2, \cdots, n\}$ such that $A \subseteq \cup U(x_j)$. Then

$$f(A) \subseteq f(\cup U(x_j)) \subseteq \cup f(U(x_j)) \subseteq \cup Cl(V_{i(x_j)}),$$

$1 \leq j \leq n$.

Thus $f(A)$ is quasi H-closed relative to (Y, μ). □

References

[1] P. Bhattacharyya, B.K. Lahiri, Semi-generalized closed sets in topology, *Indian J. Math.*, **29**, No. 3 (1987), 375-382.

[2] Á. Császár, Generalized topology, generalized continuity, *Acta Math. Hungar.*, **96** (2002), 351-357.

[3] J. Dontchev, H. Maki, On sg-closed sets and semi-λ-closed sets, *Q&A in General Topology*, **15** (1997), 259-266.

[4] D.C. Kent, W.K. Min, Neighborhood spaces, *International Journal of Mathematics and Mathematical Sciences*, **32**, No. 7 (2002), 387-399.

[5] N. Levine, Generalized closed sets in topology, *Rend. Cir. Mat. Palermo*, **19** (1970), 89-96.

[6] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Ams. Math. Monthly*, **70** (1963), 36-41.

[7] H. Maki, J. Umehara, T. Noiri, Every topological space is pre-$T_{\frac{1}{2}}$, *Mem. Fac. Sci. Kochi Univ. Ser. A*, **17** (1996), 33-42.

[8] H. Maki, R. Devi, K. Balachandran, Associated topologies of generalized α-closed sets and α-generalized closed sets, *Mem. Fac. Sci. Kochi Univ. Ser. A*, **15** (1994), 51-63.

[9] Maki, R. Devi, K. Balachandran, Generalized α-closed maps and α-generalized closed maps, *Indian J. Pure Appl. Math.*, **29**, No. 1 (1998), 37-49.

[10] A.S. Mashhour, M.E. Abd El-Monsef, S.N. El-Deeb, On precontinuous and weak precontinuous mappings, *Proc. Math. and Phys. Soc. Egypt*, **53** (1982), 47-53.

[11] W.K. Min, Some results on generalized topological spaces and generalized systems, *Acta Math. Hungar.*, **108** (1-2) (2005), 171-181.

[12] W.K. Min, On weak neighborhood systems and weak neighborhood spaces, *Acta Math. Hungar.*, (2008), **doi:** 10.1007/s10474-008-7205-8.

[13] W.K. Min, Y.K. Kim, On weak structures and w-spaces, *Far East Journal of Mathematical Sciences*, Accepted.

[14] W.K. Min, Y.K. Kim, WO-continuity and WK-continuity on associated w-spaces, *International Journal of Pure and Applied Mathematics*, Accepted.

[15] W.K. Min, Y.K. Kim, On product weak structures and product w-spaces, *International Journal of Mathematical Analysis*, Accepted.

[16] O. Njastad, On some classes of nearly open sets, *Pacific Journal of Mathematics*, **15**, No. 3 (1964), 961-970.

[17] F. Siwiec, On defining a space by a weak base, *Pacific Journal of Mathematics*, **52**, No. 1 (1974), 351-357.

[18] N.V. Velicko, H-closed topological spaces, *Amer. Math. Soc. Transl.*, **78** (1968), 103-118.