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EXISTENCE AND OSGOOD TYPE UNIQUENESS OF MILD SOLUTIONS OF NONLINEAR INTEGRODIFFERENTIAL EQUATION WITH NONLOCAL CONDITION

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Abstract: The aim of the present paper is to study existence, Osgood type uniqueness and qualitative properties of mild solutions of nonlinear integrodifferential equation with nonlocal condition. The main tools employed in the analysis are based on the applications of the Leray-Schauder alternative, rely on a priori bounds of solutions and the well known Bihari's integral inequality.

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1. Introduction

Using Tychonov's fixed point theorem, the method of successive approximations,

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and the comparison method, S. Sugiyama [18] studied the existence and uniqueness of solutions of the following problem:

$$\frac{dx(t)}{dt} = f\big(t, x(t), x(t-1)\big), \tag{1.1}$$

for $0 \leq t \leq t_1$, with the conditions

$$x(t-1) = \phi(t) \quad (0 \leq t < 1), \tag{1.2}$$

$$x(0) = x_0, \tag{1.3}$$

where x and f represent n-dimensional vectors (see [18] for details) and Stokes [17] has discussed the same problems as above for nonlinear differential equations.

In [20], S. Sugiyama proved the existence, stability, and boundedness of solutions of the difference-differential problem (1.1)–(1.3) by making use of Tychonov's fixed point theorem with additional condition on f and we also refer the papers of S. Sugiyama [19, 21]. Subsequently some authors have been studied the problems of existence, uniqueness and other properties of solutions of (1.1) or their special forms by using different techniques, see [1, 2, 3, 8, 12, 14, 15] and the references cited therein. We also refer some papers and monographs [11, 12, 18], [1, p. 342], [6, p.308], [10, p. 18].

Recently, in the interesting paper [16], B. G. Pachpatte has studied the existence and uniqueness of solutions (1.1)–(1.3) by the Leray-Schauder alternative and well known Bihari's integral inequality.

From the above works, we can see a fact, although the integrodifferential problems have been investigated by some authors. However, to our knowledge, the integrodifferential equation with nonlocal conditions and an infinitesimal generator of operators has not been discussed extensively. So motivated by all the works above, the aim of this paper is to prove the existence and uniqueness of solutions of the integrodifferential of the form:

$$x'(t) + Ax(t) = f\Big(t, x(t), \int_0^t k(t,s,x(s))ds, x(t-1)\Big), \tag{1.4}$$

for $t \in J = [0,b]$, $(b > 0)$ under the conditions

$$x(t-1) = \phi(t) \quad (0 \leq t < 1), \tag{1.5}$$

$$x(0) + g(x) = x_0, \tag{1.6}$$

where $-A$ is an infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in X, $f \in C(J \times X \times X, X)$, $g \in C(C(J,X),X)$

and $\phi(t)$ is a continuous function for $0 \leq t < 1$, $\lim_{t \to 1-0} \phi(t)$ exists, for which we denote by $\phi(1-0) = c_0$. If we consider the solutions of (1.4) for $t \in J$, we obtain a function $x(t-1)$ which is unable to define as solution for $0 \leq t < 1$. Hence, we have to impose some condition, for example the condition (1.5). We note that, if $0 \leq t < 1$, the problem is reduced to integrodifferential equation

$$x'(t) + Ax(t) = f\big(t, x(t), \int_0^t k(t, s, x(s))ds, \phi(t)\big),$$

with initial condition $x(0) + g(x) = x_0$. Here, it is essential to obtain the solutions of (1.4)–(1.6) for $0 \leq t \leq b$, so that, we suppose in the sequel b is not less than 1.

Our main objective here is to investigate the global existence of solution to (1.4)-(1.6) by using the topological transversality theorem of Granas [[7], p. 61], also known as Leray-Schauder alternative. Osgood type uniqueness result for the solutions of (1.4)-(1.6) is established by using the well known Bihari's integral inequality. Our general formulation of (1.4)-(1.6) is an attempt to generalize the results of [5, 16].

The paper is organized as follows. In Section 2, we present the preliminaries and hypotheses. Section 3 deals with existence and Osgood type uniqueness of the solutions. Section 4 discuss the boundedness of solutions. Finally, in Section 5 we discuss result on continuous dependence of solutions on initial data.

2. Preliminaries and Hypotheses

Before proceeding to the statement of our main results, we shall set forth some preliminaries and hypotheses that will be used in our subsequent discussion.

Let X be the Banach space with norm $\|\cdot\|$. Let $B = C(J, X)$ be the space of all continuous functions from J into X endowed with the supremum norm

$$\|x\|_B = \sup\{\|x(t)\| : t \in J\}.$$

Definition 2.1. Let $-A$ is the infinitesimal generator of a C_0-semigroup $T(t),\ t \geq 0$, on a Banach space X. The function $x \in B$ given by

$$x(t) = T(t)[x_0 - g(x)] + \int_0^t T(t-s) f\big(s, x(s), \int_0^s k(s, \sigma, x(\sigma))d\sigma, \phi(s)\big) ds, \tag{2.1}$$

for $0 \le t < 1$, and

$$x(t) = T(t)[x_0 - g(x)] + \int_0^1 T(t-s) f\big(s, x(s), \int_0^s k(s,\sigma,x(\sigma))d\sigma, \phi(s)\big)\,ds$$
$$+ \int_1^t T(t-s) f\big(s, x(s), \int_0^s k(s,\sigma,x(\sigma))d\sigma, x(s-1)\big)ds, \tag{2.2}$$

for $1 \le t \le b$, is called the mild solution of the problem (1.4)–(1.6).

For completeness, we state here the fixed point result by Granas in ([7], p. 61).

Lemma 2.2. (Leray-Schauder Alternative). *Let S be a convex subset of a normed linear space E and assume $0 \in S$. Let $F : S \to S$ be a completely continuous operator and let $U(F) = \{x \in S : x = \lambda Fx\}$ for some $0 < \lambda < 1$. Then either $U(F)$ is unbounded or F has a fixed point.*

We also need the following integral inequality, often referred to as Bihari's inequality [[13], p. 107].

Lemma 2.3. *Let $u(t), p(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\mathbb{R}_+ = [0,\infty)$. Let $w(u)$ be a continuous, nondecreasing function defined on $\mathbb{R}_+$, $w(u) > 0$ for $u > 0$ and $w(0) = 0$. If*

$$u(t) \le c + \int_0^t p(s) w(u(s)) ds,$$

for $t \in \mathbb{R}_+$, where $c \ge 0$ is a constant, then for $0 \le t \le t_1$,

$$u(t) \le W^{-1}\Big[W(c) + \int_0^t p(s) ds\Big],$$

where

$$W(r) = \int_{r_0}^r \frac{ds}{w(s)}, \quad r > 0,\ r_0 > 0,$$

W^{-1} is the inverse function of W and $t_1 \in \mathbb{R}_+$ be chosen so that

$$W(c) + \int_0^t p(s) ds \in \mathrm{Dom}(W^{-1}),$$

for all $t \in \mathbb{R}_+$ lying in the interval $0 \le t \le t_1$.

We list the following hypotheses:

(H_1) $-A$ is the infinitesimal generator of a semigroup of bounded linear operators $T(t)$ in X, which is compact for $t > 0$, and there exist constant $M \ge 1$ such that $\|T(t)\| \le M, \quad t \ge 0$.

(H_2) The function f in (1.4) satisfies the condition

$$\|f(t,x,y,z)\| \leq p(t)[\Omega(\|x\|) + \Omega(\|y\|) + \Omega(\|z\|)],$$

for every $x,y,z \in X$, where $p \in C(J,\mathbb{R}_+)$ and $\Omega : \mathbb{R}_+ \to (0,\infty)$ is continuous and increasing function satisfying $\Omega(\alpha(t)\|x\|) \leq \alpha(t)\Omega(\|x\|)$, where α is defined as the function p.

(H_3) There exists a continuous function $q : J \to \mathbb{R}_+$ such that

$$\| \int_0^t k(t,s,x(s))ds\| \leq q(t)\|x\|$$

for every $t \geq s \geq 0$ and $x \in X$.

(H_4) There exist constant $G > 0$ such that

$$\|g(x)\| \leq G \quad \text{for every} \quad x \in B.$$

(H_5) For each $t \in J$, the function $f(t,\cdot,\cdot,\cdot) : J \times X \times X \times X \to X$ is continuous and for each $(x,y,z) \in X \times X \times X$, the function $f(\cdot,x,y,z) : J \times X \times X \times X \to X$ is strongly measurable.

(H_6) For each $t,s \in J$, the function $k(t,s,\cdot) : J \times J \times X \to X$ is continuous and for each $x \in X$, the function $k(\cdot,\cdot,x) : J \times J \times X \to X$ is strongly measurable.

(H_7) For every positive integer m there exists $\alpha_m \in L^1(J)$ such that

$$\sup_{\|x\|\leq m,\ \|y\|\leq m,\ \|z\|\leq m} \|f(t,x,y,z)\| \leq \alpha_m(t) \quad \text{for} \quad t \in J \text{ a.e.}$$

(H_8) The function f in (1.4) satisfies the condition

$$\|f(t,x,y,z) - f(t,\bar{x},\bar{y},\bar{z})\| \leq \bar{p}(t)[\bar{\Omega}(\|x-\bar{x}\|) + \bar{\Omega}(\|y-\bar{y}\| + \bar{\Omega}(\|z-\bar{z}\|)],$$

for every $x,y,z,\bar{x},\bar{y},\bar{z} \in X$, where $\bar{p} \in C(\mathbb{R}_+,\mathbb{R}_+)$ and $\bar{\Omega}(u)$ is a continuous and increasing function for $u \geq 0$, $\bar{\Omega}(0) = 0$.

(H_9) The function k in (1.4) satisfies the condition

$$\| \int_0^t \big[k(t,s,x) - k(t,s,\bar{x})\big] ds\| \leq \bar{q}(t)\|x-\bar{x}\|,$$

for every $x,\bar{x} \in X$, where $\bar{q} \in C(\mathbb{R}_+,\mathbb{R}_+)$.

(H_{10}) There exist constant $\bar{G} > 0$ such that

$$\|g(x) - g(\bar{x})\| \leq \bar{G}\|x - \bar{x}\| \quad \text{for every} \quad x, \bar{x} \in B$$

and $M\bar{G} < 1$.

3. Existence and Uniqueness Results

The following theorem deals with the Wintner type global existence result for the solution of (1.4)-(1.6).

Theorem 3.1. *Suppose that $(H_1) - (H_7)$ hold. Then (1.4)-(1.6) has a solution $x(t)$ defined on J provided b satisfies*

$$\int_0^b M[p(s)(1 + q(s)) + p(s+1)]ds < \int_c^\infty \frac{ds}{\Omega(s)}, \tag{3.1}$$

where

$$c = M[\|x_0 + G]\| + M\int_0^1 p(s)\Omega(\|\phi(s)\|)ds. \tag{3.2}$$

Proof. We define an operator $F : B \to B$ by

$$(Fx)(t) = T(t)[x_0 - g(x)] + \int_0^t T(t-s)f\big(s, x(s), \int_0^s k(s, \sigma, x(\sigma))d\sigma, \phi(s)\big)\,ds, \tag{3.3}$$

for $0 \leq t < 1$, and

$$\begin{aligned}(Fx)(t) = T(t)[x_0 - g(x)] &+ \int_0^1 T(t-s)f\big(s, x(s), \int_0^s k(s, \sigma, x(\sigma))d\sigma, \phi(s)\big)\,ds \\ &+ \int_1^t T(t-s)f\big(s, x(s), \int_0^s k(s, \sigma, x(\sigma))d\sigma, x(s-1)\big)\,ds,\end{aligned} \tag{3.4}$$

for $1 \leq t \leq b$.

In order to use 2.2, we establish the priori bounds on the solutions of the problem

$$x'(t) + Ax(t) = \lambda f\big(t, x(t), \int_0^t k(t, s, x(s))ds, x(t-1)\big), \tag{3.5}$$

under the initial conditions (1.5)-(1.6)for $\lambda \in (0,1)$. Let $x(t)$ be a solution of (3.5) with (1.5)-(1.6), then we consider the following two cases.

Case I: $0 \le t < 1$. From the hypotheses, we have

$$\begin{aligned}
\|x(t)\| &= \Big\|T(t)[x_0 - g(x)] + \int_0^t \lambda T(t-s) f(s, x(s), \int_0^s k(s,\sigma,x(\sigma))d\sigma, \phi(s))ds\Big\| \\
&\le \|T(t)\|\|[x_0 - g(x)]\| + \int_0^t Mp(s)[\Omega(\|x(s)\|) + \Omega(\|\int_0^s k(s,\sigma,x(\sigma))d\sigma\|) \\
&\quad +\Omega(\|\phi(s)\|)]ds \\
&\le M[\|x_0\| + G] + \int_0^t Mp(s)\Omega(\|\phi(s)\|)ds + \int_0^t Mp(s)[\Omega(\|x(s)\|) \\
&\quad + \Omega(q(s)\|x(s)\|)]ds \\
&\le M[\|x_0\| + G] + \int_0^1 Mp(s)\Omega(\|\phi(s)\|)ds \\
&\quad + \int_0^t Mp(s)[\Omega(\|x(s)\|) + q(s)\Omega(\|x(s)\|)]ds \\
&= c + \int_0^t Mp(s)[1+q(s)]\Omega(\|x(s)\|)ds. \qquad (3.6)
\end{aligned}$$

Let $u(t)$ be defined by the right hand side of (3.6), then $u(0) = c$, $\|x(t)\| \le u(t)$ and

$$u'(t) = Mp(t)[1+q(t)]\Omega(\|x(t)\|) \le Mp(t)[1+q(t)]\Omega(u(t));$$

that is,

$$\frac{u'(t)}{\Omega(u(t))} \le Mp(t)[1+q(t)]. \qquad (3.7)$$

Integration of (3.7) from 0 to t $(0 \le t < 1)$, the change of variable $t \to s = u(t)$, and the condition (3.1) gives

$$\begin{aligned}
\int_c^{u(t)} \frac{ds}{\Omega(s)} \le \int_0^t Mp(s)[1+q(s)]ds \le \int_0^1 Mp(s)[1+q(s)]ds \\
< \int_c^\infty \frac{ds}{\Omega(s)}. \qquad (3.8)
\end{aligned}$$

From this inequality and the mean value theorem, we observe that, there is a constant γ_1 independent of $\lambda \in (0,1)$ such that $u(t) \le \gamma_1$ for $0 \le t < 1$ and hence $\|x(t)\| \le \gamma_1$.

Case II: $1 \leq t \leq b$. From the hypotheses, we have

$$\begin{aligned}
\|x(t)\| &= \Big\|T(t)[x_0 - g(x)] + \int_0^1 \lambda T(t-s) f(s, x(s), \int_0^s k(s,\sigma,x(\sigma))d\sigma, \phi(s))ds \\
&\quad + \int_1^t \lambda T(t-s) f(s, x(s), \int_0^s k(s,\sigma,x(\sigma))d\sigma, x(s-1))ds\Big\| \\
&\leq \|T(t)\| \|[x_0 - g(x)]\| \\
&\quad + \int_0^1 Mp(s)[\Omega(\|x(s)\|) + q(s)\Omega(\|x(s)\|) + \Omega(\|\phi(s)\|)]ds \\
&\quad + \int_1^t Mp(s)[\Omega(\|x(s)\|) + q(s)\Omega(\|x(s)\|) + \Omega(\|x(s-1)\|)]ds \\
&\leq M[\|x_0\| + G] \\
&\quad + \int_0^1 Mp(s)\Omega(\|\phi(s)\|)ds + \int_0^1 Mp(s)[1+q(s)]\Omega(\|x(s)\|)ds \\
&\quad + \int_1^t Mp(s)[1+q(s)]\Omega(\|x(s)\|)ds + \int_1^t Mp(s)\Omega(\|x(s-1)\|)ds \\
&= c + \int_0^t Mp(s)[1+q(s)]\Omega(\|x(s)\|)ds + I_1 \qquad (3.9)
\end{aligned}$$

where

$$I_1 = \int_1^t Mp(s)\Omega(\|x(s-1)\|)ds. \qquad (3.10)$$

By making the change of variable, from (3.10), we obtain

$$I_1 = \int_0^{t-1} Mp(\sigma+1)\Omega(\|x(\sigma)\|)d\sigma \leq \int_0^t Mp(\sigma+1)\Omega(\|x(\sigma)\|)d\sigma. \qquad (3.11)$$

Using this inequality in (3.9), we obtain

$$\|x(t)\| \leq c + \int_0^t M[p(s)(1+q(s)) + p(s+1)]\Omega(\|x(s)\|)ds. \qquad (3.12)$$

Let $v(t)$ be defined by the right hand side of (3.12), then $v(0) = c$, $\|x(t)\| \leq v(t)$ and

$$\begin{aligned}
v'(t) &= M[p(t)(1+q(t)) + p(t+1)]\Omega(\|x(t)\|) \\
&\qquad \leq M[p(t)(1+q(t)) + p(t+1)]\Omega(v(t));
\end{aligned}$$

that is,

$$\frac{v'(t)}{\Omega(v(t))} \leq M[p(t)(1+q(t)) + p(t+1)]. \qquad (3.13)$$

Integration of (3.13) from 0 to t, $1 \leq t \leq b$, the change of variable, and the condition (3.1) give

$$\begin{aligned}\int_c^{v(t)} \frac{ds}{\Omega(s)} &\leq \int_0^t M[p(s)(1+q(s)) + p(s+1)]ds \\ &\leq \int_0^b M[p(s)(1+q(s)) + p(s+1)]ds < \int_c^\infty \frac{ds}{\Omega(s)}. \end{aligned} \tag{3.14}$$

From (3.14) we conclude that there is a constant γ_2 independent of $\lambda \in (0,1)$ such that $v(t) \leq \gamma_2$ and hence $\|x(t)\| \leq \gamma_2$ for $1 \leq t \leq b$. Let $\gamma = \max\{\gamma_1, \gamma_2\}$. Obviously, $\|x(t)\| \leq \gamma$ for $t \in J$ and consequently, $\|x\| = \sup\{\|x(t)\| : t \in J\} \leq \gamma$.

Next we prove that F is completely continuous. Let $B_m = \{x \in B : \|x(t)\| \leq m,\ t \in J\}$, for some $m \geq 1$. Then for each $m \geq 1$, the set B_m is clearly closed, convex and bounded subset of B. First we show that FB_m is uniformly bounded. We have to consider the two cases.

Case I: $0 \leq t < 1$. From the definition of the operator F as in (3.3) , hypotheses and the fact that $x \in B_m$, we obtain

$$\begin{aligned}\|(Fx)(t)\| &\leq \|T(t)\|[\|x_0\| + G] \\ &\quad + \int_0^t \|T(t-s)f(s, x(s), \int_0^s k(s,\sigma,x(\sigma))d\sigma, \phi(s))|ds \\ &\leq M[\|x_0\| + G] + \int_0^1 M\alpha_m(s)ds \\ &\leq M[\|x_0\| + G] + M\|\alpha_m\|_{L^1}. \end{aligned} \tag{3.15}$$

Case II: $1 \leq t \leq b$. From (3.4) , hypotheses and the fact that $x \in B_m$, then looking at

Case I immediately we have

$$\begin{aligned}|(Fx)(t)| &\leq \|T(t)\|[\|x_0\| + G] + \int_0^1 M\|f(s, x(s), \int_0^s k(s,\sigma,x(\sigma))d\sigma, \phi(s))\|ds \\ &\quad + \int_1^t M\|f(s, x(s), \int_0^s k(s,\sigma,x(\sigma))d\sigma, x(s-1))\|ds \\ &\leq M[\|x_0\| + G] + \int_0^1 M\alpha_m(s)ds + \int_1^t M\alpha_m(s)ds \\ &= M[\|x_0\| + G] + \int_0^t M\alpha_m(s)ds \end{aligned}$$

$$\leq M[\|x_0\| + G] + M\|\alpha_m\|_{L^1}. \tag{3.16}$$

From (3.15) and (3.16), it follows that $\{FB_m\}$ is uniformly bounded.

We now show that F maps B_m into an equicontinuous family. Let $x \in B_m$. We must consider three cases.

Case I: t_1 and t_2 are contained in $0 \leq t < 1$. From (3.3), it follows that

$$\begin{aligned}
&(Fx)(t_1) - (Fx)(t_2) \\
&= T(t_1)[x_0 - g(x)] + \int_0^{t_1} T(t_1 - s) f\big(s, x(s), \int_0^s k(s, \sigma, x(\sigma)) d\sigma, \phi(s)\big) ds \\
&\quad - T(t_2)[x_0 - g(x)] + \int_0^{t_2} T(t_2 - s) f\big(s, x(s), \int_0^s k(s, \sigma, x(\sigma)) d\sigma, \phi(s)\big) ds \\
&= [T(t_1) - T(t_2)](x_0 - g(x)) \\
&\quad + \int_0^{t_1} [T(t_1 - s) - T(t_2 - s)] f\big(s, x(s), \int_0^s k(s, \sigma, x(\sigma)) d\sigma, \phi(s)\big) ds \\
&\quad + \int_{t_1}^{t_2} T(t_2 - s) f\big(s, x(s), \int_0^s k(s, \sigma, x(\sigma)) d\sigma, \phi(s)\big) ds.
\end{aligned} \tag{3.17}$$

From the above equality and hypotheses, we have

$$\begin{aligned}
&\|(Fx)(t_1) - (Fx)(t_2)\| \leq \|T(t_1) - T(t_2)\| [\|x_0\| + G] \\
&\quad + \int_0^{t_1} \|T(t_1 - s) - T(t_2 - s)\| \|f\big(s, x(s), \int_0^s k(s, \sigma, x(\sigma)) d\sigma, \phi(s)\big)\| ds \\
&\quad + \int_{t_1}^{t_2} \|T(t_2 - s)\| \|f\big(s, x(s), \int_0^s k(s, \sigma, x(\sigma)) d\sigma, \phi(s)\big)\| ds \\
&\leq \|T(t_1) - T(t_2)\| [\|x_0\| + G] + \int_0^{t_1} \|T(t_1 - s) - T(t_2 - s)\| \alpha_m(s) ds \\
&\quad + \int_{t_1}^{t_2} \|T(t_2 - s)\| \alpha_m(s) ds.
\end{aligned} \tag{3.18}$$

Case II: t_1 and t_2 are contained in $1 \leq t \leq b$. From (3.4), it follows that

$$\begin{aligned}
&(Fx)(t_1) - (Fx)(t_2) \\
&= [T(t_1) - T(t_2)](x_0 - g(x)) \\
&\quad + \int_0^1 [T(t_1 - s) - T(t_2 - s)] f\big(s, x(s), \int_0^s k(s, \sigma, x(\sigma)) d\sigma, \phi(s)\big) ds \\
&\quad + \int_1^{t_1} [T(t_1 - s) - T(t_2 - s)] f\big(s, x(s), \int_0^s k(s, \sigma, x(\sigma)) d\sigma, x(s-1)\big) ds
\end{aligned}$$

$$+\int_{t_1}^{t_2} T(t_2-s)f\big(s,x(s),\int_0^s k(s,\sigma,x(\sigma))d\sigma,x(s-1)\big)ds. \tag{3.19}$$

From the above equality and hypotheses, we have

$$\begin{aligned}
&\|(Fx)(t_1)-(Fx)(t_2)\| \le \|T(t_1)-T(t_2)\|[\|x_0\|+G]\\
&\quad+\int_0^1 \|T(t_1-s)-T(t_2-s)\|\|f\big(s,x(s),\int_0^s k(s,\sigma,x(\sigma))d\sigma,\phi(s)\big)\|ds\\
&\quad+\int_1^{t_1} \|T(t_1-s)-T(t_2-s)\|\|f\big(s,x(s),\int_0^s k(s,\sigma,x(\sigma))d\sigma,x(s-1)\big)\|ds\\
&\quad+\int_{t_1}^{t_2} \|T(t_2-s)\|\|f\big(s,x(s),\int_0^s k(s,\sigma,x(\sigma))d\sigma,x(s-1)\big)\|ds\\
&\le \|T(t_1)-T(t_2)\|[\|x_0\|+G]+\int_0^1 \|T(t_1-s)-T(t_2-s)\|\alpha_m(s)ds\\
&\quad+\int_1^{t_1} \|T(t_1-s)-T(t_2-s)\|\alpha_m(s)ds+\int_{t_1}^{t_2} \|T(t_2-s)\|\alpha_m(s)ds\\
&\le \|T(t_1)-T(t_2)\|[\|x_0\|+G]+\int_0^{t_1} \|T(t_1-s)-T(t_2-s)\|\alpha_m(s)ds\\
&\quad+\int_{t_1}^{t_2} \|T(t_2-s)\|\alpha_m(s)ds.
\end{aligned} \tag{3.20}$$

Case III: t_1 and t_2 are respectively contained in $[0,1)$ and $[1,b]$. From (3.3) and (3.4), it follows that

$$\begin{aligned}
&(Fx)(t_1)-(Fx)(t_2) = T(t_1)(x_0-g(x))\\
&\quad+\int_0^{t_1} T(t_1-s)f\big(s,x(s),\int_0^s k(s,\sigma,x(\sigma))d\sigma,\phi(s)\big)ds\\
&\quad-T(t_2)(x_0-g(x))-\int_0^1 T(t_2-s)f\big(s,x(s),\int_0^s k(s,\sigma,x(\sigma))d\sigma,\phi(s)\big)ds\\
&\quad-\int_1^{t_2} T(t_2-s)f\big(s,x(s),\int_0^s k(s,\sigma,x(\sigma))d\sigma,x(s-1)\big)ds\\
&= [T(t_1)-T(t_2)](x_0-g(x))+\\
&\quad\int_0^{t_1} T(t_1-s)f\big(s,x(s),\int_0^s k(s,\sigma,x(\sigma))d\sigma,\phi(s)\big)ds\\
&\quad-\int_0^{t_1} T(t_2-s)f\big(s,x(s),\int_0^s k(s,\sigma,x(\sigma))d\sigma,\phi(s)\big)ds\\
&\quad-\int_{t_1}^{1} T(t_2-s)f\big(s,x(s),\int_0^s k(s,\sigma,x(\sigma))d\sigma,\phi(s)\big)ds
\end{aligned}$$

$$
\begin{aligned}
&-\int_1^{t_2} T(t_2-s)f\big(s,x(s),\int_0^s k(s,\sigma,x(\sigma))d\sigma, x(s-1)\big)ds\\
&=[T(t_1)-T(t_2)](x_0-g(x))\\
&\quad+\int_0^{t_1}[T(t_1-s)-T(t_2-s)]f\big(s,x(s),\int_0^s k(s,\sigma,x(\sigma))d\sigma,\phi(s)\big)ds\\
&\quad-\Big[\int_{t_1}^{1} T(t_2-s)f\big(s,x(s),\int_0^s k(s,\sigma,x(\sigma))d\sigma,\phi(s)\big)ds\\
&\qquad+\int_1^{t_2} T(t_2-s)f\big(s,x(s),\int_0^s k(s,\sigma,x(\sigma))d\sigma,x(s-1)\big)ds\Big]. \qquad (3.21)
\end{aligned}
$$

From the above equality and hypotheses, we have

$$
\begin{aligned}
&\|(Fx)(t_1)-(Fx)(t_2)\|\\
&\le \|T(t_1)-T(t_2)\|[\|x_0\|+G]\\
&\quad+\int_0^{t_1}\|T(t_1-s)-T(t_2-s)\|\|f\big(s,x(s),\int_0^s k(s,\sigma,x(\sigma))d\sigma,\phi(s)\big)\|ds\\
&\quad+\int_{t_1}^{1}\|T(t_2-s)\|\|f\big(s,x(s),\int_0^s k(s,\sigma,x(\sigma))d\sigma,\phi(s)\big)\|ds\\
&\quad+\int_1^{t_2}\|T(t_2-s)\|\|f\big(s,x(s),\int_0^s k(s,\sigma,x(\sigma))d\sigma,x(s-1)\big)\|ds\\
&\le \|T(t_1)-T(t_2)\|[\|x_0\|+G]+\int_0^{t_1}\|T(t_1-s)-T(t_2-s)\|\alpha_m(s)ds\\
&\quad+\int_{t_1}^{1}\|T(t_2-s)\|\alpha_m(s)ds+\int_1^{t_2}\|T(t_2-s)\|\alpha_m(s)ds\\
&= \|T(t_1)-T(t_2)\|[\|x_0\|+G]+\int_0^{t_1}\|T(t_1-s)-T(t_2-s)\|\alpha_m(s)ds\\
&\quad+\int_{t_1}^{t_2}\|T(t_2-s)\|\alpha_m(s)ds \qquad (3.22)
\end{aligned}
$$

From (3.18), (3.20), (3.22), the right side of each one is independent of $x \in B_m$ and tends to zero as $t_1 - t_2 \to 0$; since the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology. Thus F maps B_m into an equicontinuous family of functions. It is easy to see that the family B_m is uniformly bounded.

Next, we show that $\overline{FB_m}$ is compact. Since we have shown that FB_m is an equicontinuous and uniformly bounded collection, it is sufficient by Arzela-Ascoli theorem (see [[4, 9]]) to show that F maps B_m into a precompact set in X. We have to consider the following two cases.

Case I: Let $0 < t < 1$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $x \in B_m$ we define

$$\begin{aligned}(F_\epsilon x)(t) &= T(t)[x_0 - g(x)] + \int_0^{t-\epsilon} T(t-s) f\big(s, x(s), \int_0^s k(s,\sigma,x(\sigma))d\sigma, \phi(s)\big) ds \\ &= T(t)[x_0 - g(x)] \\ &\quad + T(\epsilon) \int_0^{t-\epsilon} T(t-s-\epsilon) f\big(s, x(s), \int_0^s k(s,\sigma,x(\sigma))d\sigma, \phi(s)\big) ds.\end{aligned}$$

Since $T(t)$ is a compact operator, the set $Y_\epsilon(t) = \{(F_\epsilon x)(t) : x \in B_m\}$ is precompact in X for every ϵ, $0 < \epsilon < t$ and $0 < t < 1$. Moreover, for every $x \in B_m$, we get

$$\begin{aligned}\|(Fx)(t) - (F_\epsilon x)(t)\| &\le \int_{t-\epsilon}^t \|T(t-s)\| \|f\big(s, x(s), \int_0^s k(s,\sigma,x(\sigma))d\sigma, \phi(s)\big)\| ds \\ &\le M \int_{t-\epsilon}^t \alpha_m(s) ds. \qquad (3.23)\end{aligned}$$

Therefore there are precompact sets arbitrary close to the set $\{(Fx)(t) : x \in B_m\}$ for $0 < t < 1$. Hence the set $\{(Fx)(t) : x \in B_m\}$ is precompact in X for $0 < t < 1$.

Case II: Let $1 < t < b$ be fixed and ϵ a real number satisfying $1 < \epsilon < t$. For $x \in B_m$ we define

$$\begin{aligned}(F_\epsilon x)(t) = T(t)[x_0 - g(x)] &+ \int_0^1 T(t-s) f\big(s, x(s), \int_0^s k(s,\sigma,x(\sigma))d\sigma, \phi(s)\big) ds \\ &+ \int_1^{t-\epsilon} T(t-s) f\big(s, x(s), \int_0^s k(s,\sigma,x(\sigma))d\sigma, x(s-1)\big) ds.\end{aligned}$$

Since $T(t)$ is a compact operator, the set $\overline{Y}_\epsilon(t) = \{(F_\epsilon x)(t) : x \in B_m\}$ is precompact in X for every ϵ, $1 < \epsilon < t$ and $1 < t < b$. Moreover, for every $x \in B_m$, we get

$$\begin{aligned}\|(Fx)(t) - (F_\epsilon x)(t)\| &\le \int_{t-\epsilon}^t \|T(t-s)\| \|f\big(s, x(s), \int_0^s k(s,\sigma,x(\sigma))d\sigma, x(s-1)\big)\| ds \\ &\le M \int_{t-\epsilon}^t \alpha_m(s) ds. \qquad (3.24)\end{aligned}$$

Therefore there are precompact sets arbitrary close to the set $\{(Fx)(t) : x \in B_m\}$ for $1 < t < b$. Hence the set $\{(Fx)(t) : x \in B_m\}$ is precompact in

X for $1 < t < b$. On combining these two cases we conclude that the set $\{(Fx)(t) : x \in B_m\}$ is precompact in X for $t \in J$.

It remains to show that $F : B \to B$ is continuous. Let $\{v_n\}$ be a sequence of elements of B converging to v in B. Then there exists an integer q such that $\|v_n(t)\| \leq q$ for all n and $t \in J$, so $v_n \in B_q$ and $u \in B_q$. We consider the following two cases.

Case I:For each $t \in [0, 1)$ and by $(H_4) - (H_7)$, we have

$$f\big(s, v_n(s), \int_0^s k(s, \sigma, v_n(\sigma))d\sigma, \phi(s)\big) \to f\big(s, v(s), \int_0^s k(s, \sigma, v(\sigma))d\sigma, \phi(s)\big).$$

Thus, since

$$\|f\big(s, v_n(s), \int_0^s k(s, \sigma, v_n(\sigma))d\sigma, \phi(s)\big) - f\big(s, v(s), \int_0^s k(s, \sigma, v(\sigma))d\sigma, \phi(s)\big)\| \leq 2\alpha_q,$$

we have by dominated convergence

$$\begin{aligned}\|Fv_n - Fv\| &= \sup_{t\in[0,1)} \Big\|T(t)[g(v_n) - g(v)]) \\ &\quad + \int_0^t T(t-s)\big[f\big(s, v_n(s), \int_0^s k(s, \sigma, v_n(\sigma))d\sigma, \phi(s)\big) \\ &\quad - f\big(s, v(s), \int_0^s k(s, \sigma, v(\sigma))d\sigma, \phi(s)\big)\big]ds\Big\| \\ &\leq \quad M\|g(v_n) - g(v)\| \\ &\quad + M\int_0^t \Big\|\big[f\big(s, v_n(s), \int_0^s k(s, \sigma, v_n(\sigma))d\sigma, \phi(s)\big) \\ &\quad - f\big(s, v(s), \int_0^s k(s, \sigma, v(\sigma))d\sigma, \phi(s)\big)\big]ds\Big\| \to 0. \end{aligned} \tag{3.25}$$

Case II:For each $t \in [1, b]$ and by $(H_4) - (H_7)$, we have

$$f\big(s, v_n(s), \int_0^s k(s, \sigma, v_n(\sigma))d\sigma, v_n(s-1)\big) \to f\big(s, v(s), \int_0^s k(s, \sigma, v(\sigma))d\sigma, v(s-1)\big).$$

Thus, since

$$\|f\big(s, v_n(s), \int_0^s k(s, \sigma, v_n(\sigma))d\sigma, v_n(s-1)\big) - f\big(s, v(s), \int_0^s k(s, \sigma, v(\sigma))d\sigma, v(s-1)\big)\| \leq 2\alpha_q,$$

we have by dominated convergence

$$\begin{aligned}
\|Fv_n - Fv\| = \sup_{t\in[1,b]} \Big\| & T(t)[g(v_n) - g(v)]) \\
& + \int_0^1 T(t-s)\big[f\big(s, v_n(s), \int_0^s k(s,\sigma,v_n(\sigma))d\sigma, \phi(s)\big) \\
& - f\big(s, v(s), \int_0^s k(s,\sigma,v(\sigma))d\sigma, \phi(s)\big)\big]ds \\
& + \int_1^t T(t-s)\big[f\big(s, v_n(s), \int_0^s k(s,\sigma,v_n x(\sigma))d\sigma, v_n(s-1)\big) \\
& - f\big(s, v(s), \int_0^s k(s,\sigma,v(\sigma))d\sigma, v(s-1)\big)\big]ds\Big\| \\
\leq \quad & M\|g(v_n) - g(v)\| \\
& + M\int_0^1 \Big\|\big[f\big(s, v_n(s), \int_0^s k(s,\sigma,v_n(\sigma))d\sigma, \phi(s)\big) \\
& - f\big(s, v(s), \int_0^s k(s,\sigma,v(\sigma))d\sigma, \phi(s)\big)\big]ds\Big\| \\
& + M\int_0^t \Big\|\big[f\big(s, v_n(s), \int_0^s k(s,\sigma,v_n(\sigma))d\sigma, v_n(s-1)\big) \\
& - f\big(s, v(s), \int_0^s k(s,\sigma,v(\sigma))d\sigma, v(s-1)\big)\big]ds\Big\| \to 0. \qquad (3.26)
\end{aligned}$$

From (3.25) and (3.26), we conclude that the operator F is continuous. This completes the proof that F is completely continuous.

Finally, the set $U(F) = \{x \in B : x = \lambda Fx, \lambda \in (0,1)\}$ is bounded which was proved in the first part. Consequently, by Lemma 2.2, the operator F has a fixed point in B. This means that the problem (1.4)-(1.6) has a solution. This completes the proof of theorem. □

Remark 3.2. We note that the advantage of our approach here is that, it yields simutaneously the existence of solution of (1.4)-(1.6) and maximal interval of existence. In the special case, if we take $p(t) = 1$ in (3.1) and the integral on the right hand side in (3.1) is assumed to diverge, then the solution of (1.4)-(1.6) exists for every $b < \infty$; that is, on the entire interval $\mathbb{R}_+$. Our result in Theorem 3.1 yields existence of solution of (1.4)-(1.6) on $\mathbb{R}_+$, if the integral on the right hand side in (3.1) is divergent i.e., $\int_c^\infty \frac{ds}{\Omega(s)} = \infty$. Thus Theorem 3.1 can be considered as a further extension of the well known theorem on global existence of solution of ordinary differential equation due to Wintner given in [22]

The next theorem deals with the Osgood type uniqueness result for the solutions of (1.4)-(1.6).

Theorem 3.3. *Suppose that* $(H_1), (H_8) - (H_{10})$ *hold. Let*

$$\Upsilon(r) = \int_{r_0}^{r} \frac{ds}{\bar{\Omega}(s)}, \quad (0 < r_0 \leq r),$$

with Υ^{-1} *being the inverse function of* Υ *and assume that* $\lim_{r_0 \to +0} \Upsilon(r) = \infty$, *for any fixed* r. *Then* (1.4)-(1.6) *has at most one solution on* $\mathbb{R}_+$.

Proof. Let $x(t), y(t)$ be two solutions of equation (1.4), under the initial conditions

$$\begin{aligned} &x(t-1) = y(t-1) = \phi(t), \quad (0 \leq t < 1), \\ &x(0) + g(x) = x_0 \quad \text{and} \quad y(0) + g(y) = x_0, \end{aligned} \tag{3.27}$$

and let $u(t) = \|x(t) - y(t)\|$, $t \in \mathbb{R}_+$. We consider the following two cases.

Case I: $0 \leq t < 1$. From the hypotheses, we have

$$\begin{aligned} u(t) \leq{}& \|T(t)\|\|g(x) - g(y)\| \\ &+ \int_0^t \|T(t-s)\|\|f(s, x(s), \int_0^s k(s, \sigma, x(\sigma))d\sigma, \phi(s)) \\ &- f(s, y(s), \int_0^s k(s, \sigma, y(\sigma))d\sigma, \phi(s))\|ds \\ &\leq M\bar{G}\|x - y\| + \int_0^t M\bar{p}(s)[\bar{\Omega}(\|x(s) - y(s)\|) + \bar{\Omega}(\bar{q}(s)\|x(s) - y(s)\|)]ds \\ &= M\bar{G}u(t) + \int_0^t M\bar{p}(s)(1 + \bar{q}(s))\bar{\Omega}(u(s))ds, \end{aligned}$$

which implies

$$\begin{aligned} u(t) &\leq \int_0^t \frac{M\bar{p}(s)(1 + \bar{q}(s))}{1 - M\bar{G}}\bar{\Omega}(u(s))ds \\ &\leq \varepsilon_1 + \int_0^t \frac{M\bar{p}(s)(1 + \bar{q}(s))}{1 - M\bar{G}}\bar{\Omega}(u(s))ds, \end{aligned} \tag{3.28}$$

where $\varepsilon_1 > 0$ is sufficiently small constant. Now, an application of Lemma 2.3 to (3.28) yields

$$\|x(t) - y(t)\| \leq \Upsilon^{-1}\Big[\Upsilon(\varepsilon_1) + \int_0^t \frac{M\bar{p}(s)(1+\bar{q}(s))}{1 - M\bar{G}} ds\Big]. \tag{3.29}$$

Case II: $1 \leq t < \infty$. From the hypotheses, we have

$$\begin{aligned} u(t) &\leq M\bar{G}u(t) + \int_0^1 M\bar{p}(s)(1+\bar{q}(s))\bar{\Omega}(u(s))ds + \int_1^t M\bar{p}(s)(1+\bar{q}(s))\bar{\Omega}(u(s))ds \\ &\quad + \int_1^t M\bar{p}(s)\bar{\Omega}(u(s-1))ds \\ &= M\bar{G}u(t) + \int_0^t M\bar{p}(s)(1+\bar{q}(s))\bar{\Omega}(u(s))ds + I_2, \end{aligned} \tag{3.30}$$

where

$$I_2 = \int_1^t M\bar{p}(s)\bar{\Omega}(u(s-1))ds.$$

By the change of variable, we observe that

$$I_2 \leq \int_0^t M\bar{p}(s+1)\bar{\Omega}(u(s))ds. \tag{3.31}$$

Using (3.31) in (3.30), we obtain

$$u(t) \leq M\bar{G}u(t) + \int_0^t M[\bar{p}(s)(1+\bar{q}(s)) + \bar{p}(s+1)]\bar{\Omega}(u(s))ds,$$

which implies

$$\begin{aligned} u(t) &\leq \int_0^t \frac{M[\bar{p}(s)(1+\bar{q}(s)) + \bar{p}(s+1)]}{1 - M\bar{G}}\bar{\Omega}(u(s))ds, \\ &\leq \varepsilon_2 + \int_0^t \frac{M[\bar{p}(s)(1+\bar{q}(s)) + \bar{p}(s+1)]}{1 - M\bar{G}}\bar{\Omega}(u(s))ds, \end{aligned} \tag{3.32}$$

where $\varepsilon_2 > 0$ is sufficiently small constant. Now, an application of Lemma 2.3 to (3.32) yields

$$\|x(t) - y(t)\| \leq \Upsilon^{-1}\Big[\Upsilon(\varepsilon_2) + \int_0^t \frac{M\{\bar{p}(s)(1+\bar{q}(s)) + \bar{p}(s+1)\}}{1 - M\bar{G}} ds\Big]. \tag{3.33}$$

To apply the estimations in (3.29), (3.33) to the uniqueness problem, we use the notation $\Upsilon(r, r_0)$ instead of $\Upsilon(r)$ and impose the assumption $\lim_{r_0 \to +0} \Upsilon(r, r_0) = +\infty$, for fixed r, then we obtain $\lim_{r_0 \to +0} \Upsilon^{-1}(r, r_0) = 0$, see [[21], p. 77]. From (3.29), (3.33), it follows that $\|x(t) - y(t)\| \leq 0$ for $t \in \mathbb{R}_+$ and hence $x(t) = y(t)$ on $\mathbb{R}_+$. Thus, there is at most one solution to (1.4)-(1.6) on $\mathbb{R}_+$. □

Remark 3.4. We note that the hypothesis (H_8) corresponds to the Osgood type condition concerning the uniqueness of solutions in the theory of differential equations (see [[4], p. 35]).

4. Boundedness of Solutions

In this section, we obtain estimates on the solutions of equations (1.4)-(1.6) under some suitable assumptions on the functions involved therein.

The following theorem concerning the estimate on the solution of equation (1.4).

Theorem 4.1. *Suppose that $(H_1), (H_3), (H_6)$ hold. Let*

$$d_1 = \sup_{t \in \mathbb{R}_+} \| \int_0^t f(s, 0, 0, 0) ds \| < \infty.$$

If $x(t),\ t \in \mathbb{R}_+$, is any solution of equation (1.4)-(1.6), then

$$\|x(t)\| \leq \Upsilon^{-1}\Big[\Upsilon(d_2) + \int_0^t M\bar{p}(s)(1 + \bar{q}(s))ds\Big], \tag{4.1}$$

for $0 \leq t < 1$ and

$$\|x(t)\| \leq \Upsilon^{-1}\Big[\Upsilon(d_2) + \int_0^t M\big(\bar{p}(s)(1 + \bar{q}(s)) + \bar{p}(s+1)\big)ds\Big], \tag{4.2}$$

for $1 \leq t < \infty$, where

$$d_2 = M[\|x_0\| + G] + Md_1 + \int_0^1 M\bar{p}(s)\bar{\Omega}(\|\phi(s)\|)ds.$$

In particular, if $\int_0^t [\bar{p}(s) + \bar{p}(s+1)]ds$ is bounded on $\mathbb{R}_+$, then every solution to the problem (1.4)-(1.6) is bounded on $\mathbb{R}_+$.

Proof. Let $x(t)$ be a solution of the problem (1.4)-(1.4). We consider the following two cases.

Case I: $0 \leq t < 1$. Using the fact that the solution $x(t)$ of the problem (1.4)-(1.6) and the hypotheses, we have

$$\begin{aligned}
\|x(t)\| &= \Big\|T(t)[x_0 - g(x)] + \int_0^t T(t-s)f(s, x(s), \int_0^s k(s,\sigma,x(\sigma))d\sigma, \phi(s))ds\Big\| \\
&\leq M[\|x_0\| + G] + M\|\int_0^t f(s,0,0,0)ds\| + \int_0^t M\bar{p}(s)\bar{\Omega}(\|\phi(s)\|)ds \\
&\quad + \int_0^t M\bar{p}(s)(1+\bar{q}(s))\bar{\Omega}(\|x(s)\|)ds \\
&\leq M[\|x_0\| + G] + Md_1 \\
&\quad + \int_0^1 M\bar{p}(s)\bar{\Omega}(\|\phi(s)\|)ds + \int_0^t M\bar{p}(s)(1+\bar{q}(s))\bar{\Omega}(\|x(s)\|)ds \\
&= d_2 + \int_0^t M\bar{p}(s)(1+\bar{q}(s))\bar{\Omega}(\|x(s)\|)ds. \qquad (4.3)
\end{aligned}$$

Now, an application of Lemma 2.3 to (4.3) yields (4.1).

Case II: $1 \leq t < \infty$. Using the fact that the solution $x(t)$ of the problem (1.4)-(1.6) and the hypotheses, we have

$$\begin{aligned}
\|x(t)\| &= \Big\|T(t)[x_0 - g(x)] + \int_0^1 T(t-s)f(s, x(s), \int_0^s k(s,\sigma,x(\sigma))d\sigma, \phi(s))ds \\
&\quad + \int_1^t T(t-s)f(s, x(s), \int_0^s k(s,\sigma,x(\sigma))d\sigma, x(s-1))ds\Big\| \\
&\leq M[\|x_0\| + G] \\
&\quad + \|\int_0^1 T(t-s)\Big[f(s, x(s), \int_0^s k(s,\sigma,x(\sigma))d\sigma, \phi(s)) \\
&\quad - f(s,0,0,0) + f(s,0,0,0)\Big]ds\| \\
&\quad + \|\int_1^t T(t-s)\Big[f(s, x(s), x(s-1)) - f(s,0,0,0) + f(s,0,0,0)\Big]ds\| \\
&\leq M[\|x_0\| + G] + M\|\int_0^t f(s,0,0,0)ds\| + \int_0^1 M\bar{p}(s)\bar{\Omega}(\|\phi(s)\|)ds \\
&\quad + \int_0^1 M\bar{p}(1+\bar{q}(s))(s)\bar{\Omega}(\|x(s)\|)ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_1^t M\bar{p}(s)(1+\bar{q}(s))\bar{\Omega}(\|x(s)\|)ds+\int_1^t M\bar{p(s)}\bar{\Omega}(\|x(s-1)\|)ds\\
&\leq M[\|x_0\|+G]+M\|d_1+\int_0^1 M\bar{p}(s)\bar{\Omega}(\|\phi(s)\|)ds\\
&+\int_0^t M\bar{p}(s)(1+\bar{q}(s))\bar{\Omega}(\|x(s)\|)ds+\int_1^t M\bar{p(s)}\bar{\Omega}(\|x(s-1)\|)ds\\
&=d_2+\int_0^t M\bar{p}(s)(1+\bar{q}(s))\bar{\Omega}(\|x(s)\|)ds+I_3, \qquad (4.4)
\end{aligned}
$$

where

$$I_3=\int_1^t M\bar{p}(s)\bar{\Omega}(\|x(s-1)\|)ds.$$

By the change of variable, we observe that

$$I_3\leq\int_0^t M\bar{p}(s+1)\bar{\Omega}(\|x(s)\|)ds. \qquad (4.5)$$

Using (4.5) in (4.4), we obtain

$$\|x(t)\|\leq d_2+\int_0^t M[\bar{p}(s)(1+\bar{q}(s))+\bar{p}(s+1)]\bar{\Omega}(\|x(s)\|)ds. \qquad (4.6)$$

Now, an application of Lemma 2.3 to (4.6) yields (4.2).

Moreover, if $\int_0^t[\bar{p}(s)(1+\bar{q}(s))+\bar{p}(s+1)]ds$ is bounded on $\mathbb{R}_+$, then from (4.1) and (4.2) we observe that every solution to the problem (1.4)-(1.6) is bounded on $\mathbb{R}_+$. □

5. Continuous Dependence

In this section, we shall deals with the continuous dependence of solutions of (1.4) on the given initial data.

Theorem 5.1. *Suppose that the hypotheses $(H_1),(H_8)-(H_{10})$ hold and let $x_1(t),\ x_2(t)$ be the solutions of (1.4) with the initial conditions*

$$x_1(t-1)=\phi_1(t)\quad(0\leq t<1),\quad x_1(0)+g(x_1)=x_0, \qquad (5.1)$$

$$x_2(t-1)=\phi_2(t)\quad(0\leq t<1),\quad x_2(0)+g(x_2)=\bar{x}_0, \qquad (5.2)$$

respectively, where $x_0, \bar{x}_0$ are elements of X. Then

$$\|x_1(t) - x_2(t)\| \le \Upsilon^{-1}\Big[\Upsilon(d_3) + \int_0^t \frac{M\bar{p}(s)(1+\bar{q}(s))}{1-M\bar{G}}ds\Big], \tag{5.3}$$

for $0 \le t < 1$ and

$$\|x_1(t) - x_2(t)\| \le \Upsilon^{-1}\Big[\Upsilon(d_3) + \int_0^t \frac{M\big(\bar{p}(s)(1+\bar{q}(s)) + \bar{p}(s+1)\big)}{1-M\bar{G}}ds\Big], \tag{5.4}$$

for $1 \le t < \infty$, where

$$d_3 = \frac{M\|x_0 - \bar{x}_0\|}{1-M\bar{G}} + \int_0^1 \frac{M\bar{p}(s)}{1-M\bar{G}}\bar{\Omega}(\|\phi_1(s) - \phi_2(s)\|)ds. \tag{5.5}$$

Proof. Let $u(t) = \|x_1(t) - x_2(t)\|$ for $t \in \mathbb{R}_+$. We consider the following two cases. **Case I:** $0 \le t < 1$**.** From the hypotheses, it follows that

$$\begin{aligned}
u(t) &\le M[\|x_0 - \bar{x}_0\| + \bar{G}u(t)] \\
&\quad + \int_0^t M\|f(s, x_1(s), \int_0^s k(s,\sigma,x_1(\sigma))d\sigma, \phi_1(s)) \\
&\quad - f(s, x_2(s), \int_0^s k(s,\sigma,x_2(\sigma))d\sigma, \phi_2(s))\|ds \\
&\le M[\|x_0 - \bar{x}_0\| + \bar{G}u(t)] \\
&\quad + \int_0^t M\bar{p}(s)\Big[\bar{\Omega}(u(s)) + \bar{\Omega}(\bar{q}(s)u(s)) + \bar{\Omega}(\|\phi_1(s) - \phi_2(s)\|)\Big]ds \\
&\le M[\|x_0 - \bar{x}_0\| + \bar{G}u(t)] \\
&\quad + \int_0^1 M\bar{p}(s)\bar{\Omega}(\|\phi_1(s) - \phi_2(s)\|)ds + \int_0^t M\bar{p}(s)(1+\bar{q}(s))\bar{\Omega}(u(s))ds,
\end{aligned} \tag{5.6}$$

which implies

$$\begin{aligned}
u(t) &\le \frac{M\|x_0 - \bar{x}_0\|}{1-M\bar{G}} + \int_0^1 \frac{M\bar{p}(s)}{1-M\bar{G}}\bar{\Omega}(\|\phi_1(s) - \phi_2(s)\|)ds \\
&\quad + \int_0^t \frac{M\bar{p}(s)(1+\bar{q}(s))}{1-M\bar{G}}\bar{\Omega}(u(s))ds \\
&= d_3 + \int_0^t \frac{M\bar{p}(s)(1+\bar{q}(s))}{1-M\bar{G}}\bar{\Omega}(u(s))ds.
\end{aligned} \tag{5.7}$$

Now an application of Lemma 2.3 (with $c = d_3$) to (5.7), yields (5.3).

Case II: $1 \leq t < \infty$**.** By following a similar arguments as in **Case II** of the proof of Theorem 3.3 and from the hypotheses, it follows that

$$u(t) \leq M[\|x_0 - \bar{x}_0\| + \bar{G}u(t)] + \int_0^1 M\bar{p}(s)\bar{\Omega}(\|\phi_1(s) - \phi_2(s)\|)ds + \int_0^t M\bar{p}(s)(1 + \bar{q}(s))\bar{\Omega}(u(s))ds + I_4, \tag{5.8}$$

where

$$I_4 = \int_1^t M\bar{p}(s)\bar{\Omega}(u(s-1))ds.$$

By the change of variable, we observe that

$$I_4 \leq \int_0^t M\bar{p}(s+1)\bar{\Omega}(u(s))ds. \tag{5.9}$$

Using (5.9) in (5.8), we obtain

$$u(t) \leq M[\|x_0 - \bar{x}_0\| + \bar{G}u(t)] + \int_0^1 M\bar{p}(s)\bar{\Omega}(\|\phi_1(s) - \phi_2(s)\|)ds + \int_0^t M[\bar{p}(s)(1 + \bar{q}(s)) + \bar{p}(s+1)]\bar{\Omega}(u(s))ds$$

which implies

$$\begin{aligned} u(t) &\leq \frac{M\|x_0 - \bar{x}_0\|}{1 - M\bar{G}} + \int_0^1 \frac{M\bar{p}(s)}{1 - M\bar{G}}\bar{\Omega}(\|\phi_1(s) - \phi_2(s)\|)ds \\ &\quad + \int_0^t \frac{M[\bar{p}(s)(1 + \bar{q}(s)) + \bar{p}(s+1)]}{1 - M\bar{G}}\bar{\Omega}(u(s))ds \\ &= d_3 + \int_0^t \frac{M[\bar{p}(s)(1 + \bar{q}(s)) + \bar{p}(s+1)]}{1 - M\bar{G}}\bar{\Omega}(u(s))ds. \end{aligned} \tag{5.10}$$

Now an application of Lemma 2.3 (with $c = d_3$), to (5.10), yields (5.4). From (5.3) and (5.4), it follows that the solutions of equation (1.4) depends on the given initial data. □

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