International Journal of Pure and Applied Mathematics
Volume 104 No. 3 2015, 461-470
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v104i3.13

QUASI-COMPLETION OF FILTER SPACES

Nandita Rath
School of Mathematics and Statistics
University of Western Australia
Crawley, W.A. 6009, AUSTRALIA

Abstract: The category FIL of filter spaces being isomorphic to the category of grill-determined nearness spaces has become significant in the later part of the twentieth century. During that period, a substantial completion theory has been developed using the equivalence classes of filters in a filter space. However, that completion was quite general in nature, and did not allow the finest such completion. As a result, a completion functor could not be defined on FIL. In this paper, this issue is partially addressed by constructing a completion that is finer than the existing completions. Also, a completion functor is defined on a subcategory of FIL comprising all filter spaces as objects.

AMS Subject Classification: 54D35, 54A20, 54C20
Key Words: filter space, Cauchy map, convergence structure, s-map, stable completion, completion in standard form

1. Introduction

In 1990, Bently et al. [1] formalised the concept of filter spaces for being isomorphic to Katetov's [2] filter merotopic spaces. Since then these spaces have been studied by several topologists (see [3], [4], [5], [6], [7]) in the context of their applications to category theory and algebra . Kent and Rath [3] defined

Received: August 16, 2015

an equivalence relation on a filter space (X, C), which led to the construction of its T_2 Wyler completion. However, soon they realised that unlike the completion of Cauchy spaces, there is no finest completion when there are infinite number of equivalence classes (see Proposition 2.4 [3]). Attempts have been made in this paper to construct a certain type of weaker completion, called *quasi completion* of a filter space which may yield a finest such completion in a subcategory of FIL which has all filter spaces as objects.

Also, the well-known completion theory for Cauchy spaces was extended to obtain a completion without the T_2-restriction on the space by the author [9]. An s-map was introduced to form a special class of morphisms which led to a completion functor on a subcategory of CHY (the category of Cauchy spaces with Cauchy maps) with respect to a new class of morphisms. In this paper, a modified form of s-maps is used to build a completion functor on a subcategory of FIL (the category of filter spaces with Cauchy maps) without the T_2 restriction.

Reed [12] introduced a special type of completion for T_2 Cauchy spaces, namely *completion in standard form*, which was very interesting in the sense that it led to a powerful result: any T_2 Wyler completion is equivalent to one in standard form. However, as pointed out via a counter example by the author in an earlier paper [9, Example 3.2], this is not the case for all Cauchy spaces in general, that is, it fails to preserve the equivalence of completions in standard form, since it is not a categorical equivalence in the sense of Preuss [8]. Since Cauchy spaces are special cases of filter spaces, Reed's completion will also fail to preserve the equivalence, for non-T_2 filter spaces in general. This motivates the introduction of *quasi-stable completion.*

2. Preliminaries

Let X be a nonempty set and $\mathbf{F}(X)$ be the set of filters on X. If $\mathcal{F}$ and $\mathcal{G} \in F(X)$ and $F \cap G \neq \phi$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$, then $\mathcal{F} \vee \mathcal{G}$ denotes the filter generated by $\{F \cap G : F \in \mathcal{F}$ and $G \in \mathcal{G}\}$. If there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \cap G = \phi$, then we say that $\mathcal{F} \vee \mathcal{G}$ *fails to exist.* For each $x \in X$, $\dot{x}$ denotes the ultrafilter generated by $\{x\}$. If $C \subset \mathbf{F}(X)$ satisfies the following conditions:

$c_{1.}$ $\dot{x} \in C$, for all $x \in X$,

$c_{2.}$ $\mathcal{F} \in C$ and $\mathcal{G} \geq \mathcal{F}$ imply that $\mathcal{G} \in C$,

then the pair (X, C) is called a *filter space* and C is called a *pre-Cauchy structure* on X. If C and D are two pre-Cauchy structures on X, and $C \subseteq D$

then C is *finer than* D, written $C \geq D$. Associated with each pre-Cauchy structure C on a set X, there is a convergence structure q_c, defined as

$$\mathcal{F} \xrightarrow{q_c} x \quad \text{if and only if} \quad \mathcal{F} \cap \dot{x} \in C.$$

The two filters $\mathcal{F}$ and $\mathcal{G} \in \mathbf{F}(X)$ are said to be $C-$*linked* [3], if there exist a finite number of filters $\mathcal{H}_1$, $\mathcal{H}_2$,. . . , $\mathcal{H}_n \in C$ such that $\mathcal{F} \vee \mathcal{H}_1$, $\mathcal{H}_1 \vee \mathcal{H}_2$,. . . , $\mathcal{H}_{n-1} \vee \mathcal{H}_n$ all exist. In particular, if $\mathcal{F}$ and $\mathcal{G} \in C$, we write $\mathcal{F} \sim_c \mathcal{G}$ iff $\mathcal{F}$, $\mathcal{G}$ are C-linked. A filter space is said to be a *c-filter space* (respectively, *Cauchy space*), if $\mathcal{F} \cap \dot{x} \in C$ whenever $\mathcal{F} \sim_c \dot{x}$ (respectively, $\mathcal{F} \cap \mathcal{G} \in C$ whenever $\mathcal{F} \sim_c \mathcal{G}$). Note that '$\sim_c$' defines an equivalence relation on C. For $\mathcal{F} \in C$, let $[\mathcal{F}]_c$ denote the equivalence class containing $\mathcal{F}$. There is a *pre-convergence* structure [4] p_c associated with C in a natural way: $\mathcal{F} \xrightarrow{p_c} x$ iff $\mathcal{F} \sim_c \dot{x}$. Note that $p_c \leq q_c$ [3], since $\mathcal{F} \xrightarrow{q_c} x$ implies $\mathcal{F} \sim_c \dot{x}$, but $p_c \neq q_c$ in general, as illustrated in the following example.

Example 1. Let $X = R$, the set of real numbers and

$$C = \{\dot{x} \mid x \in X\} \cup \{\mathcal{F} \mid \mathcal{F} \geq \mathcal{G}\} \cup \{\text{all free filters}\}.$$

Clearly, C is a pre-Cauchy structure on X. Consider the filter $\mathcal{H} = \{R \setminus F \mid F \text{ is a finite subset of } R\}$. Since $\mathcal{F}$ is a free filter, it is in C. Observe that for the filter $\mathcal{G} = [\{[0, 1/n] \mid n \in N\}]$, $\mathcal{H} \vee \mathcal{G}$ and $\mathcal{G} \vee \dot{0}$ exist, which imply that $\mathcal{H} \xrightarrow{p_c} 0$. However $\mathcal{F} \cap \dot{0} \notin C$.

Lemma 1. *For a filter space* (X, C), $p_c = q_c$ *if and only if it is a c-filter space.*

A filter space (X, C) is said to be *quasi-* T_1 (respectively, *quasi-* T_2) iff $\dot{x} \cap \dot{y} \in C \Rightarrow x = y$ (respectively, $\mathcal{F} \cap \dot{x}, \mathcal{F} \cap \dot{y} \in C \Rightarrow x = y$). Henceforth, the term "quasi" associated with any property for a filter space will be abbreviated to q-property, for example, quasi-T_1 will be referred to as q-T_1. The filter space is *q-regular* iff $cl_{q_c}\mathcal{F} \in C$ whenever $\mathcal{F} \in C$ and q-T_3 iff it is q-T_1 and q-regular.

Note that the properties such as T_1, T_2, T_3 and regularity of a filter space $(X, C$ are stronger than the properties q-T_1, q-T_2, q-T_3 and q-regularity, respectively. It follows from Lemma 1 that these properties are equivalent only when (X, C) is a c-filter space.

However, these properties shouldn't be undermined, since the quasi-properties of (X, C) guarantee the corresponding properties of the convergence space (X, q_c). For instance, (X, C) is q-T_1 (respectively, q-T_2, q-T_3 and q-regular) implies that (X, q_c) is T_1 (respectively, T_2, T_3 and regular). Moreover, if (X, q_c) is regular, and every filter in C q_c-converges, then (X, C) is regular.

One of the remarkable differences between these properties and the weaker quasi-properties is that the T_1 and T_2 properties are equivalent [9] for a filter space, whereas this is not true in general for q-T_1 and q-T_2. The following example shows that there is a filter space which is q-T_1, but not necessarily q-T_2.

Example 2. Let (X, C) be an infinite set and $a, b \in X$ such that $a \neq b$. Let

$$C = \{\dot{x} \mid x \in X\} \cup \{\mathcal{G} \mid \mathcal{G} \geq \mathcal{H} \text{ or } \mathcal{H} \cap \dot{a} \text{ or } \mathcal{H} \cap \dot{b}\},$$

where $\mathcal{H}$ is any filter on X. Clearly, (X, C) is a filter space. For any x, y in X, $\dot{x} \cap \dot{y} \in C \Rightarrow \dot{x} \cap \dot{y}$ is a fixed ultra-filter generated by a single element in X, which means $x = y$. So, X is q-T_1. However, it is not q-T_2 since $\mathcal{H} \cap \dot{a}$ and $\mathcal{H} \cap \dot{b} \in C$, but $a \neq b$.

Note that $q - T_1$ and $q - T_2$ properties are equivalent for a c-filter space.

3. Quasi-Completion

Though the completion obtained by Kent and Rath [3] is the most general one, one of its drawback is that it does not have the finest such completion. As a result, a completion functor could not be defined on the category FIL of all filter spaces. In this section, a different completion is constructed, which yields such a functor on a subcategory of FIL. The T_2 Wyler completion of a filter space (X, C) that was constructed by Kent and Rath [3] had the property that if (X, C) was a c-filter space (Respectively, Cauchy space), then its completion was also c-filter space (respectively, Cauchy space). However, this is not the case for a quasi-completion.

For two filter spaces (X, C) and (Y, D), a mapping $f : (X, C) \longrightarrow (Y, D)$ is called a *Cauchy map*, if $\mathcal{F} \in C$ implies $f(\mathcal{F}) \in D$ for all $\mathcal{F} \in C$, and it is called a *Cauchy embedding* if $f : (X, C) \longrightarrow (f(X), D_{f(X)})$ is bijective and both f and f^{-1} are Cauchy maps.

A filter space (X, C) is said to be *quasi-complete* (respectively, complete) iff each $\mathcal{F} \in C$ q_c-converges (respectively, p_c-converges). In view of Example 1, it follows that every quasi-complete filter space is complete, but not conversely. A *quasi-completion* of a filter space (X, C) is a pair $((Y, D), \psi)$ consisting of a quasi-complete filter space (Y, D) and a Cauchy embedding map $\psi : (X, C) \longrightarrow (Y, D)$ satisfying $cl_{q_D} \psi(X) = Y$. A quasi-completion $((Y, D), \psi)$ is said to be a *quasi-$\mathcal{P}$ completion*, if (Y, D) has the property $\mathcal{P}$ whenever (X, C) has the same property. It is said to be *q-proper*, if images of any two equivalent filters in C

q_D-converge to the same point in Y.

Proposition 1. *Any q-T_2 quasi-completion of a q-T_2 filter space is q-proper.*

We construct a quasi-completion of a filter space (X, C) as follows:

$X_1^* = X \cup \{[\mathcal{F}] \mid \mathcal{F} \in C\}, \mathcal{F} \nsim_c \dot{x}$ for any $x \in X\}$,

$j : X \longrightarrow X_1^*$ is the inclusion map,

$C_1^* = j(C) \cup \{\mathcal{A} \in \mathbf{F}(X_1^*) \mid$ there exists a filter $\mathcal{F} \in C$ such that $\mathcal{A} \geq j(\mathcal{F}) \cap [\dot{\mathcal{F}}]\}$.

Proposition 2. *$((X_1^*, C_1^*), j)$ is a quasi-completion of (X, C).*

Proof. Clearly (X_1^*, C_1^*) is a filter space and j is a Cauchy embedding. To show that it is quasi-complete, let $\mathcal{A} \in C_1^*$. Then either $\mathcal{A} \geq j(\mathcal{F})$, for some $\mathcal{F}$ that is q_c-convergent or $\mathcal{A} \geq j(\mathcal{G}) \cap [\dot{\mathcal{G}}]$, for some $\mathcal{G}$ non-q_c-convergent. If $\mathcal{F} \overset{q_c}{\to} x$, then $j(\mathcal{F}) \overset{q_{C_1^*}}{\to} j(x)$. On the other hand, if $\mathcal{F}$ is non-q_c-convergent, then $j(\mathcal{F}) \cap [\dot{\mathcal{F}}] \in C_1^*$, which implies that $\mathcal{A} \overset{q_{C_1 *}}{\to} [\mathcal{F}]$. Therefore, $((X_1^*, C_1^*))$ is quasi-complete. Next, let $[\mathcal{F}] \in X_1 * \backslash j(X)$. This implies that $j(\mathcal{F}) \cap [\dot{\mathcal{F}}] \in C_1^*$, that is, $j(\mathcal{F}) \overset{q_{C_1^*}}{\to} [\mathcal{F}]$. Therefore, $[\mathcal{F}] \in cl_{q_{C_1^*}}(j(X))$. This proves that $((X_1^*, C_1^*), j)$ is a quasi-completion of (X, C), and this completes the proof. $\square$

This completion will be referred to as *quasi-Wyler completion*. Note that if (X, C) is a c-filter space (respectively, Cauchy space), then $((X_1^*, C_1^*), j)$ is a c-filter space (respectively, Cauchy space). If we identify each $x \in X$ with the equivalence class $[\dot{x}]$ of all filters which are p_c-convergent to x, then the quasi-Wyler completion coincides with $((X^*, C^*), j)$ in [3]. We will refer to the latter completion as the T_2 Wyler completion of (X, C). Unlike T_2 completions of a filter space, the quasi-completion $((X_1^*, C_1^*), j)$ is not a quasi-T_2 completion, in general, even if (X, C) is $q - T_2$. The following proposition gives a condition which guarantees that a q-T_2 filter space has a quasi-T_2 completion.

Proposition 3. *A q-T_2 filter space has a quasi-T_2 completion if and only if (X, C) is a c-filter space.*

Proof. Let $((Y, K), \phi)$ be a q-T_2 completion of (X, C). Let $\mathcal{F} \in C$ and $\mathcal{F} \sim_C \dot{x}$. From Proposition 1, it follows that $\phi(\mathcal{F}) \overset{q_k}{\to} \phi(\dot{x})$, that is, $\phi(\mathcal{F}) \cap \phi(\dot{x}) \in K$. Since ϕ is an embedding, $\mathcal{F} \cap \dot{x} \in C$, which shows that (X, C) is a c-filter space.

Next, let (X, C) be a q-T_2 c-filter space. Then, as shown in Proposition 2, $((X_1^*, C_1^*), j)$ is a quasi-completion of (X, C). Let $\dot{y_1} \cap \dot{y_2} \in C_1^*$. If y_1, $y_2 \in X$, then $y_1 = y_2$, since (X, C) is q-T_2. If at least one of y_1 or y_2 is in $X_1^* \setminus X$, then by the definition of C_1^*, $\dot{y_1} \cap \dot{y_1} \in C_1^*$ only when $y_1 = y_2$. This completes the proof. □

A quasi-completion $((Y, K), \phi)$ is said to be in *standard form* if $Y = X_1^*$ and $\phi = j$, satisfying the condition $j(\mathcal{F}) \overset{q_{c_{1*}}}{\to} [\mathcal{F}]$ for all non-q_c-convergent filters in C. A similar property was introduced by Reed [12] to establish that a T_2 Cauchy completion can be made equivalent to one in standard form. However, since this is not the case for all Cauchy spaces in general (see Example 3.2 [9]), the *stable completions* were introduced [9]. Since Cauchy spaces are special cases of filter spaces, Reed's result will also fail for non-T_2 filter spaces in general. This leads to the notion of *quasi-stable* completion for filter-spaces.

A quasi-completion $((Y, D), \phi)$ of a filter space (X, C) is said to be *quasi-stable* if for each non-q_c-convergent filter $\mathcal{F} \in C$, $\phi(\mathcal{F}) \cap \dot{[\mathcal{F}]} \in D$. This property of a completion is stronger than the property of being stable introduced in [3], since quasi-stable implies that it is stable. However, there exist stable completion of some filter spaces which are not quasi-stable. Two quasi-stable completions of a filter space (X, C) can be compared to each other in the obvious way: A quasi-stable completion $((Y_1, K_1), \varphi_1)$ is said to be *finer* than another quasi-stable completion $((Y_2, K_2), \varphi_2)$, if there is a continuous map $h : (Y_1, K_1) \to (Y_2, K_2)$ such that $h \circ \phi_1 = \phi_2$, and they are *equivalent* if each is finer than the other. Note that the map h is a unique homeomorphism, when the quasi-stable completions are equivalent.

Proposition 4. *The quasi-Wyler completion is the finest quasi-stable completion in standard form.*

Proof. Let $((Y, K), \phi)$ be a quasi-stable completion of the filter space (X, C) and $h : Y \longrightarrow X_1^*$ be defined as

$$h(y) = \begin{cases} [\mathcal{F}] & \text{if } y \in Y \setminus \phi(X) \text{ and } \phi(\mathcal{F}) \xrightarrow{q_k} y, \\ y & \text{if } y = \phi(x) \text{ for some } x \in X. \end{cases}$$

To show that h is well-defined, let $y_1 = y_2 \in Y$. If $y_1 = y_2 \in \phi(X)$, then clearly $h(y_1) = h(y_2)$. If $y_1 = y_2 \in Y \setminus \phi(X)$, then $\phi(\mathcal{F}_1) \xrightarrow{q_k} y_1$ and $\phi(\mathcal{F}_2) \xrightarrow{q_k} y_2$, for which $\mathcal{G}_1 = \phi^{-1}(\phi(\mathcal{F}_1) \cap \dot{y}_1)$ and $\mathcal{G}_2 = \phi^{-1}(\phi(\mathcal{F}_2) \cap \dot{y}_2)$ are in C. This implies that $\mathcal{F}_1 \vee \mathcal{G}_1$, $\mathcal{G}_1 \vee \mathcal{G}_2$ and $\mathcal{G}_2 \vee \mathcal{F}_2$ exist, which yields $[\mathcal{F}_1] = [\mathcal{F}_2]$. Therefore, $h(y_1) = h(y_2)$.

Next, let $h(y_1) = h(y_2)$. If $h(y_1) = x_1$ and $h(y_2) = x_2$ for $x_1, x_2 \in X$, then $y_1 = y_2$. On the other hand, if $h(y_1) = [\mathcal{F}]$ and $h(y_2) = [\mathcal{G}]$ for some $\mathcal{F}, \mathcal{G} \in C$, then $\mathcal{F} \sim_c \mathcal{G}$, which leads to $\phi(\mathcal{F}) \sim_c \phi(\mathcal{G})$. Therefore, $\phi(\mathcal{F}) \xrightarrow{q_k} y_1$

and $\phi(\mathcal{G}) \xrightarrow{q_k} y_2$, which imply $\phi(\mathcal{F}) \xrightarrow{q_k} y_1, y_2$. But, since $((Y,K),\phi)$ is a quasi-stable completion of (X,C), it follows that $y_1 = y_2$. Hence, h is bijective and $h\phi = j$.

Let $C' = \{h(\mathcal{G}) \mid \mathcal{G} \in K\}$ be the quotient structure on X_1^* with respect to h. Obviously, both h and h^{-1} are Cauchy maps, which makes the bijective maps j and j^{-1} Cauchy maps. It is also routine to show that (X_1^*, C') is quasi-complete and $cl_{q_{c'}} j(X) = Y$. Hence, $((X_1^*, C'), j)$ is a quasi-completion of (X,C). This proves that $((Y,K),\phi) \simeq ((X_1^*, C'), j)$. Also, for a non-$q_c$-convergent filter $\mathcal{F} \in C$, $\phi(\mathcal{F}) \xrightarrow{q_k} y$ implies $j(\mathcal{F}) = h \circ \phi(\mathcal{F}) \xrightarrow{q_{c'}} h(y) = [\mathcal{F}]$, which shows that $((X_1^*, C'), j)$ is in standard form. This completes the proof. □

Note that the quasi-Wyler completion is the finest quasi-stable completion in standard form, but it is not the finest stable completion in FIL. In fact, there is no such finest one for a filter space [3], whenever $X^* \setminus j(X)$ is infinite.

4. Extension Theorem

Extension theorems for filter spaces [3], regular filter spaces [10], filter semi-groups [11] and Cauchy spaces (not necessarily T_2) [9] have led to some interesting reflective subcategories of the categories FIL and CHY with some special type of morphisms called s-maps. In case of T_2 filter spaces, an unique extension of a Cauchy map $f : (X,C) \longrightarrow (Y,D)$ to the corresponding completion space was possible only when the codomain was a c-filter space. Here, an extension theorem is established without this restriction on the codomain, which is a considerable departure from the previous results ([3], [4]).

A Cauchy map between two filter spaces $f : (X,C) \to (Y,D)$ is said to be a *quasi-s-map*, if it satisfies the following condition:

$\mathcal{F} \in C$ q_c-converges to at most one point in X implies that $f(\mathcal{F})$ is D-linked to at most one point in Y.

Note that a quasi-s map is an s-map [9]. There are several examples of quasi-s-maps. Any Cauchy map is a quasi-s-map, if the codomain of the map is a q-T_2 filter space. The identity map on a filter space and the embedding map φ for a stable completion are also quasi-s-maps. In particular, the mapping j in the quasi-Wyler completion is a quasi-s-map. Note that it follows from the definition of s-map that composition of two quasi-s-maps is a quasi-s-map. The class of all filter spaces with the quasi-s-maps as morphisms forms a category, which we call FIL'. We observe that every Cauchy map is not necessarily a quasi-s-map. For example, any mapping from a nontrivial filter space or an incomplete filter space into an indiscrete filter space containing at least two points is a Cauchy map, but not a quasi-s-map. So FIL' is not a full subcategory of FIL.

The following proposition shows that the quasi-Wyler completion $((X_1^*, C_1^*), j)$ has a property similar to the universal property of the T_2 completions [3]. A significant departure from the previous result is that we don't need to restrict the codomain of the quasi-s-map to be a c-filter space [3].

Proposition 5. *Let (X, C) and (Y, D) be two filter spaces with the quasi-Wyler completions $((X_1^*, C_1^*), j_X)$ and $((Y_1^*, D_1^*), j_Y)$, respectively. If $f : (X, C) \to (Y, D)$ is a quasi-s-map, then there is a unique extension $f^* : (X_1^*, C_1^*) \to (Y_1^*, D_1^*)$ which is also a quasi-s-map and $f^* \circ j_X = j_Y \circ f$.*

Proof. Define $f^* : (X_1^*, C^*) \to (Y_1^*, D^*)$ as follows

$$f^*(x) = f(x)$$

$$f^*([\mathcal{F}]) = \begin{cases} [f(\mathcal{F})] & \text{if } f(\mathcal{F}) \text{ not } D\text{-linked to } \dot{y} \text{ for any } y \in Y, \\ y & \text{if } f(\mathcal{F}) \xrightarrow{q_D} y \text{ for some } y \in Y. \end{cases}$$

Note that $f(\mathcal{F})$ is not D-linked to $\dot{y}$ for any $y \in Y$ implies that $f(\mathcal{F})$ is q_D-non-convergent. The mapping f^* is a well-defined map, because, if $[\mathcal{F}] = [\mathcal{G}]$, then $f(\mathcal{F}) \sim_D f(\mathcal{G})$. So either both $f(\mathcal{F})$ and $f(\mathcal{G})$ are not D-linked to any element in Y, or otherwise. In the first case, $f^*([\mathcal{F}]) = f^*([\mathcal{G}])$. Otherwise, if $f(\mathcal{F}) \sim_D \dot{y}_1$ and $f(\mathcal{G}) \sim_D \dot{y}_2$, then $f(\mathcal{F}) \sim_D \dot{y}_1,\ \dot{y}_2$. This is a contradiction, since $\mathcal{F}$ is not C-linked to $\dot{x}$ for any $x \in X$ implies $\mathcal{F}$ is q_c-non-convergent and f is a quasi-s-map. So in either case $f^*([\mathcal{F}]) = f^*([\mathcal{G}])$. Also, it can be easily verified that $f^* \circ j_X = j_Y \circ f$.

Next we show that f^* is a quasi-s-map. Let $\mathcal{A} \in C^*$. If $\mathcal{A} \geq j_X(\mathcal{F})$, then $f^*(\mathcal{A}) \geq f^* \circ j_X(\mathcal{F}) = j_Y \circ f(\mathcal{F}) \in D^*$. If $\mathcal{A} \geq j_X(\mathcal{F}) \cap [F]$, where $\mathcal{F}$ is not C-linked to any $x \in X$, then $f^*(\mathcal{A}) \geq (j_Y \circ f(\mathcal{F})) \cap f^*([\mathcal{F}])$. If $f(\mathcal{F})$ is q_D-non-convergent in Y, then $(j_Y \circ f(\mathcal{F})) \cap [f(\mathcal{F})] \in D^*$. If $f(\mathcal{F})$ q_D-converges to $y \in Y$, then, $f(\mathcal{F}) \cap \dot{y} \in D$, so it follows that $(j_Y \circ f(\mathcal{F})) \cap \dot{y} \in D^*$. Therefore, f^* is a Cauchy map. To show that it is a quasi-s-map, it suffices to show that if $\mathcal{A} \in C^*$ q_{C^*}-converges to only one point, then $f^*(\mathcal{A})$ q_{D^*}-converges to only one point in Y^*. If $\mathcal{A} \geq j_X(\mathcal{F})$, then $j_Y \circ f(\mathcal{F}) = f^* \circ j_X(\mathcal{F})$ is D^*-linked to only one point in Y^*, which implies it q_{D^*}-converges to only one point, since j_Y and f are quasi-s-maps. If $\mathcal{A} \geq j_X(\mathcal{F}) \cap [\dot{\mathcal{F}}]$, then $\mathcal{F}$ is not C-linked to any point in X, implies $\mathcal{F}$ is q_c-non-convergent. Hence, it follows from f being a quasi-s-map that $f(\mathcal{F})$ is D-linked to at most one point in Y. Therefore, $f^*(j_X(\mathcal{F}) \cap [\dot{\mathcal{F}}]) = (f^* \circ j_X(\mathcal{F})) \cap f^*([\dot{\mathcal{F}}]) = (j_Y \circ f(F)) \cap [f(\dot{\mathcal{F}})]$ or $(j_Y \circ f(\mathcal{F})) \cap \dot{y}$ according as $f(\mathcal{F})$ is not D-linked to any point (hence q_D non-convergent) or

$f(\mathcal{F})$ q_D-converges to $y \in Y$. But in either case $f^*(\mathcal{A})$ converges to only one point in Y^*.

Finally, we show that f^* is an unique extension. Let $\overline{f} : (X^*, C^*) \to (Y^*, D^*)$ be another quasi-s-map such that $\overline{f} \circ j_X = j_Y \circ f$. It is obvious that $\overline{f} \circ j_X(x) = f^* \circ j_X(x)$ for all $x \in X$. So, let $[\mathcal{F}] \in X^* \setminus j_X(X)$. Since $\mathcal{F} \in C$ is not C-linked to any point in X, $j_X(\mathcal{F}) \cap [\dot{\mathcal{F}}]$. Since f^*, $\overline{f}$ are also Cauchy maps, $f^* \circ j_X(\mathcal{F}) = \overline{f} \circ j_X(\mathcal{F}) = j_Y \circ f(\mathcal{F})$ q_{D^*}-converges to $f^*([\mathcal{F}])$, $\overline{f}([\mathcal{F}])$. Therefore $j_Y \circ f(\mathcal{F})$ is D^*-linked to both $f^*([\mathcal{F}])$ and $\overline{f}([\mathcal{F}])$. However, $\mathcal{F}$ is not C-linked, which implies it is also q_c-non-convergent, and f, j_Y are quasi-s-maps imply that $j_Y \circ f(\mathcal{F})$ can be D^*-linked to at most one point in Y^*. Hence $f^* = \overline{f}$. This completes the proof. □

The unique mapping f^* in Proposition 5 is called the *quasi-s-extension of f*.

Remark (I) If $f : (X, C) \to (Y, K)$ is a quasi-s-map, where (Y, K) is a quasi-complete filter space, then there exists a unique quasi-s-extension $f^* : (X^*, C^*) \to (Y, K)$ such that $f^* \circ J_X = f$.

(II) If (X, C) is a q-T_2 filter space, then its T_2 quasi-Wyler completion also has the extension property. Recall that if the codomain of an s-map is a q-T_2 space, then the s-map is simply a Cauchy map. If $f : (X, C) \to (Y, K)$ is a Cauchy map, where (Y, K) is a complete T_2 c-filter space [3] (or a complete T_3 filter space [10]), then there exists a unique Cauchy extension $f^* : (X^*, C^*) \to (Y, K)$ such that $f^* \circ J_X = f$.

Note that a composition of quasi-s-maps is a quasi-s-map and the identity map is a quasi-s-map. So the class of all filter spaces with quasi-s-maps as morphisms form a subcategory of FIL. We denote this category by FIL'. Since it comprises quasi-s-maps as morphisms, it is not a full subcategory of FIL. Let FIL'^* be the subcategory of FIL' consisting of the quasi-complete objects of FIL'. On the category FIL', we can define a functor $W_q : FIL' \to FIL'^*$ by $W_q(X, C) = (X_1^*, C_1^*)$ for all objects, and $W_q(f) = f^*$ for all morphisms in FIL'. Using the property of s-maps, it is a routine matter to show that W_q is a covariant functor on FIL'. The functor W_q is called the *quasi-Wyler completion functor*.

References

[1] H.L. Bently, H. Herrlich, E. Lowen-Colebunders, Convergence, *J. Pure Appl. Algebra*, **68** (1990), 27-45; **doi:** 10.1016/0022-4049(90)90130-A.

[2] M. Katetov, On continuity structures and spaces of mappings, *Comment. Math. Carolinae*, **6** (1965), 257-278; https://eudml.org/doc/16128.

[3] D.C. Kent, N. Rath, Filter spaces, *Applied Categorical Structures*, **1** (1993), 297-309; **doi:** 10.1007/BF00873992.

[4] D.C. Kent, N. Rath, On completions of filter spaces **767** , Annals of the New York Academy of Sciences (1995), 97-107; **doi:** 10.1111/j.1749-6632.1995.tb55898.x.

[5] G. Minkler, J. Minkler, G. Richardson, Extensions for filter spaces, *Acta. Math. Hungar.*, **82**, No. 4 (1999), 301-310; **doi:** 10.1023/A:1006688224938.

[6] G. Preuss, Semiuniform convergence spaces and filter spaces, (Beyond Toplogy, Contemporary mathematics Series- 486, AMS Publn., 2009, Eds: F. Maynard and E. Pearl) 333-374.

[7] G. Preuss, Improvement of Cauchy spaces, *Q&A in General Topology*, **9** (1991), 159-166.

[8] G. Preuss, *Theory of Topological Structures*, D. Reidel Publ. Co., Dordrecht (1988).

[9] N. Rath, Completion of a Cauchy space without the T_2 restriction on the space, *Int. J. Math. Math. Sci.*, **24**, No. 3 (2000), 163-172, **doi:** 10.1155/S0161171200003331.

[10] N. Rath, Regular filter spaces, *Topics in Applied Theoretical Mathematics and Computer Science*, WSES Press (2001) 249-254; http://www.wseas.us/e-library/conferences/cairns2001/papers/610.pdf.

[11] N. Rath, Completions of filter semigroups, *Acta. Math. Hungar.*, **107**, No-s: 1-2 (2005), 45-54; http://link.springer.com/article/10.1007/s10474-005-0176-0.

[12] E.E. Reed, Completions of uniform convergence spaces, *Math. Ann.*, **194** (1971), 83-108; **doi:** 10.1007/BF01362537.