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HYERS-ULAM-RASSIAS STABILITY OF LIE ∗-DERIVATIONS OF A CUBIC FUNCTIONAL EQUATION WITH THREE VARIABLES

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Abstract: We will prove the general solution of the following cubic functional equation

$$4\{f(2x+y+z)+f(x+2y+z)+f(x+y+2z)\} = 27f(x+y+z) + f(-x+y+z)+f(x-y+z)+f(x+y-z)+12\{f(x)+f(y)+f(z)\}$$

and investigate the stability of a cubic Lie ∗-derivation associated with the given functional equation.

AMS Subject Classification: 39B82, 39B62
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1. Introduction

In the theory of functional equations, a classical problem is the following: "When is it true that a function which approximately satisfies a functional equation ε must be close to an exact solution of ε?" If the problem accepts a solution, we say that the equation is ε-stable. The first stability problem about group homomorphisms was raised by Ulam [13] in 1940. We are given a group G and a metric group G' with netruc $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a

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$\delta > 0$ such that if $f : G \to G'$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $d(f(x), h(x)) < \varepsilon$ for all $x \in G$? Ulam's problem was partially solved by Hyers [5] in 1941. Let E_1 be a normed space, E_2 a Banach space and suppose that a mapping $f : E_1 \to E_2$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for all $x, y \in E_1$, where $\varepsilon > 0$ is a constant. Then the limit $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for each $x \in E_1$ and T is the unique additive mapping satisfying

$$\|f(x) - T(x)\| \le \varepsilon \tag{1}$$

for all $x \in E_1$. Also, if the function $t \longmapsto f(tx)$ from $\mathbb{R}$ to E_2 for each x is continuous on $\mathbb{R}$, then T is linear. If f is continuous at a single point of E_1, then T is continuous everywhere in E_1. Moreover (1) is sharp.

Bourgin [2] was the next author who treated this problem for additive mappings (see also [1]). In [12], Rassias provided a generalization of Hyers theorem which allows the Cauchy difference to be unbounded. Găvruta then generalized the Rassias result in [4] for the unbounded Cauchy difference. Subsequently, various approaches to the problem have been studied by a number of authors.

Also, Jang and Park [6] investigated the stability of $*$-derivations and of quadratic $*$-derivations with Cauchy functional equation and the Jensen functional equation on Banach $*$-algebra. The stability of $*$-derivations on Banach $*$-algebra by using fixed point alternative was proved by Park and Bodaghi and also Yang et al.; see [10] and [14], respectively. In particular, the stability of cubic Lie derivations was introduced by Fošner and Fošner; see [3].

Let both E_1 and E_2 real vector spaces. Jun and Kim [8] proved that a mapping $f : E_1 \to E_2$ satisfies the functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{2}$$

if and only if there exists a mapping $B : E_1 \times E_1 \times E_1 \to E_2$ such that $f(x) = B(x.x.x)$ for all $x \in E_1$, where B, defined by

$$\begin{aligned} B(x,y,z) \quad &= \quad \frac{1}{24}\big[f(x+y+z) + f(x-y-z) \\ &\quad -f(x+y-z) - f(x-y+z)\big] \end{aligned}$$

for all $x, y, z \in E_1$, is symmetric for each fixed one variable and additive for each fixed two variables. It is easy to see that the functional equation (2) is equivalent to a *cubic functional equation*

$$C(2x+y) + C(x-y) + 3C(y) = 3C(x+y) + 6C(x)$$

and every solution of the cubic functional equation is said to be a cubic mapping [11]. Najati [9] investigated the following generalized cubic functional equation:

$$f(sx+y)+f(sx-y)=sf(x+y)+sf(x-y)+2(s^3-s)f(x)$$

for a positive integers $s \geq 2$. Also, Jun and Kim [7] proved the Hyers-Ulam-Rassias stability of a Euler-Lagrange type cubic mapping as follows:

$$\begin{aligned}&f(sx+y)+f(x+sy)\\&=(s+1)(s-1)^2[f(x)+f(y)]+s(s+1)f(x+y),\end{aligned}$$

where $s \in \mathbb{Z}(s \neq 0, \pm 1)$.

In this paper, we deal with the following cubic functional equation with three variables which is equivalent to (2),

$$\begin{aligned}&4\{f(2x+y+z)+f(x+2y+z)+f(x+y+2z)\}=\\&27f(x+y+z)+f(-x+y+z)+f(x-y+z)\\&+f(x+y-z)+12\{f(x)+f(y)+f(z)\}.\end{aligned} \tag{3}$$

It is easy to see that the function $f(x)=cx^3$ is a solution of the above functional equation. We will investigate the stability of a cubic Lie $*$-derivations associated with the given functional equation on normed algebras.

2. General Solution

Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers and let both E_1 and E_2 be real vector spaces. We here present the general solution of (3).

Theorem 1. *A function $f: E_1 \to E_2$ satisfies the functional equation (2) if and only if $f: E_1 \to E_2$ satisfies the functional (3). Therefore, every solution of functional equations (3) is also a cubic function.*

Proof. Assume that $f: E_1 \to E_2$ satisfy the functional equation (3). Putting $x=y=z=0$ in (2), we get $f(0)=0$. Setting $y=z=0$ in (3), we have

$$4f(2x)=33f(x)+f(-x). \tag{4}$$

for all $x \in E_1$. Letting $y=x, z=0$ in (3), we obtain

$$f(3x)=3f(2x)+3f(x) \tag{5}$$

for all $x \in E_1$. Replacing y and z by x and x in (3) respectively, we get

$$12f(4x) = 27f(3x) + 39f(x) \tag{6}$$

for all $x \in E_1$. Using (5) and (6), we have

$$12f(4x) = 81f(2x) + 120f(x) \tag{7}$$

for all $x \in E_1$. Associating (4) and (7), we obtain

$$12f(4x) = \frac{3153}{4}f(x) + \frac{81}{4}f(-x) \tag{8}$$

for all $x \in E_1$. Also,

$$12f(4x) = 3(33f(2x) + f(-2x)) = \frac{3270}{4}f(x) + \frac{198}{4}f(-x) \tag{9}$$

for all $x \in E_1$, by using (4). So $f(-x) = -f(x)$ for all $x \in E_1$. Letting $z = 0$ in (3), we have

$$f(2x + y) + f(x + 2y) = 6f(x + y) + 3f(x) + 3f(y) \tag{10}$$

for all $x, y \in E_1$. Replacing x by $x - y$, we get

$$f(2x - y) = -f(x + y) + 3f(x - y) + 6f(x) + 3f(y) \tag{11}$$

for all $x, y \in E_1$. Taking y into $-y$ in (11) and adding (11), we have (2).

Conversely, if $f : E_1 \to E_2$ satisfies (2), then by [8], we can get a function $B : E_1 \times E_1 \times E_1 \to E_2$ such that $f(x) = B(x, x, x)$ for all $x \in E_1$, and B is symmetric for fixed one variable and B is additive for fixed two variables. It is obvious that f satisfies (3).

□

As the above result and Corollary 2.4 of [7], we obtain the following Corollary.

Corollary 2. *If a mapping $f : \mathbb{R} \to \mathbb{R}$ is measurable which satisfies (3), then f is continuous on $\mathbb{R}$ and $f(x) = f(1)x^3$ for all $x \in \mathbb{R}$.*

3. Cubic Lie $*$-Derivations

Throughout this section, we assume that A is a complex normed $*$-algebra and M is a Banach A-module. We will use the same symbol $\|\cdot\|$ as norms on a normed algebra A and a normed A-bimodule M. A mapping $f : A \to M$ is a *cubic homogeneous mapping* if $f(\mu a) = \mu^3 f(a)$, for all $a \in A$ and $\mu \in \mathbb{C}$. A cubic hoogeneous mapping $f : A \to M$ is called a *cubic derivation* if

$$f(xy) = f(x)y^3 + x^3 f(y)$$

holds for all $x, y \in A$. For all $x, y \in A$, the symbol $[x, y]$ will denote the commutator $xy - yx$. We say that a cubic homogeneous mapping $f : A \to M$ is a cubic Lie derivation if

$$f([x, y]) = [f(x), y^3] + [x^3, f(y)]$$

for all $x, y \in A$. In addition, if f satisfies in condition $f(x^*) = f(x)^*$ for all $x \in A$, then it is called *the cubic Lie $*$-derivation*. In the following, $\mathbb{T}^1$ will stand for the set of all complex units, that is,

$$\mathbb{T}^1 = \{\mu \in \mathbb{C} | |\mu| = 1\}.$$

For the given mapping $f : A \to M$, we consider

$$\begin{aligned} \triangle_\mu f(x, y, z) \ &:= \ f(2\mu x + \mu y + \mu z) + f(\mu x + 2\mu y + \mu z) \\ &\quad + f(\mu x + \mu y + 2\mu z) - \frac{27}{4}\mu^3 f(x + y + z) \\ &\quad - \frac{1}{4}\mu^3 \{ f(-x + y + z) + f(x - y + z) \\ &\quad + f(x + y - z) \} - 3\mu^3 \{ f(x) + f(y) + f(z) \}, \\ \triangle f(x, y) \ &:= \ f([x, y]) - [f(x), y^3] - [x^3, f(y)] \end{aligned}$$

for all $x, y, z \in A$, $\mu \in \mathbb{C}$.

Theorem 3. *Let $\varphi : A^3 \to \mathbb{R}^+$ and $\psi : A^3 \to \mathbb{R}^+$ be functions such that*

$$\tilde{\varphi}(x) := \sum_{n=0}^{\infty} \frac{1}{8^n} \varphi(2^n x, 0, 0) < \infty \tag{12}$$

and

$$\lim_{n\to\infty} \frac{1}{8^n} \varphi(2^n x, 2^n y, 2^n z) = 0, \tag{13}$$

$$\lim_{n\to\infty}\frac{1}{8^n}\psi(2^n x, 2^n y, 2^n z) < \infty \tag{14}$$

for all $x, y, z \in A$. Suppose that $f : A \to M$ is a mapping with $f(0) = 0$ such that

$$\|\triangle_\mu f(x, y, z)\| \le \varphi(x, y, z) \tag{15}$$

and

$$\|\triangle f(x, y) + f(z^*) - f(z)^*\| \le \psi(x, y, z) \tag{16}$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}} = \left\{e^{i\theta} | 0 \le \theta \le \frac{2\pi}{n_0}\right\}$ and all $x, y, z \in A$ in which $n_0 \in \mathbb{N}$. Also, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique cubic Lie $*$-derivation $T : A \to M$ satisfying

$$\|f(x) - T(x)\| \le \frac{1}{8}\Big(\frac{\tilde{\varphi}(x) + \tilde{\varphi}(-x)}{2}\Big) + \frac{2}{17}\Big(\frac{\tilde{\tilde{\varphi}}(x) + \tilde{\tilde{\varphi}}(-x)}{2}\Big) \tag{17}$$

for all $x \in A$, where $\tilde{\tilde{\varphi}}(x) = \sum_{n=0}^{\infty}\left(\frac{2}{17}\right)^n \varphi(2^n x, 0, 0)$.

Proof. Let $y = z = 0$ and $\mu = 1$ in (15), we have

$$\|f(2x) - \frac{33}{4}f(x) - \frac{1}{4}f(-x)\| \le \varphi(x, 0, 0) \tag{18}$$

for all $x \in A$. One can easily get

$$\|\frac{1}{8}f_o(2x) - f_o(x)\| \le \frac{1}{8}\Big(\frac{\varphi(x, 0, 0) + \varphi(-x, 0, 0)}{2}\Big)$$
$$\|\frac{2}{17}f_e(2x) - f_e(x)\| \le \frac{2}{17}\Big(\frac{\varphi(x, 0, 0) + \varphi(-x, 0, 0)}{2}\Big)$$

where $f_e(x) = \frac{f(x)+f(-x)}{2}$ and $f_o(x) = \frac{f(x)-f(-x)}{2}$ for all $x \in A$. Using the induction, it is easy to show that

$$\|\frac{1}{8^t}f_o(2^t x) - \frac{1}{8^k}f_o(2^k x)\| \tag{19}$$
$$\le \frac{1}{8}\sum_{j=k}^{t-1}\frac{1}{8^j}\Big(\frac{\varphi(2^j x, 0, 0) + \varphi(-2^j x, 0, 0)}{2}\Big)$$

$$\|\Big(\frac{2}{17}\Big)^t f_e(2^t x) - \Big(\frac{2}{17}\Big)^k f_e(2^k x)\| \tag{20}$$

$$\leq \frac{2}{17}\sum_{j=k}^{t-1}\big(\frac{2}{17}\big)^j\Big(\frac{\varphi(2^jx,0,0)+\varphi(-2^jx,0,0)}{2}\Big)$$

for all $t > k > 0$ and $x \in A$. The inequality (12) and (19), (20) imply that the sequences $\{\frac{1}{8^n}f_e(2^nx)\}$ and $\{(\frac{2}{17})^n f_e(2^nx)\}$ are Cauchy. Since M is complete, these sequences are convergent. Hence we can define two mappings T_o, T_e as

$$T_o(x) = \lim_{n\to\infty}\frac{1}{8^n}f_o(2^nx), \quad T_e(x) = \lim_{n\to\infty}\big(\frac{2}{17}\big)^n f_e(2^nx)$$

for all $x \in X$. By taking $t \to \infty$ and $k = 0$ in the inequalities (19) and (3.9), we have

$$\|f_o(x) - T_o(x)\| \leq \frac{1}{8}\Big(\frac{\tilde{\varphi}(x)+\tilde{\varphi}(-x)}{2}\Big),$$
$$\|f_e(x) - T_e(x)\| \leq \frac{2}{17}\Big(\frac{\tilde{\tilde{\varphi}}(x)+\tilde{\tilde{\varphi}}(-x)}{2}\Big)$$

for all $x \in A$. If we define $T = T_o + T_e$, we get (17).

Now, we will show that the mapping T is a unique cubic Lie $*$-derivation such that the inequality (17) holds for all $x \in A$. We note that

$$\begin{aligned}&\|\triangle_\mu T(x,y,z)\| \leq \|\triangle_\mu T_o(x,y,z)\| + \|\triangle_\mu T_e(x,y,z)\| \\ &\leq \lim_{n\to\infty}\frac{2}{8^n}\frac{\varphi(2^nx,2^ny,2^nz)+\varphi(-2^nx,-2^ny,-2^nz)}{2} = 0\end{aligned} \tag{21}$$

for all $x, y, z \in A$ and $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$. By taking $\mu = 1$ in the inequality (21), it follows that the mapping T is a cubic mapping. Also, the inequality (21) implies that $\triangle_\mu T(x,y,z) = 0$ for all $x, y, z \in A$. Hence

$$T(\mu x) = \mu^3 T(x)$$

for all $x \in A$ and $\mu \in \mu_1 \in \mathbb{T}^1_{\frac{1}{n_0}}$. Let $\mu \in \mathbb{T}^1 = \{\lambda \in \mathbb{C}||\lambda| = 1\}$. Then $\mu = e^{i\theta}$, where $0 \leq \theta \leq 2\pi$. Let $\mu_1 = \mu^{\frac{1}{n_0}} = e^{\frac{i\theta}{n_0}}$. So we have $\mu_1 \in \mathbb{T}^1_{\frac{1}{n_0}}$. Then

$$L(\mu x) = L(\mu_1^{n_0}x) = \mu^{3n_0}L(x) = \mu^3 L(x)$$

for all $\mu \in \mathbb{T}^1$ and $x \in A$. For any continuous linear functional L defined on A, let $\rho : \mathbb{R} \to \mathbb{R}$ be given by

$$\rho(t) := L[T(tx)]$$

for $t \in \mathbb{R}$, where x is fixed. Then ρ is a cubic mapping and, moreover, is also measurable since it is the pointwise limit of the sequence

$$\rho_n(t) := 8^{-n} L[f(2^n tx)].$$

Hence it has the form $\rho(t) = t^3\rho(1)$ for all $t \in \mathbb{R}$ by Corollary 2.2. Therefore one obtains that for each fixed $x \in X$ and all $t \in \mathbb{R}$

$$L[T(tx)] = \rho(t) = t^3\rho(1) = t^3 L[T(x)] = L[t^3 T(x)],$$

which implies the condition

$$T(tx) = t^3 T(x), \quad \forall x \in A, \forall t \in \mathbb{R}.$$

Let $\mu \in \mathbb{C}(\mu \neq 0)$. Then $\frac{\mu}{|\mu|} \in \mathbb{T}^1$. Hence

$$T(\mu x) = T(\frac{\mu}{|\mu|}|\mu| x) = (\frac{\mu}{|\mu|})^3 |\mu|^3 T(x) = \mu^3 T(x)$$

for all $x \in A$ and $\mu \in \mathbb{C}(\mu \neq 0)$. Since x is an arbitrary element in A, we may conclude that T is cubic homogeneous. Next, using (16), we have

$$\frac{1}{8^{2n}}\|\triangle f(2^n x, 2^n y)\| \leq \frac{\psi(2^n x, 2^n y, 0)}{8^n}$$

for all $x, y \in A$. Taking the limit as n tends to infinity, we have $\triangle T(x, y) = 0$, for all $x, y \in A$. That is, T is a cubic Lie derivation. Letting $x = y = 0$ and replacing z by $2^n x$ in the inequality (16), we get

$$\|\frac{f(2^n z^*)}{8^n} - \frac{f(2^n z)^*}{8^n}\| \leq \frac{\psi(0, 0, 2^n z)}{8^n} \tag{22}$$

for all $z \in A$. As $n \to \infty$ in the inequality (22), we have

$$T(z^*) = T(z)^*$$

for all $z \in A$. This means that T is a cubic Lie $*$-derivation. Now, assume $T' : A \to A$ is another cubic $*$-derivation satisfying the inequality (17). Then

$$\begin{aligned}
\|T(x) - T'(x)\| &= \frac{1}{8^n}\|T(2^n x) - T'(2^n x)\| \\
&\leq \frac{1}{8^n}\Big(\|T(2^n x) - f(2^n x)\| + \|f(2^n x) - T'(2^n x)\|\Big) \\
&\leq \frac{4}{8}\sum_{j=n}^{\infty}\frac{1}{8^j}\Big(\frac{\varphi(2^j x, 0, 0) + \varphi(-2^j x, 0, 0)}{2}\Big)
\end{aligned}$$

which tends to zero as $n \to \infty$, for all $x \in A$. Thus $T(x) = T'(x)$ for all $x \in A$. This proves the uniqueness of T. □

Corollary 4. *Let ε and $\varepsilon' > 0$. Suppose $f : A \to M$ is a mapping with $f(0) = 0$ such that*

$$\|\triangle_\mu f(x,y,z)\| \leq \varepsilon$$

and

$$\|\triangle f(x,y) + f(z^*) - f(z)^*\| \leq \varepsilon'$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}} = \left\{e^{i\theta} | 0 \leq \theta \leq \frac{2\pi}{n_0}\right\}$ and all $x, y, z \in A$ in which $n_0 \in \mathbb{N}$. Also, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique cubic Lie $*$-derivation $T : A \to M$ satisfying

$$\|f(x) - T(x)\| \leq \frac{29}{105}\varepsilon$$

for all $x \in A$.

Proof. Letting $\varphi(x,y,z) = \varepsilon$, $\psi(x,y) = \varepsilon'$ and applying Theorem 3, we get the desired result, as claimed. □

Corollary 5. *Let $\theta, \theta' > 0$ and $0 < r < 3$, $0 < r' < 3$. Suppose $f : A \to M$ is a mapping such that*

$$\|\triangle_\mu f(x,y,z)\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r)$$

and

$$\|\triangle f(x,y) + f(z^*) - f(z)^*\| \leq \theta'(\|x\|^{r'} + \|y\|^{r'} + \|z\|^{r'})$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}} = \left\{e^{i\theta} | 0 \leq \theta \leq \frac{2\pi}{n_0}\right\}$ and all $a, b, c, x, y, z \in A$ in which $n_0 \in \mathbb{N}$. Also, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique cubic Lie $*$-derivation $T : A \to M$ satisfying

$$\|f(x) - T(x)\| \leq \frac{\theta\|x\|^r}{8 - 2^r} + \frac{\theta\|x\|^r}{\frac{17}{2} - 2^r}$$

for all $x \in A$.

Proof. Letting $\varphi(x,y,z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$, $\psi(x,y) = \theta'(\|x\|^{r'} + \|y\|^{r'} + \|z\|^{r'})$ and applying Theorem 3, we get the desired result, as claimed. □

Theorem 6. *Let $\phi : A^3 \to \mathbb{R}^+$ and $\psi' : A^3 \to \mathbb{R}^+$ be functions such that*

$$\tilde{\tilde{\phi}}(x) := \sum_{n=1}^{\infty} \big(\frac{17}{2}\big)^n \phi(\frac{x}{2^n}, 0, 0) < \infty$$

and

$$\lim_{n\to\infty} \big(\frac{17}{2}\big)^n \phi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}) = 0,$$

$$\lim_{n\to\infty} 8^{2n} \psi'(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}) < \infty$$

for all $x, y, z \in A$. Suppose that $f : A \to M$ is a mapping with $f(0) = 0$ such that

$$\|\triangle_\mu f(x, y, z)\| \le \phi(x, y, z)$$

and

$$\|\triangle f(x, y) + f(z^*) - f(z)^*\| \le \psi'(x, y, z)$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}} = \{e^{i\theta} | 0 \le \theta \le \frac{2\pi}{n_0}\}$ and all $x, y, z \in A$ in which $n_0 \in \mathbb{N}$. Also, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique cubic Lie $*$-derivation $T : A \to M$ satisfying

$$\|f(x) - T(x)\| \le \frac{1}{8}\Big(\frac{\tilde{\phi}(x) + \tilde{\phi}(-x)}{2}\Big) + \frac{2}{17}\Big(\frac{\tilde{\tilde{\phi}}(x) + \tilde{\tilde{\phi}}(-x)}{2}\Big) \tag{23}$$

for all $x \in A$, where $\tilde{\phi}(x) = \sum_{n=0}^{\infty} 8^n \phi(\frac{x}{2^n}, 0, 0)$.

Proof. Using (18), we have

$$\|f_o(x) - 8 f_o(\frac{x}{2})\| \le \Big(\frac{\phi(\frac{x}{2^j}, 0, 0) + \varphi(-\frac{x}{2}, 0, 0)}{2}\Big),$$
$$\|f_e(x) - \frac{17}{2} f_e(\frac{x}{2})\| \le \Big(\frac{\phi(\frac{x}{2}, 0, 0) + \phi(-\frac{x}{2}, 0, 0)}{2}\Big)$$

for all $x \in A$. And using induction, we get

$$\|8^k f_o(\frac{x}{2^k}) - 8^t f_o(\frac{x}{2^t})\| \le \frac{1}{8} \sum_{j=k+1}^{t} 8^j \Big(\frac{\phi(\frac{x}{2^j}, 0, 0) + \phi(\frac{x}{2^j}, 0, 0)}{2}\Big)$$

$$\|(\frac{17}{2})^k f_e(\frac{x}{2^k}) - (\frac{17}{2})^t f_e(\frac{x}{2^t})\| \leq \frac{2}{17} \sum_{j=k+1}^{t} (\frac{17}{2})^j \Big(\frac{\phi(\frac{x}{2^j},0,0) + \phi(\frac{x}{2^j},0,0)}{2}\Big)$$

for all $t > k > 0$ and $x \in A$. Thus the sequences $\{8^n f_o(\frac{x}{2^n})\}$ and $\{(\frac{17}{2})^n f_e(\frac{x}{2^n})\}$ are Cauchy. So, we can define

$$T_o(x) = \lim_{n\to\infty} 8^n f_o(\frac{x}{2^n}), \quad T_e(x) = \lim_{n\to\infty} \{\frac{17}{2}\}^n f_e(\frac{x}{2^n})$$

for all $x \in X$. If we take $T = T_e + T_o$, then we have (23). And

$$\begin{aligned} \|\triangle_\mu T(x,y,z)\| &\leq \|\triangle_\mu T_o(x,y,z)\| + \|\triangle_\mu T_e(x,y,z)\| \\ &\leq \lim_{n\to\infty} 2 \cdot \Big(\frac{17}{2}\Big)^n \frac{\phi(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}) + \phi(-\frac{x}{2^n},-\frac{y}{2^n},-\frac{z}{2^n})}{2} = 0 \end{aligned}$$

for all $x,y,z \in A$ and all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$. By the similar way of proof of Theorem 3, we get the mapping T is a unique cubic homogeneous. And we also have

$$8^{2n}\|\triangle f(\frac{x}{2^n},\frac{y}{2^n})\| \leq 8^{2n}\psi'(\frac{x}{2^n},\frac{y}{2^n},0)$$

$$\|8^n f(\frac{z^*}{2^n}) - 8^n f(\frac{z}{2^n})^*\| \leq 8^n \psi'(0,0,\frac{z}{2^n})$$

for all $x,y,z \in A$ and all $n \in \mathbb{N}$. The rest proof is similar to that of Theorem 3. □

Corollary 7. *Let $\theta, \theta' > 0$ and $r > \log_2 \frac{17}{2}$, $r' > 6$. Suppose $f : A \to M$ is a mapping such that*

$$\|\triangle_\mu f(x,y,z)\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r)$$

and

$$\|\triangle f(x,y) + f(z^*) - f(z)^*\| \leq \theta'(\|x\|^{r'} + \|y\|^{r'} + \|z\|^{r'})$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}} = \{e^{i\theta} | 0 \leq \theta \leq \frac{2\pi}{n_0}\}$ and all $x,y,z \in A$ in which $n_0 \in \mathbb{N}$. Also, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique cubic Lie $*$-derivation $T : A \to M$ satisfying

$$\|f(x) - T(x)\| \leq \frac{\theta\|x\|^r}{2^r - 8} + \frac{\theta\|x\|^r}{2^r - \frac{17}{2}}$$

for all $x \in A$.

Proof. Letting $\phi(x,y,z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$, $\psi'(x,y) = \theta'(\|x\|^{r'} + \|y\|^{r'} + \|z\|^{r'})$ and applying Theorem 6, we get the desired result, as claimed. □

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References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 222-224.

[2] D. G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc., **57** (1951), 223-237.

[3] A. Fošner and M. Fošner, *Approximate cubic Lie derivations*, Abstract and Applied Analysis Vol. 2013, Article ID 425784, 5page, (2013)

[4] P. Găvruta *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431-436

[5] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222-224.

[6] S. Jang and C. Park, *Approximate $*$-derivations and approximate quadratic $*$-derivations on C^*-algebra*, J. Inequal. bf 2011 Article ID 55 (2011),

[7] K.W. Jun, H.M. Kim, *On the stability of Euler-Lagrange type cubic functional equations in quasi-Banach spaces*, J. Math. Anal. Appl. **332** (2007), 1335-1350.

[8] K.W. Jun, H.M. Kim, *The generalized Hyers-Ulam-Rassias stability of a cubic functional equation*, J. Math. Anal. Appl. **274** (2002), 867-878.

[9] A. Najati, *The generalized Hyers-Ulam-Rassias stability of a cubic functional equation* , Turk. J. Math. **31** (2007), 395-408

[10] C. Park and A. Bodaghi, *On the stability of $*$-derivations on Banach $*$-algebras*, Adv. Diff. Equat. **2012** 2012:138(2012)

[11] J.M. Rassias, *Solution of the Ulam stability problem for cubic mappings*, Glasnik Matem. **36** (2001), 63-72.

[12] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297-300

[13] S. M. Ulam, *Problems in Modern Mathematics*, Chapter 6 Wiley Interscience, New York, 1964.

[14] S.Y. Yang, A. Bodaghi, K.A.M. Atan, *Approximate cubic $*$-derivations on Banach $*$-algebra*, Abstract and Applied Analysis, vol. 2012, Article ID 684179, 12pages, (2012).