ON $\phi$-PSEUDO SYMMETRIES OF GENERALIZED SASAKIAN-SPACE-FORMS

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Abstract: In this paper we study $\phi$-pseudo symmetric and $\phi$-pseudo Ricci symmetric generalized Sasakian-space-forms with various geometric properties.

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1. Introduction

The notion of generalized Sasakian-space-forms was first studied by Alegre et al. [1]. A generalized Sasakian-space-form is an almost contact metric manifold $(M, g)$ whose curvature tensor is given by

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}$$

where $f_1, f_2, f_3$ are differentiable functions on $M$ and $X, Y, Z$ are vector fields on $M$. In such case we will write the manifold as $M(f_1, f_2, f_3)$. This kind of manifolds appears as a natural generalization of the Sasakian-space-forms by taking:

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\[ f_1 = \frac{c+3}{4} \quad \text{and} \quad f_2 = f_3 = \frac{c-1}{4}, \]
where \( c \) denotes constant \( \phi \)-sectional curvature.

The \( \phi \)-sectional curvature of generalized Sasakian-space-form \( M(f_1, f_2, f_3) \) is \( f_1 + 3f_2 \). Moreover, cosymplectic space-form and Kenmotsu space-form are also considered as particular types of generalized Sasakian-space-form \([1, 2, 3, 4, 11, 16, 18, 19, 20]\).

The geometry of a space mainly depends on the curvature of the space. One of the most important geometric property of a space is symmetry. The study of symmetry of a manifold began with the works of Cartan \([7]\) and then this notion has been weakened by many authors by giving several defining conditions and some curvature restrictions in different directions.

In 1987, Chaki \([8]\) introduced the notion of pseudo symmetric manifold, which is weaker than the local symmetry. A non-flat Riemannian manifold \((M^n, g)(n > 2)\) is called pseudo symmetric in the sense of Chaki if it satisfies the relation

\[
\]  (1.2)

for any vector fields \( X, Y, Z, U \) and \( W \), where \( R \) is the Riemannian curvature tensor of the manifold. \( A \) is a non-zero 1-form such that \( g(X, \rho) = A(X) \) for every vector field \( X \). Such an \( n \)-dimensional manifold is denoted by \((PS)_n\).

In 1988, Chaki \([9]\) introduced the notion of pseudo Ricci symmetric manifolds defined as follows:

A non-flat Riemannian manifold \((M^n, g)\) is said to be pseudo Ricci symmetric \([9]\) if its Ricci tensor \( S \) of type (0,2) is not identically zero and satisfies the condition

\[
(\nabla_W S)(X, Y) = 2A(W)S(X, Y) + A(X)S(W, Y) + A(Y)S(X, W)
\]  (1.3)

for any vector field \( X, Y \) and \( Z \), where \( A \) is a nowhere vanishing 1-form and \( \nabla \) denotes the operator of covariant differentiation with respect to the metric tensor \( g \). Such an \( n \)-dimensional manifold is denoted by \((PRS)_n\).

The relation (1.3) can be written as

\[
(\nabla_W Q)(X) = 2A(W)Q(X) + A(X)Q(W) + S(X, W)\rho
\]  (1.4)

where \( \rho \) is the vector field associated 1-form \( A \) such that \( A(X) = g(X, \rho) \) and \( Q \) is the Ricci operator i.e., \( g(QX, Y) = S(X, Y) \) for all \( X \) and \( Y \).
The object of the paper is to study $\phi$-pseudo symmetric and $\phi$-pseudo Ricci symmetric generalized Sasakian-space-forms. The paper is organized as follows: Section 2 is devoted to preliminaries on generalized Sasakian-space-forms. Section 3 deals with a study of $\phi$-pseudo symmetric generalized Sasakian-space-forms. In Section 4, we study $\phi$-pseudo symmetric generalized Sasakian-space-forms. It is proved that, a $\phi$-pseudo symmetric trans-Sasakian generalized Sasakian-space-forms is cosymplectic provided $f_1 \neq f_3$. Also, we prove that a $\phi$-pseudo symmetric generalized Sasakian-space-form, which is $\beta$-Kenmotsu is an $\eta$-Einstein manifold. Section 5 is concerned on $\phi$-pseudo Ricci symmetric generalized Sasakian-space-forms.

2. Preliminaries

Let $M$ be an almost contact metric manifold [5] with an almost contact structure $(\phi, \xi, \eta, g)$, where $\phi$ is a (1, 1)-tensor field, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is a compatible Riemannian metric such that,

\begin{align*}
(a) \quad \phi^2(X) &= -X + \eta(X)\xi, \quad (b) \quad \phi(\xi) = 0, \quad (c) \quad \eta(\phi X) = 0, \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \\
(a) \quad g(X, \phi Y) &= -g(\phi X, Y), \quad (b) \quad g(X, \xi) = \eta(X),
\end{align*}

for all vector fields $X, Y$ on $M$.

An almost contact structure $(\phi, \xi, \eta, g)$, on $M$ is called trans-Sasakian structure [17] if there exist two functions $\alpha$ and $\beta$ on $M$ such that

\begin{equation}
(\nabla_X \phi)(Y) = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi) - \eta(Y)\phi X)
\end{equation}

for some smooth functions $\alpha$ and $\beta$ on $M$. From (2.4), it follows that

\begin{align*}
\nabla_X \xi &= -\alpha \phi X + \beta(X - \eta(X)\xi), \\
(\nabla_X \eta)Y &= -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).
\end{align*}

Again in [10] the authors have introduced two subclasses of trans-Sasakian structures, the $C_5$ and $C_6$ structures which contain the Kenmotsu and Sasakian structure respectively. The subclass $C_5$ and $C_6$ are respectively called $\alpha$-Sasakian and $\beta$-Kenmotsu structure. A trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic [5], $\beta$-Kenmotsu [12] and $\alpha$-Sasakian [12] respectively.
In addition to the relation (1.1), for an \((2n + 1)\)-dimensional \((n > 1)\) generalized Sasakian-space-form \(M^{2n+1}(f_1, f_2, f_3)\) the following relation holds [1]

\[
S(X, Y) = \left[ (2n)f_1 + 3f_2 - f_3 \right] g(X, Y) - [3f_2 + (2n - 1)f_3] \eta(X)\eta(Y), \tag{2.7}
\]

\[
QX = \left[ (2n)f_1 + 3f_2 - f_3 \right] X - [3f_2 + (2n - 1)f_3] \eta(X)\eta(\xi), \tag{2.8}
\]

\[
S(X, \xi) = (2n)(f_1 - f_3) \eta(X), \tag{2.9}
\]

\[
Q\xi = 2n(2n + 1)f_1 + 6nf_2 - 4nf_3, \tag{2.10}
\]

\[
r = 2n(2n + 1)f_1 + 6nf_2 - 4nf_3, \tag{2.11}
\]

\[
R(X, Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}, \tag{2.12}
\]

\[
R(\xi, X)Y = (f_1 - f_3)\{g(X, Y)\xi - \eta(Y)X\}, \tag{2.13}
\]

\[
S(\phi X, \phi Y) = S(X, Y) - 2n(f_1 - f_3) \eta(X)\eta(Y). \tag{2.14}
\]

Where \(R, S\) and \(Q\) are the curvature tensor, Ricci tensor and the Ricci operator of the space-form, respectively.

### 3. \(\phi\)-Pseudo Symmetric Generalized Sasakian-Space-Forms

**Definition 1.** A generalized Sasakian-space-form \(M^{2n+1}(f_1, f_2, f_3)(n > 2)\) is said to be a \(\phi\)-pseudo symmetric if the curvature tensor satisfies

\[
\phi^2((\nabla_W R)(X, Y)Z) = 2A(W)R(X, Y)Z + A(X)R(W, Y)Z + A(Y)R(X, W)Z + A(Z)R(X, Y)W + g(R(X, Y)Z, W)\rho, \tag{3.1}
\]

for any vector fields \(X, Y, Z\) and \(W\), where \(A\) is a non zero 1-form. If, in particular, \(A = 0\), then the manifold is called a \(\phi\)-symmetric [21]. The author, Hui have been studied the above notions to Kenmotsu manifolds [13], \((LCS)\_n\)-manifolds [15] and para-Sasakian manifolds [14].

We now consider a generalized Sasakian-space-form \(M^{2n+1}(f_1, f_2, f_3)(n > 1)\) which is \(\phi\)-pseudo symmetric. Then, by virtue of (2.1), it follows from (3.1) that

\[
-(\nabla_W R)(X, Y)Z + \eta((\nabla_W R)(X, Y)Z)\xi
= 2A(W)R(X, Y)Z + A(X)R(W, Y)Z + A(Y)R(X, W)Z
+ A(Z)R(X, Y)W + g(R(X, Y)Z, W)\rho. \tag{3.2}
\]

Taking \(Z = \xi\) in (3.2) and then using (2.12), we have Then

\[
-(\nabla_W R)(X, Y)\xi + \eta((\nabla_W R)(X, Y)\xi)\xi
\]
From (3.4), we have

\[ -(\nabla_W R)(\xi, Y)\xi + \eta((\nabla_W R)(\xi, Y)\xi) \]

\[ = (f_1 - f_3)[2A(W)\{\eta(Y)\xi - Y\}] + A(\xi)\{\eta(Y)W - \eta(W)Y\} \]

\[ + A(Y)\{\eta(W)\xi - W\}] + A(\xi)(f_1 - f_3)\{g(Y, W)\xi - \eta(W)Y\} \]

\[ + (f_1 - f_3)\{\eta(Y)\eta(W)\rho - g(Y, W)\rho\}. \]  \hspace{1cm} (3.3)

For \( X = \xi \) in (3.3), we have

\[ (\nabla_W R)(\xi, Y)\xi = \nabla_W R(\xi, Y)\xi - R(\nabla_W \xi, Y)\xi \]

\[ - R(\xi, \nabla_W Y)\xi - R(\xi, Y)\nabla_W \xi. \]  \hspace{1cm} (3.5)

Using (2.12) (2.13) and (2.5) in (3.5), we have

\[ (\nabla_W R)(\xi, Y)\xi = \nabla_W [(f_1 - f_3)(\eta(Y)\xi - Y)] - (f_1 - f_3)[\eta(\nabla_W Y)\xi - \nabla_W Y] \]

\[ -(f_1 - f_3)[\alpha\eta(Y)\phi W + \beta\eta(Y)W - \beta\eta(W)\eta(Y)\xi] \]

\[ -(f_1 - f_3)[\alpha g(Y, \phi W)\xi \]

\[ + \beta(g(Y, W)\xi) - \eta(W)\eta(Y)\xi)]. \hspace{1cm} (3.6)

Using (2.5) and (2.6), equation (3.6) reduces to

\[ (\nabla_W R)(\xi, Y)\xi = d(f_1 - f_3)(W)(\eta(Y)\xi - Y) + (f_1 - f_3)\nabla_W Y. \] \hspace{1cm} (3.7)

From (3.4) and (3.7), we have

\[ d(f_1 - f_3)(W)(Y - \eta(Y)\xi) - (f_1 - f_3)\nabla_W Y \]

\[ = (f_1 - f_3)[2A(W)\{\eta(Y)\xi - Y\}] \]

\[ + A(\xi)\{\eta(Y)W - \eta(W)Y\}] + A(Y)\{\eta(W)\xi - W\}] \]

\[ + (f_1 - f_3)A(\xi)\{g(Y, W)\xi - \eta(W)Y\} \]

\[ + (f_1 - f_3)\{\eta(Y)\eta(W)\rho - g(Y, W)\rho\}. \]  \hspace{1cm} (3.8)

Taking \( Y = \xi \) in (3.8), we obtain

\[ (f_1 - f_3)\nabla_W \xi = 0. \] \hspace{1cm} (3.9)

From (3.9), we get that \( M \) is co-symplectic provided \( f_1 \neq f_3 \). Thus, we have
Theorem 1. A trans-Sasakian generalized Sasakian-space-form which is \( \phi \)-pseudosymmetric, is cosymplectic provided \( f_1 \neq f_3 \).

Taking inner product of (3.3) with \( U \) and then taking contraction over \( X \) and \( U \), we get

\[
-(\nabla_W S)(Y, \xi) + g((\nabla_W R)(\xi, Y)\xi, \xi) = (f_1 - f_3)[2(2n + 1)A(W)\eta(Y) + (2n - 1)A(Y)\eta(W)] \\
-(f_1 - f_3)A(\xi)g(Y, W) + A(\xi)S(Y, W).
\]

Using (2.12), we have

\[
g((\nabla_W R)(\xi, Y)\xi, \xi) = 0.
\]

By definition of covariant derivative, we have

\[
(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi).
\]

Using (2.5) and (2.9) in (3.12), we get

\[
(\nabla_W S)(Y, \xi) = 2n[d(f_1 - f_3)(W)\eta(Y) - (f_1 - f_3)\{\alpha g(\phi W, Y) - \beta g(W, Y)\}] \\
+ \alpha S(Y, \phi W) - \beta g(Y, W).
\]

Plugging (3.11) and (3.13) in (3.10), we have

\[
-2n[d(f_1 - f_3)(W)\eta(Y) - (f_1 - f_3)\{\alpha g(\phi W, Y) - \beta g(W, Y)\}] \\
-\alpha S(Y, \phi W) + \beta g(Y, W) = (f_1 - f_3)[2(2n + 1)A(W)\eta(Y) + (2n - 1)A(Y)\eta(W)] \\
-(f_1 - f_3)A(\xi)g(Y, W) + A(\xi)S(Y, W).
\]

Taking \( Y = \xi \) in (3.14) and then using (2.1) and (2.9), we get

\[
d(f_1 - f_3)(W) = -\frac{(f_1 - f_3)}{2n}[2(2n + 1)A(W) + 2A(\xi)\eta(W)].
\]

Using (3.15) in (3.14), we have

\[
[\beta - A(\xi)]S(Y, W) - \alpha[S(Y, \phi W) - 2n(f_1 - f_3)g(Y, \phi W)] \\
= (f_1 - f_3)[2n\beta - A(\xi)]g(Y, W) + (2n - 1)(f_1 - f_3)[A(Y)\eta(W)] \\
-2A(\xi)\eta(Y)\eta(W).
\]

For \( W = \xi \), the equation (3.16) reduces to

\[
A(Y) = A(\xi)\eta(Y).
\]
By taking account of (3.17) in (3.16), one can get

\[ \beta - A(\xi)S(Y, W) = \alpha[S(Y, \phi W) - 2n(f_1 - f_3)g(Y, \phi W)] \\
+ (f_1 - f_3)[(2n\beta - A(\xi))g(Y, W) - (2n - 1)A(\xi)\eta(Y)\eta(W)]. \quad (3.18) \]

Changing \( W \) by \( \phi W \) in (3.18), and using (2.1) and (2.9), we obtain

\[ S(Y, \phi W) = \frac{\alpha}{\beta - A(\xi)} \{-S(Y, W) + 2n(f_1 - f_3)g(Y, W)\} \\
+ \frac{2n\beta - A(\xi)}{\beta - A(\xi)}g(Y, \phi W). \quad (3.19) \]

Using (3.19), (3.18) gives

\[ [\alpha^2 + (\beta - A(\xi)^2)S(Y, W) = [2n\alpha^2 + (2n\beta - A(\xi))(\beta - A(\xi))](f_1 - f_3)g(Y, W) \\
- (2n - 1)(f_1 - f_3)[\beta - A(\xi)]A(\xi)\eta(Y)\eta(W) \\
+ (2n - 1)(f_1 - f_3)\alpha A(\xi)g(Y, \phi W). \quad (3.20) \]

Thus, we can able to state the following:

**Theorem 2.** In a \( \phi \)-pseudosymmetric trans-Sasakian generalized Sasakian-space-form, the Ricci tensor \( S \) is of the form (3.20).

If we taking \( \alpha = 0 \) in (3.20), then

\[ S(Y, W) = \frac{(2n\beta - A(\xi))}{\beta - A(\xi)}(f_1 - f_3)g(Y, W) - \frac{(2n - 1)A(\xi)}{\beta - A(\xi)}\eta(Y)\eta(W). \quad (3.21) \]

That is, \( M^{2n+1}(f_1, f_2, f_3) \) is an \( \eta \)-Einstein manifold.Hence, we state.

**Corollary 1.** A \( \phi \)-pseudosymmetric generalized Sasakian-space-form

\[ M^{2n+1}(f_1, f_2, f_3) \quad (n > 1), \]

which is \( \beta \)- Kenmotsu is an \( \eta \)-Einstein manifold.

4. \( \phi \)-Pseudo Ricci Symmetric Generalized Sasakian-Space-Forms

**Definition 2.** A generalized Sasakian-space-form \( M^{2n+1}(f_1, f_2, f_3) \) is said to be a \( \phi \)-pseudo Ricci symmetric if the Ricci operator \( Q \) satisfies

\[ \phi^2((\nabla_W Q)(X)) = 2A(W)QX + A(X)QW + S(X, W)\rho, \quad (4.1) \]
for any vector field \(X\) and \(W\), where \(A\) is a non-zero 1-form. If, in particular, \(A = 0\), then (4.1) turns into the notion of \(\phi\)-Ricci symmetric generalized Sasakian-space-form.

Now, consider a generalized Sasakian-space-form \(M^{2n+1}(f_1, f_2, f_3)\) which is \(\phi\)-pseudo Ricci symmetric. Then by, virtue of (2.1), it follows from (4.1) that

\[-(\nabla_WQ)(X) + \eta((\nabla_WQ)(X))\xi = 2A(W)QX + A(X)QW + S(W, X)\rho.\]

Then

\[-g(\nabla_WQ(X), Y) + S(\nabla_WX, Y) + \eta(\nabla_WQ)(X)\eta(Y) = 2A(W)S(X, Y) + A(X)S(W, Y) + A(Y)S(X, W).\quad (4.2)\]

Putting \(Y = \xi\) in (4.2) and using (2.9), we have

\[-g(\nabla_WQ\xi, Y) + S(\nabla_W\xi, Y) + \eta(\nabla_WQ)(\xi)\eta(Y) = 2n(f_1 - f_3)[2A(W)\eta(Y) + A(Y)\eta(W)] + A(\xi)S(W, Y).\quad (4.3)\]

Using (2.5) and (2.9) in (4.3), we have

\[2n\{-(d\{f_1 - f_3\}(W)\eta(Y) + (f_1 - f_3)(\alpha g(\phi W, Y) - \beta g(W, Y))\}\}
\[= 2n(f_1 - f_3)[2A(W)\eta(Y) + A(Y)\eta(W)] + A(\xi)S(W, Y)
\[+ \alpha S(\phi W, Y) - \beta S(W, Y).\quad (4.4)\]

Taking \(Y = \xi\) in (4.4) and using (2.1) and (2.9), we obtain

\[d(f_1 - f_3)(W) = (f_1 - f_3)[(\beta - 2A(\xi))\eta(W) - 2A(W)].\quad (4.5)\]

Using (4.5) in (4.4), gives

\[(n - 1)(f_1 - f_3)A(Y)\eta(W) + A(\xi)S(W, Y)
\[= 2n(f_1 - f_3)[-((\beta - 2A(\xi))\eta(W)\eta(Y) + \alpha g(\phi W, Y) - \beta g(W, Y)]
\[+ \alpha S(\phi W, Y) + \beta S(W, Y).\quad (4.6)\]

Putting \(W = \xi\) in (4.6), we get

\[A(Y) = -((\beta - A(\xi))\eta(Y).\quad (4.7)\]

By taking account of (4.7) in (4.6), one can get

\[(\beta - A(\xi))S(W, Y) = 2n(f_1 - f_3)[\beta g(W, Y) - A(\xi)\eta(W)\eta(Y)]
\[+ \alpha[S(\phi W, Y) - 2n(f_1 - f_3)g(\phi W, Y)].\quad (4.8)\]
Changing $W$ by $\phi W$ in (4.8), and using (2.1) and (2.9), we have

$$S(Y, \phi W) = \frac{\alpha}{(\beta - A(\xi))}[-S(Y, W) + 2n(f_1 - f_3)g(Y, W)]$$

$$+ \frac{\beta}{(\beta - A(\xi))}2n(f_1 - f_3)g(Y, \phi W). \quad (4.9)$$

Using (4.9), (4.8) reduces to

$$[\alpha^2 + (\beta - A(\xi))^2]S(Y, W) = 2n(f_1 - f_3)[\alpha^2 + \beta(\beta - A(\xi))g(Y, W)$$

$$+ 2n(f_1 - f_3)[(\beta - A(\xi))A(\xi)]\eta(Y)\eta(W)$$

$$+ 2n(f_1 - f_3)\alpha A(\xi)g(Y, \phi W). \quad (4.10)$$

Thus, we state the following;

**Theorem 3.** In a $\phi$-pseudosymmetric trans-Sasakian generalized Sasakian-space-form, the Ricci tensor $S$ is of the form (4.11).

If we taking $\alpha = 0$ in (4.10), then

$$S(Y, W) = \frac{\beta}{(\beta - A(\xi))}2n(f_1 - f_3)g(Y, W)$$

$$- \frac{A(\xi)}{(\beta - A(\xi))}2n(f_1 - f_3)\eta(Y)\eta(W). \quad (4.11)$$

That is, $M^{2n+1}(f_1, f_2, f_3)$ is an $\eta$-Einstein manifold. Hence, we state

**Corollary 2.** A $\phi$-pseudu Ricci symmetric generalized Sasakian-space-form, which is $\beta$-Kenmotsu is an $\eta$-Einstein manifold.

References


