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# ON $(a, b, c)$-TRICHOTOMY AND $(d, b)$-DICHOTOMY FOR LINEAR DISCRETE-TIME SYSTEMS

Marioara Lăpădat
Department of Mathematics
West University of Timişoara
V. Pârvan Boulevard No. 4, 300223, Timişoara, ROMANIA

**Abstract:** The paper considers two general concepts of trichotomy and dichotomy for linear discrete-time systems in Banach spaces. It is shown that the $(a, b, c) - trichotomy$ of a discrete-time system is equivalent with the $(d, b) - dichotomy$ of two associated systems.

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## 1. Introduction

Among the asymptotic behaviors of linear discrete -time systems, an important role is played by the dichotomy and trichotomy properties. These concepts were studied in an extensive manner from the point of view of both the exponential and polynomial behavior (see [1], [4]-[7], [10], [12], [13]-[15], [17]).

As natural generalizations of both of the above behaviors (in the uniform and nonuniform case) are successfully modeled by $(h, k) - dichotomy$ and respectively $(h, k) - trichotomy$, where a large number of papers containing many interesting results were published from which we mention [2], [3], [8], [9], [11], [16], [18].

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In this paper we introduce two general concepts of $(h,k)-$ *dichotomy* and $(a,b,c)-$ *trichotomy* for dynamical systems defined by a sequence of bounded linear operator in a Banach space.

We consider simultaneously the general cases of *nonautonomous, noninvertible and nonuniform dynamics with arbitrary growth rates.*

These concepts of $(h,k)-$ *dichotomy* respectively $(a,b,c)-$ *trichotomy* can be considered as some general types of uniform (nonuniform) dichotomic respectively uniform (nonuniform) tricho-tomies. As important particular cases of these properties are some concepts of uniform (nouniform) dichotomy respectively trichotomy, as uniform (nonuniform) exponential and uniform (nonuniform) polynomial dichotomy respectively trichotomy concepts.

In the present paper, our main objective is to prove the equivalence between $(a,b,c)-$ *trichotomy* of a discrete-time linear system and $(d,b)-$ *dichotomy* of two associated systems.

This result may be used in the robustness theory of dichotomies and trichotomies for difference equations. In this sense, we remark that in the particular case of linear discrete-time systems defined by a sequence of invertible linear operators a robustness was obtained by L. Barreira and C. Valls in [7] for some stronger concepts of nonuniform exponential dichotomy and nonuniform exponential trichotomy.

## 2. $(h,k)$-Dichotomy

We recall that an operator valued function $P:\mathbb{N}\to B(X)$, $P(n)=P_n$ is called a **sequence of projections**, if

$$P_n^2 = P_n, \text{ for every } n\in\mathbb{N}.$$

**Definition 1.** If $P_1,P_2:\mathbb{N}\to B(X), P_1(n)=P_n^1, P_2(n)=P_n^2$ are two sequences of projections on $X$ then we say that $\mathcal{P}=\{(P_n^1),(P_n^2)\}$ is *compatible* with the linear discrete-time system

$$x_{n+1}=A_n x_n,\ n\in\mathbb{N}, \tag{$\mathcal{A}$}$$

where $A:\mathbb{N}\to B(X), A(n)=A_n$, if:

$c_1$) $P_n^1+P_n^2=I$, for every $n\in\mathbb{N}$;

$c_2$) $A_nP_n^j=P_{n+1}^jA_n$, for all $n\in\mathbb{N}$ and all $j\in\{1,2\}$.

Let $h.k : \mathbb{N} \to \mathbb{R}_+^*$, $h(n) = h_n$, $k(n) = k_n$ be two nondecreasing sequences of positive real numbers and let $\mathcal{P} = \{(P_n^1), (P_n^2)\}$ be a family of two sequences of projections on $X$ which is compatible with $(\mathcal{A})$.

**Definition 2.** We say that *the pair* $(\mathcal{P}, \mathcal{A})$ *is* $(h,k)$*-dichotomic*, if there exists $N \geq 1$ such that:

$(d_1)$ $\quad h_m \|A_m^n P_n^1 x\| \leq N h_n k_n \|P_n^1 x\|$;

$(d_2)$ $\quad h_m \|P_n^2 x\| \leq N h_n k_m \|A_m^n P_n^2 x\|$;

for all $(m,n,x) \in T = \Delta \times X$, where

$$\Delta = \{(m,n) \in \mathbb{N}^2 : m \geq n\}.$$

**Remark 3.** In the classical dichotomy theory it is studied the case when *$h$ is a growth rate*, i.e. $h$ is nondecreasing and $\lim\limits_{n \to +\infty} h(n) = +\infty$.

Thus, as important particular cases of $(h,k) - $ *dichotomy* are for:

- *(i)* $h_n = e^{n\alpha}$ with $\alpha > 0$ when the $(h,k) - $ *dichotomy* is a concept of *nonuniform exponential dichotomy*;
- *(ii)* $h_n = e^{n\alpha}$ with $\alpha > 0$ and $(k_n)$ is a constant sequence, we obtain a concept of *uniform exponential dichotomy*;
- *(iii)* $h_n = (n+1)^\alpha$ with $\alpha > 0$ it obtains a concept of *nonuniform polynomial dichotomy*;
- *(iv)* $h_n = (n+1)^\alpha$ with $\alpha > 0$ and $(k_n)$ is a constant sequence it results a concept of *uniform polynomial dichotomy*.

**Example 4.** On $X = \mathbb{R}^3$ with the norm

$$\|(x_1, x_2, x_3)\| = |x_1| + |x_2| + |x_3|.$$

We consider the projections sequences

$$P_n^1(x_1, x_2, x_3) = (x_1, 0, 0),$$

$$P_n^2(x_1, x_2, x_3) = (0, x_2, x_3).$$

Let $h, k : \mathbb{N} \to [1, +\infty)$ be two nondecreasing sequences of positive real numbers, with $h(n) = h_n$ and $k(n) = k_n$ for every $n \in \mathbb{N}$.

Let $(\mathcal{A})$ be the linear discrete-time system defined by

$$A_n(x_1,x_2,x_3)=\left(\frac{h_nkn}{h_{n+1}k_{n+1}}x_1,\frac{h^3_{n+1}k_n}{h^3_nk_{n+1}}x_2,\frac{h_{n+1}k_n}{h_nk_{n+1}}x_3\right).$$

Then

$$A^n_m(x_1,x_2,x_3)=\left(\frac{h_nkn}{h_mk_m}x_1,\frac{h^3_mk_n}{h^3_nk_m}x_2,\frac{h_mk_n}{h_nk_m}x_3\right),$$

for all $(m,n)\in\Delta$.

We have that

$$h_m\|A^n_mP^1_nx\|=\frac{h_nk_n}{k_m}\|P^1_nx\|\leq h_nk_n\|P^1_nx\|,$$

and

$$h_m\|P^2_nx\|=h_m(|x_2+|x_3|)\leq h_mk_n\left(\frac{h^2_m}{h^2_n}|x_2|+|x_3|\right)$$
$$=h_nk_m\left(\frac{h^3_mk_n}{h^3_nk_m}|x_2|+\frac{h_mk_n}{h_nk_m}|x_3|\right)=h_nk_m\|A^n_mP^3_nx\|,$$

for all $(m,n)\in\Delta$ and all $x=(x_1,x_2,x_3)\in X$.

Finally, we obtain that the pair $(\mathcal{A},\mathcal{P})$ with $\mathcal{P}=\{(P^1_n),(P^2_n)\}$ is $(h,k)-$ *dichotomic*.

**Remark 5.** The previous example shows that for $X=\mathbb{R}^3$ we have that for every two nondecreasing sequences $h.k:\mathbb{N}\to\mathbb{R}^*_+$ it is a linear discrete time-system $(\mathcal{A})$ and a family of projections $\mathcal{P}=\{(P^1_n),(P^2_n)\}$ compatible with $(\mathcal{A})$ such that $(\mathcal{A},\mathcal{P})$ is $(h,k)-$ *dichotomic*.

## 3. $(a,b,c)$-Trichotomy

Let $P_1,P_2,P_3:\mathbb{N}\to B(X)$ be three sequences of projections on $X$ with

$$P_j(n)=P^j_n,\ for\ all\ n\in\mathbb{N}\ and\ j\in\{1,2,3\}.$$

**Definition 6.** We say that $\mathcal{P}=\{P_1,P_2,P_3\}$ is *compatible* with the system $(\mathcal{A})$ if:

$c_1)$ $P^1_n+P^2_n+P^3_n=I$, for every $n\in\mathbb{N}$;

$c_2)$ $A_nP^j_n=P^j_{n+1}A_n$, for all $n\in\mathbb{N}$ and all $j\in\{1,2,3\}$.

Let $a, b, c : \mathbb{N} \to [1, +\infty)$ be three nondecreasing sequences of positive real numbers and let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a family of three sequences of projections on $X$ which is compatible with the linear discrete-time system $(\mathcal{A})$.

**Definition 7.** We say that the pair $(\mathcal{A}, \mathcal{P})$ is $(a, b, c)-$*trichotomic* if there is $N \geq 1$ such that:

$t_1)$ $a_m \|A_m^n P_n^1 x\| \leq N a_n b_n \|P_n^1 x\|$;

$t_2)$ $a_m \|P_n^2 x\| \leq N a_n b_m \|A_m^n P_n^2 x\|$;

$t_3)$ $c_n \|A_m^n P_n^3 x\| \leq N b_n c_m \|P_n^3 x\|$;

$t_4)$ $c_n \|P_n^3 x\| \leq N b_m c_m \|A_m^n P_n^3 x\|$;

for all $(m, n, x) \in T$.

**Example 8.** Let $a, b, c : \mathbb{N} \to [1, +\infty)$ be three nonndecreasing sequences of positive real numbers with

$$a(n) = a_n, b(n) = b_n \textit{ and } c(n) = c_n, \text{ for every } n \in \mathbb{N}.$$

On $X = \mathbb{R}^3$ with the norm

$$\|(x_1, x_2, x_3)\| = |x_1| + |x_2| + |x_3|,$$

we consider the projections seguences

$$P_n^1(x_1, x_2, x_3) = (x_1, 0, 0), \quad P_n^2(x_1, x_2, x_3) = (0, x_2, 0),$$
$$P_n^3(x_1, x_2, x_3) = (0, 0, x_3) \quad \text{for all } n \in \mathbb{N}\text{and } \textit{all } x = (x_1, x_2, x_3) \in X.$$

Consider the linear discrete-time system $(\mathcal{A})$ defined by the sequence $(A_n)$ defined by

$$A_n = \frac{a_n b_n}{a_{n+1} b_{n+1}} P_n^1 + \frac{a_{n+1} b_n}{a_n b_{n+1}} P_n^2 + \frac{c_{n+1} b_n}{c_n b_{n+1}} P_n^3, \text{ for all } n \in \mathbb{N}.$$

Then $\mathcal{P} = \{P_1, P_2, P_3\}$ is compatible with $(\mathcal{A})$ and

$$A_m^n = \frac{a_n b_n}{a_m b_m} P_n^1 + \frac{a_m b_n}{a_n b_m} P_n^2 + \frac{c_m b_n}{c_n b_m} P_n^3, \text{ for all } (m, n) \in \Delta.$$

We observe that

$$a_m \|A_m^n P_n^1 x\| = \frac{a_n b_n}{b_m} \|P_n^1 x\| \leq a_n b_n \|P_n^1 x\|,$$

$$a_m\|P_n^2x\| = \frac{a_n b_m}{b_n}\|A_m^n P_n^2 x\| \leq a_n b_m \|A_m^n P_n^2 x\|,$$
$$c_n\|A_m^n P_n^3 x\| = \frac{c_m b_n}{b_m}\|P_n^3 x\| \leq b_n c_m \|P_n^3 x\|,$$
$$c_n\|P_n^3 x\| = \frac{c_n^2 b_m}{c_m b_n}\|A_m^n P_n^3 x\| \leq b_m c_m \|A_m^n P_n^3 x\|,$$

for all $(m, n, x) \in T = \Delta \times X$.

Thus the pair $(\mathcal{A}, \mathcal{P})$ with $\mathcal{P} = \{P_1, P_2, P_3\}$ is $(a, b, c) -$ *trichotomic.*

**Remark 9.** The previous example shows that in the case $X = \mathbb{R}^3$ for every $a, b, c : \mathbb{N} \to [1, +\infty)$ there are a linear discrete-time system $(\mathcal{A})$ and a projections family $\mathcal{P} = \{P_1, P_2\}$ such that $(\mathcal{A}, \mathcal{P})$ is $(a, b, c) -$ *trichotomic.*

**Remark 10.** In the clasical trichotomy theory it is studied the case when the sequence $(a_n)$ and $(c_n)$ are growth rates.

Thus as important particular cases of $(a, b, c) -$ *trichotomy* we obtain

$(i)$ for $a_n = e^{n\alpha}$ and $c_n = e^{n\beta}$ with $\alpha > 0$ and $\beta \geq 0$ a concept of *nonuniform exponential trichotomy*;

$(ii)$ for $a_n = e^{n\alpha}$ with $\alpha > 0$ and $(b_n)$ constant and $c_n = e^{n\beta}$ with $\beta \geq 0$ a concept of *uniform exponential trichotomy*;

$(iii)$ for $a_n = (n+1)^\alpha$ with $\alpha > 0$ and $c_n = (n+1)^\beta$ with $\beta \geq 0$ a concept of *nonuniform polynomial trichotomy*;

$(iv)$ for $a_n = (n+1)^\alpha$ with $\alpha > 0$ and $(b_n)$ is a constant and $c_n = (n+1)^\beta$ with $\beta \geq 0$ a concept of *uniform polynomial trichotomy.*

Another important particular case of $(a, b, c) -$ *trichotomy* for a pair $(\mathcal{A}, \mathcal{P})$ is the case when $\mathcal{P}$ satisfies the following:

**Definition 11.** We say that a family of three sequences of projections $\mathcal{P} = \{P_1, P_2, P_3\}$ is *orthogonal* if

$(o_1)$ $P_i P_j = O$ for all $i, j \in \{1, 2, 3\}$ with $i \neq j$;

$(o_1)$ $\|P_i^n x + P_j^n x\|^2 = \|P_i^n x\|^2 + \|P_j^n x\|^2$

for all $n \in \mathbb{N}$ and all $i, j \in \{1, 2, 3\}$ with $i \neq j$.

A property of orthogonal projections families is given by

**Proposition 12.** *If $\mathcal{P} = \{P_1, P_2, P_3\}$ is orthogonal and compatible with linear discrete-time systems $(\mathcal{A})$ then $\mathcal{Q} = \{Q_1, Q_2\}$ respectively $\mathcal{R} = \{R_1, R_2\}$ defined by*

$$Q_1 = P_1,\ Q_2 = P_2 + P_3 \text{ respectively } R_1 = P_1 + P_3,\ R_2 = P_2$$

*satisfy the following properties:*

$(i)$ *$\mathcal{Q}$ and $\mathcal{R}$ are compatible with $(\mathcal{A})$;*

$(ii)$ $Q_n^1 R_n^2 = R_n^2 Q_n^1 = O$;

$(iii)$ $Q_n^2 R_n^1 = R_n^1 Q_n^2 = Q_n^2 - R_n^2 = R_n^1 - Q_n^1$;

$(iv)$ $\|Q_n^1 x + R_n^2 x\|^2 = \|Q_n^1 x\|^2 + \|R_n^2 x\|^2$;

$(v)$ $\|Q_n^2 x - R_n^2 x\|^2 = \|Q_n^2 x\|^2 - \|R_n^2 x\|^2$;

$(vi)$ $\|R_n^1 x - Q_n^1 x\|^2 = \|R_n^1 x\|^2 - \|Q_n^1 x\|^2$;

*for all $(n, x) \in \mathbb{N} \times X$.*

*Proof.* $(i)$ It is an immediate consequence of Definition 1 and 6.

$(ii)$ $Q_n^1 R_n^2 = P_n^1 P_n^2 = O = P_n^2 P_n^1 = R_n^2 Q_n^1$ for all $n \in \mathbb{N}$.

$(iii)$ We have that:

$$Q_n^2 R_n^1 = (P_n^2 + P_n^3)(P_n^1 + P_n^3) = P_n^3 = Q_n^2 - R_n^2,$$

$$R_n^1 Q_n^2 = (P_n^1 + P_n^3)(P_n^2 + P_n^3) = P_n^3 = R_n^1 - Q_n^1,$$

for all $n \in \mathbb{N}$.

$(iv)$ $\|Q_n^1 x + R_n^2 x\|^2 = \|P_n^1 x + P_n^2 x\|^2 = \|P_n^1 x\|^2 + \|P_n^2 x\|^2 = \|Q_n^1 x\|^2 + \|R_n^2 x\|^2$, for all $n \in \mathbb{N}$.

$(v)$ $\|Q_n^2 x - R_n^2 x\|^2 = \|P_n^3 x\|^2 = \|P_n^2 x\|^2 + \|P_n^3 x\|^2 - \|P_n^2 x\|^2 = \|Q_n^2 x\|^2 - \|R_n^2 x\|^2$, for all $(n, x) \in \mathbb{N} \times X$.

$(vi)$ $\|R_n^1 x - Q_n^2 x\|^2 = \|P_n^1 x\|^2 - \|P_n^2 x\|^2 = \|P_n^1 x\|^2 + \|P_n^3 x\|^2 - \|P_n^2 x\|^2 - \|P_n^3 x\|^2 = \|R_n^1 x\|^2 - \|Q_n^2 x\|^2$, for all $(n, x) \in \mathbb{N} \times X$. □

The previous results yields to the following

**Definition 13.** Let $\mathcal{Q} = \{Q_1, Q_2\}$ and $\mathcal{R} = \{R_1, R_2\}$ be two projections families which are compatible with the system $(\mathcal{A})$. We say that $(\mathcal{Q})$ and $(\mathcal{R})$ *are orthogonal* if it satisfy the properties $(ii) - (vi)$ from Proposition 12 .

A converse of Proposition 12 is given by

**Proposition 14.** *Let $\mathcal{Q} = \{Q_1, Q_2\}$ and $\mathcal{R} = \{R_1, R_2\}$ be two orthogonal projections families which are compatible with the system $(\mathcal{A})$.*
*Then $\mathcal{P} = \{P_1, P_2, P_3\}$ where*

$$P_1 = Q_1,\ P_2 = R_2,\ P_3 = R_1 - Q_1$$

*is orthogonal and compatible with $(\mathcal{A})$.*

*Proof.* If $\mathcal{Q}$ and $\mathcal{R}$ are compatible with $(\mathcal{A})$ then by Definition 1 and Proposition 12 we obtain

$c_1$) $P_1 + P_2 + P_3 = Q_1 + R_2 + R_1 - Q_1 = R_1 + R_2 = I$;

$c_2$) $A_n P_n^1 = A_n Q_n^1 = Q_{n+1}^1 A_n = P_{n+1}^1 A_n$;
$A_n P_n^2 = A_n R_n^2 = R_{n+1}^2 A_n = P_{n+1}^2 A_n$;
$A_n P_n^3 = A_n(R_n^1 - Q_n^1) = R_{n+1}^1 A_n - Q_{n+1}^1 A_n = P_{n+1}^3 A_n$

for all $n \in \mathbb{N}$.

Thus $\mathcal{P}$ is compatible with $(\mathcal{A})$. It remains to prove that $\mathcal{P}$ is orthogonal. Using the properties $(ii)$ and $(iii)$ we obtain

$(o_1)$ $P_1P_2 = Q_1R_2 = P_2P_1 = R_2Q_1 = O$;
$P_1P_3 = Q_1Q_2R_1 = O = R_1Q_2Q_1 = P_3P_1$;
$P_2P_3 = R_2R_1Q_2 = O = Q_2R_1R_2 = P_3P_2$.

$(o_2)$ The condition $(iv), (v)$ and $(vi)$ from Proposition 12 imply that

$$\begin{aligned}\|P_n^1x + P_n^2x\|^2 =& \|Q_n^1x + R_n^2x\|^2 = \|Q_n^1x\|^2 + \|R_n^2x\|^2\\ =& \|P_n^1x\|^2 + \|P_n^2x\|^2 = \|P_n^2x\|^2 + \|P_n^3x\|^2\\ =& \|R_n^2x\|^2 + \|Q_n^2x - R_n^2x\|^2\\ =& \|R_n^2x\|^2 + \|Q_n^2x\|^2 - \|R_n^2x\|^2 = \|Q_n^2\|^2\\ =& \|P_n^2x + P_n^3x\|^2,\\ \|P_n^1x + P_n^3x\|^2 =& \|R_n^1x\|^2 = \|Q_n^1x\|^2 + \|R_n^1x\|^2 - \|Q_n^1x\|^2\\ =& \|P_n^1x\|^2 + \|P_n^3x\|^2,\end{aligned}$$

for all $(n, x) \in \mathbb{N} \times X$.

Finally, we obtain that $\mathcal{Q}$ and $\mathcal{R}$ are orthogonal. □

## 4. The Main Results

Let $a,b,c:\mathbb{N}\to[1,\infty)$ be three nondecreasing sequences of positive real numbers and let $\mathcal{P}=\{P_n^1,P_n^2,P_n^3\}$ be a set of projection families which is compatible with the linear discrete-time system $(\mathcal{A})$.

Let $h:\mathbb{N}\to\mathbb{R}^*=(0,\infty)$ be the sequence defined by

$$h(n)=h_n=\left(\frac{a_n}{c_n}\right)^{\frac{1}{2}}.$$

We associate to the linear discrete-time system $(\mathcal{A})$ the systems $(\mathcal{B})$ and $(\mathcal{C})$ defined by

$$y_{n+1}=B_ny_n,\ n\in\mathbb{N} \tag{$\mathcal{B}$}$$

and respectively

$$z_{n+1}=C_nz_n,\ n\in\mathbb{N}, \tag{$\mathcal{C}$}$$

where

$$B_n=\left(\frac{a_{n+1}c_{n+1}}{a_nc_n}\right)^{\frac{1}{2}}A_n \textit{ and } C_n=\left(\frac{a_nc_n}{a_{n+1}c_{n+1}}\right)^{\frac{1}{2}}A_n.$$

Then every solution of $(\mathcal{B})$ respectively $(\mathcal{C})$ is given by

$$y_m=B_m^ny_n \text{ respectively } z_m=C_m^nz_n,$$

where

$$B_m^n=b_m^nA_m^n \text{ respectively } C_m^n=c_m^nA_m^n,$$

with

$$b_m^n=\left(\frac{a_mc_m}{a_nc_n}\right)^{\frac{1}{2}} \text{ respectively } c_m^n=\frac{1}{b_m^n},$$

for every $(m,n)\in\Delta$.

**Theorem 15.** *Let $\mathcal{P}=\{P_n^1,P_n^2,P_n^3\}$ be a set of projections families which is orthogonal and compatible with $(\mathcal{A})$ and let $\mathcal{Q}=\{Q_n^1,Q_n^2\}$, $\mathcal{R}=\{R_n^1,R_n^2\}$ be the projections families defined in Proposition 12 .*

*If the pair $(\mathcal{A},\mathcal{P})$ is $(a,b,c)-$trichotomic then $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ are $(h,b)-$ dichotomic.*

*Proof.* $(i)$ Firstly, we prove that $(\mathcal{B},\mathcal{Q})$ is $(h,b)-$ *dichotomic.*

$(d_1^{'})$ From the definitions of $(h_n)$; $(B_m^n)$; $(Q_n^1)$ and the property $(t_1)$ we have that

$$h_m\|B_m^nQ_n^1x\| = \frac{a_m}{(a_nc_n)^{\frac{1}{2}}}\|A_m^nP_n^1x\| \leq N\frac{a_nb_n}{(a_nc_n)^{\frac{1}{2}}}\|P_n^1x\|$$
$$= Nb_n\left(\frac{a_n}{c_n}\right)^{\frac{1}{2}}\|P_n^1x\| = Nh_nb_n\|P_n^1x\|.$$

for all $(m, n, x) \in T$.

$(d_2^{'})$ Similarly, using the definitions of $(h_n)$; $(C_m^n)$; $(Q_n^2)$, the orthogonality of $\mathcal{P}$ and the conditions $(t_2), (t_3)$ it results that

$$\begin{aligned}
h_m^2\|Q_n^2x\|^2 =& h_m^2(\|P_n^2x\|^2 + \|P_n^3x\|^2)\\
\leq& N^2b_m^2\left(\frac{a_n^2}{a_mc_m}\|A_m^nP_n^2x\|^2 + \frac{a_mc_m}{c_n^2}\|A_m^nP_n^3x\|^2\right)\\
=& N_2b_m^2h_n^2\left(\frac{a_n^2c_n^2}{a_m^2c_m^2}\|B_m^nP_n^2x\|^2 + \|B_m^nP_n^3x\|^2\right)\\
\leq& N^2b_m^2h_n^2(\|P_m^2B_m^nx\|^2 + \|B_m^nP_m^3x\|^2)\\
=& N_2b_m^2h_n^2\|Q_m^2B_m^nx\|^2\\
=& N_2b_m^2h_n^2\|B_m^nQ_n^2x\|^2,
\end{aligned}$$

and then

$$h_m\|Q_n^2x\| \leq Nb_mh_n\|B_m^nQ_n^2x\|,$$

for all $(m, n, x) \in T$.

$(ii)$ It remais to prove that $(\mathcal{C}, \mathcal{R})$ is $(h, b) - dichotomic$.

$(d_1^{"})$ Similarly, using the definitions of $(h_n)$; $(C_m^n)$; $(R_n^1)$, the orthogonality of $\mathcal{P}$ and the conditions $(t_1), (t_3)$ it results that

$$\begin{aligned}
h_m^2\|C_m^nR_n^1x\|^2 =& \frac{a_nc_n}{c_m^2}\|A_m^n(P_n^1x + P_n^3x)\|^2\\
=& \frac{a_mc_n}{c_m^2}\|P_m^1A_m^nx + P_m^3A_m^nx\|^2\\
=& \frac{a_mc_n}{c_m^2}(\|P_m^1A_m^nx\|^2 + \|P_m^3A_m^nx\|^2)\\
=& \frac{a_mc_n}{c_m^2}(\|A_m^nP_n^1x\|^2 + \|A_m^nP_n^3x\|^2)\\
\leq& N^2h_n^2b_n^2\left(\frac{a_n^2}{a_m^2}\frac{c_n^2}{c_m^2}\|P_n^1x\|^2 + \|P_n^3x\|^2\right)\\
\leq& N^2b_n^2h_n^2\left(\|P_n^1x\|^2 + \|P_n^3x\|^2\right)
\end{aligned}$$

$$=N^2 b_n^2 h_n^2 \|R_n^1 x\|^2,$$

and then

$$h_m \|C_m^n R_n^1 x\| \leq N b_n h_n \|R_n^1 x\|,$$

for all $(m,n,x) \in T$.

$(d_2^{"})$ From $(t_2)$ and the definitions of $(h_n)$, $(R_n^2)$ and $(C_m^n)$ we obtain

$$h_m \|R_n^2 x\| = h_m \|P_n^2 x\| \leq N \frac{h_m a_n b_m}{a_m} \|A_m^n P_n^2 x\| = N b_m h_n \|C_m^n R_n^2 x\|,$$

for all $(m,n,x) \in T$. □

Now, we prove as a converse if Theorem 15 the following

**Theorem 16.** *Let $\mathcal{Q} = \{Q_n^1, Q_n^2\}$ and $\mathcal{R} = \{R_n^1, R_n^2\}$ be two orthogonal projections families which are compatible with the system $(\mathcal{A})$.*

*If the pairs $(\mathcal{B}, \mathcal{Q})$ and $(\mathcal{C}, \mathcal{R})$ are $(h,b)-$dichotomic then $(\mathcal{A}, \mathcal{P})$ is $(a,b,c)-$trichotomic, where $(\mathcal{P})$ is defined in Proposition 14.*

*Proof.* Let $\mathcal{P} = \{P_n^1, P_n^2, P_n^3\}$ be family of projections defined in Proposition 14.

$(t_1)$ Using the definitions of $(P_n^1)$, $(h_n)$, $(B_m^n)$ and the property $(d_1^{'})$ we have that

$$a_m \|A_m^n P_n^1 x\| = a_m \left(\frac{a_n c_n}{a_m c_m}\right)^{\frac{1}{2}} \|C_m^n Q_n^1 x\| \leq N \left(\frac{a_n c_n}{a_m c_m}\right)^{\frac{1}{2}} \frac{h_n b_n}{h_m} \|Q_n^1 x\|$$
$$= N a_n b_n \|P_n^1 x\|,$$

for all $(m,n,x) \in T$.

$(t_2)$ Similary, from $(d_2^{"})$ we obtain

$$a_m \|P_n^2 x\| = a_m \|R_n^2 x\| \leq N \frac{a_m b_m h_n}{h_m} \|C_m^n R_n^2 x\| = N a_n b_m \|A_m^n P_n^2 x\|,$$

for all $(m,n,x) \in T$.

$(t_3)$ From $P_3 = R_1 Q_2$ and $(d_1^{"})$ it results that

$$c_n \|A_m^n P_n^3 x\| = c_n \left(\frac{a_m c_m}{a_n c_n}\right)^{\frac{1}{2}} \|C_m^n R_n^1 Q_n^2 x\| \leq N b_n \frac{c_n h_n}{h_m} \left(\frac{a_m c_m}{a_n c_n}\right)^{\frac{1}{2}} \|R_n^1 Q_n^2 x\|$$
$$= N b_n c_m \|P_n^3 x\|,$$

for all $(m,n,x) \in T$.

$(t_4)$ Similarly, from $P_3 = Q_2R_1$ and $(d_2")$ we have

$$c_n\|P_n^3 x\| = c_n\|Q_n^2 R_n^1 x\| \leq N b_m c_n \frac{h_n}{h_m}\|C_m^n Q_n^2 R_n^1 x\| = N b_m c_m \|A_m^n P_n^3 x\|$$

for all $(m,n,x) \in T$.

Finally, we obtain that $(\mathcal{A},\mathcal{P})$ is $(a,b,c)-$ *trichotomic.* □

The main result of this paper is

**Theorem 17.** *Let $a,b,c,h : \mathbb{N} \to \mathbb{R}^*$ be four sequences of positive real number and let $\mathcal{P} = \{P_n^1, P_n^2, P_n^3\}$, $\mathcal{Q} = \{Q_n^1, Q_n^2\}$, $\mathcal{R} = \{R_n^1, R_n^2\}$ be families of projections considered in Theorem 15.*

*Then the pair $(\mathcal{A},\mathcal{P})$ is $(a,b,c)-$trichotomic if and only if $(\mathcal{B},\mathcal{Q})$ and $(\mathcal{C},\mathcal{R})$ are $(h,b)-$ dichotomic.*

*Proof.* It results from Theorem 15 and 16 . □

**Remark 18.** The previous result is important in the case when $a, c$ and $h$ are growth rates. This property is verified in the following two particular cases:

$(i)$ $a_n = e^{n\alpha}$, $c_n = e^{n\beta}$ with $\alpha > \beta$;

$(ii)$ $a_n = n+1^{\alpha}$, $c_n = n+1^{\beta}$ with $\alpha > \beta$.

Thus Theorem 17 in these particular cases gives characterizations of exponential respectively polynomial trichotomies of the system $(\mathcal{A})$ in terms of exponential respectively polynomial dichotomies of the associated systems $(\mathcal{B})$ and $(\mathcal{C})$.

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