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CERTAIN CLASSES OF MEROMORPHIC MULTIVALENT FUNCTIONS WITH RESPECT TO (j, k)-SYMMETRIC POINTS

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Abstract: The author introduces new class of $p-$valent meromorphic multivalent analytic function with respect (j, k)-symmetric points. Integral representation and inclusion relations for functions in these classes are obtained.

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1. Introduction

Let $\mathcal{M}_p$ denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p}, \quad (p \geq 1), \tag{1.1}$$

which are analytic in the punctured open unit disk

$$\mathcal{U}^{\star} = \{z : z \in \mathbb{C} \,\text{and}\, 0 <| z |< 1\} = \mathcal{U} \setminus \{0\}.$$

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For the functions $f(z)$ of the form (1.1) and

$$g(z) = z^{-p} + \sum_{n=1}^{\infty} b_n z^{n-p},$$

the Hadamard product (or convloution) of f and g is defined by $(f * g)(z) = z^{-p} + \sum_{n=1}^{\infty} a_n\, b_n\, z^{n-p}$.

For complex parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ ($\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- = 0, -1, -2, \ldots; j = 1, \ldots, s$), we define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by

$${}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ \ldots\ (\alpha_q)_n}{(\beta_1)_n \ \ldots\ (\beta_s)_n} \frac{z^n}{n!}$$

$$(q \le s+1;\ q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ z \in \mathcal{U}),$$

where $(x)_k$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$\begin{aligned}(x)_k &= \frac{\Gamma(x+k)}{\Gamma(x)} \\ &= \begin{cases} 1 & if\, k = 0 \\ x(x+1)(x+2)\ \ldots\ (x+k-1) & if\, k \in \mathbb{N} = \{1, 2, , \ldots\}. \end{cases}\end{aligned}$$

Corresponding to a function $\mathcal{G}^p_{q,\,s}(\alpha_1, \beta_1; z)$ defined by

$$\mathcal{G}^p_{q,\,s}(\alpha_1, \beta_1; z) := z^{-p}\, {}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z), \tag{1.2}$$

we consider the operator $H^p(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s)f : \mathcal{U} \longrightarrow \mathcal{U}$ which is defined by means of Hadamard product (or convolution):

$$H^p(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s)f(z) = \mathcal{G}^p_{q,\,s}(\alpha_1, \beta_1; z) * f(z). \tag{1.3}$$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$\begin{aligned}&H^p(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s)f(z) \\ &\qquad = z^{-p} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \ \ldots\ (\alpha_q)_n}{(\beta_1)_n \ \ldots\ (\beta_s)_n} \frac{a_n z^{n-p}}{n!}.\end{aligned} \tag{1.4}$$

For convenience, we write

$$H^p(\alpha_1) = H^p(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s).$$

It can be easily seen from (1.3) that

$$z(H^p(\alpha_1)f(z))' = \alpha_1 H^p(\alpha_1+1)f(z) - (\alpha_1+p)H^p(\alpha_1)f(z). \tag{1.5}$$

The linear operator $H^p(\alpha_1)f(z)$ was introduced and investigated by Liu and Srivastava [3].

Let k be an arbitrarily fixed integer not smaller than 2, then the set $\mathcal{G}_k$ of all roots of k-th degree of unity has the form

$$\mathcal{G}_k = \left\{\varepsilon^0, \varepsilon^1, \ldots, \varepsilon^{k-1}\right\} \tag{1.6}$$

where $\varepsilon = \exp(2\pi i/k)$.

A function $f \in \mathcal{M}_p$ is said to be (j, k)-symmetrical if for each $z \in \mathcal{U}$

$$f(\varepsilon z) = \varepsilon^{-pj} f(z), \tag{1.7}$$

$$(k = 1, 2, \ldots; j = 0, 1, 2, \ldots (k-1)).$$

The family of (j, k)-symmetrical functions will be denoted by $\mathcal{F}_k^j$. We observe that $\mathcal{F}_2^1$, $\mathcal{F}_2^0$ and $\mathcal{F}_k^1$ are well-known families of odd functions, even functions and k-symmetrical functions respectively.

Also let $f_{j,k}(z)$ be defined by the following equality

$$f_{j,k}(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \frac{f(\varepsilon^\nu z)}{\varepsilon^{-\nu pj}}, \tag{1.8}$$

$$(f \in \mathcal{M}_p;\ k = 1, 2, \ldots; j = 0, 1, 2, \ldots (k-1)).$$

It is obvious that $f_{j,k}(z)$ is linear from $\mathcal{U}$ into $\mathcal{U}$. The notion of (j, k)-symmetric functions was introduced and studied by P. Liczberski and J. Połubi´-nski in [2].

If ν is an integer, then the following identities follow directly from (1.8):

$$\begin{aligned} f_{j,k}(\varepsilon^\nu z) &= \varepsilon^{-\nu pj} f_{j,k}(z), \\ f'_{j,k}(\varepsilon^\nu z) &= \varepsilon^{-\nu pj-\nu} f'_{j,k}(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{\nu+\nu pj} f'(\varepsilon^\nu z), \\ f''_{j,k}(\varepsilon^\nu z) &= \varepsilon^{-\nu pj-2\nu} f''_{j,k}(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{2\nu+\nu pj} f''(\varepsilon^\nu z). \end{aligned} \tag{1.9}$$

Throughout this paper, we assume that p, $k \in \mathbb{N}$, $\varepsilon = \exp(2\pi i/k)$ and

$$f^{p}_{j,k}(\alpha_1;\, z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{\nu j p}\left(H^p(\alpha_1) f(\varepsilon^{\nu} z)\right) = z^{-p} + \cdots, \tag{1.10}$$

$$(f \in \mathcal{M}_p;\ k = 1, 2, \ldots;\ j = 0, 1, 2, \ldots (k-1)).$$

Clearly, for $j = k = 1$, we have $f^{p}_{j,k}(\alpha_1;\, z) = H^p(\alpha_1) f(z)$.

Motivated by the concept introduced by K. Sakaguchi in [6], recently several subclasses of analytic functions with respect to k-symmetric points were introduced and studied by various authors (see [7, 8, 9, 10]). In this paper, new class of functions in $\mathcal{M}_p$ with respect to $(j,\, k)$-symmetric points is introduced.

For fixed parameters A, B and λ $(-1 \le B < A \le 1;\, 0 \le \lambda < p;\, p \in \mathbb{N})$, we say that a function $f(z) \in \mathcal{M}_p$ is in the class $\mathcal{S}^{p}_{j,k}(\alpha_1;\, A,\, B;\, \lambda)$ of meromorphically p-valent functions in $\mathcal{U}$, if it also satisfies the inequality:

$$\left| \frac{\frac{z(H^p(\alpha_1) f(z))'}{f^{p}_{j,k}(\alpha_1;\, z)} + p}{B\frac{z(H^p(\alpha_1) f(z))'}{f^{p}_{j,k}(\alpha_1;\, z)} + [pB + (A-B)(p-\lambda)]} \right| < 1, \quad (z \in \mathcal{U}). \tag{1.11}$$

If we let $j = k = 1$ in (1.11), the class $\mathcal{S}^{p}_{j,k}(\alpha_1;\, A,\, B;\, \lambda)$ reduces to the function class $\mathcal{Q}_{p,q,s}(\alpha_1;\, A,\, B;\, \lambda)$ introduced and studied by Aouf [1].

2. Integral Representation

Theorem 2.1. *If $f \in \mathcal{S}^{p}_{j,k}(\alpha_1;\, A,\, B;\, \lambda)$, then*

$$f_{j,k}(z) \in \mathcal{Q}_{p,q,s}(\alpha_1;\, A,\, B;\, \lambda).$$

Proof. Let $f \in \mathcal{S}^{p}_{j,k}(\alpha_1;\, A,\, B;\, \lambda)$, then equivalently the condition (1.11) can be rewritten as have

$$\frac{z\,(H^p(\alpha_1) f(z))'}{f^{p}_{j,k}(\alpha_1;\, z)} = -\frac{p + [pB + (A-B)(p-\lambda)]w(z)}{1 + Bw(z)} \tag{2.1}$$

where the function $w(z)$ is either analytic or meromorphic in $\mathcal{U}$, with $w(0) = 0$. If we replace z by $\varepsilon^{\nu} z$ in (2.1), then (2.1) will be of the form

$$\frac{\varepsilon^{\nu} z\,(H^p(\alpha_1) f(\varepsilon^{\nu} z))'}{f^{p}_{j,k}(\alpha_1;\, \varepsilon^{\nu} z)} = -\frac{p + [pB + (A-B)(p-\lambda)]w(\varepsilon^{\nu} z)}{1 + Bw(\varepsilon^{\nu} z)} \tag{2.2}$$

Using (1.8) in (2.2), we get

$$\frac{\varepsilon^{\nu+\nu pj} z\left(H^p(\alpha_1)f(\varepsilon^\nu z)\right)'}{f^p_{j,k}(\alpha_1;\, z)} = -\frac{p+[pB+(A-B)(p-\lambda)]w(\varepsilon^\nu z)}{1+Bw(\varepsilon^\nu z)} \tag{2.3}$$

Let $\nu = 0, 1, 2, \ldots, k-1$ in (2.3) respectively and summing them, we get

$$\frac{\sum_{\nu=0}^{k-1} \varepsilon^{\nu+\nu pj} z\left(H^p(\alpha_1)f(\varepsilon^\nu z)\right)'}{f^p_{j,k}(\alpha_1;\, z)} = -\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{p+[pB+(A-B)(p-\lambda)]w(\varepsilon^\nu z)}{1+Bw(\varepsilon^\nu z)}.$$

Or equivalently,

$$\frac{z\left(f^p_{j,k}(\alpha_1;\, z)\right)'}{f^p_{j,k}(\alpha_1;\, z)} = -\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{p+[pB+(A-B)(p-\lambda)]w(\varepsilon^\nu z)}{1+Bw(\varepsilon^\nu z)},$$

that is, $f_{j,k}(z) \in \mathcal{Q}_{p,q,s}(\alpha_1;\, A,\, B;\, \lambda)$. □

Theorem 2.2. *Let $f \in \mathcal{S}^p_{j,k}(\alpha_1;\, A,\, B;\, \lambda)$, then we have*

$$f^p_{j,k}(\alpha_1;\, z) = z\exp\left\{\int_0^{\varepsilon^\nu z}\left(-\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{p+[pB+(A-B)(p-\lambda)]w(t)}{t\,(1+Bw(t))}-\frac{1}{t}\right)dt\right\}, \tag{2.4}$$

where $f_{j\,k}(z)$ defined by equality (1.8), $w(z)$ is analytic in $\mathcal{U}$ and $w(0) = 0$, $\mid w(z) \mid < 1$.

Proof. Let $f \in \mathcal{S}^p_{j,k}(\alpha_1;\, A,\, B;\, \lambda)$. In view of (1.11), we have

$$\frac{z\left(H^p(\alpha_1)f(z)\right)'}{f^p_{j,k}(\alpha_1;\, z)} = -\frac{p+[pB+(A-B)(p-\lambda)]w(z)}{1+Bw(z)}, \tag{2.5}$$

where $w(z)$ is analytic in $\mathcal{U}$ and $w(0) = 0$, $\mid w(z) \mid < 1$. Substituting z by $\varepsilon^\nu z$ in the equality (2.5) respectively ($\nu = 0, 1, 2, \ldots, k-1$, $\varepsilon^k = 1$), we have

$$\frac{\varepsilon^\nu z\left(H^p(\alpha_1)f(\varepsilon^\nu z)\right)'}{f^p_{j,k}(\alpha_1;\, \varepsilon^\nu z)} = -\frac{p+[pB+(A-B)(p-\lambda)]w(\varepsilon^\nu z)}{1+Bw(\varepsilon^\nu z)}. \tag{2.6}$$

Using(1.8), equation (2.6) can be rewritten in the form

$$\frac{z\varepsilon^{\nu+\nu pj}\left(H^p(\alpha_1)f(\varepsilon^\nu z)\right)'}{f^p_{j,k}(\alpha_1;\, z)} = -\frac{p+[pB+(A-B)(p-\lambda)]w(\varepsilon^\nu z)}{1+Bw(\varepsilon^\nu z)}. \tag{2.7}$$

Let $\nu = 0,\, 1,\, 2,\, \ldots,\, k-1$ in (2.7) respectively and summing them we get,

$$\frac{z\left(f^p_{j,k}(\alpha_1;\, z)\right)'}{f^p_{j,k}(\alpha_1;\, z)} = -\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{p+[pB+(A-B)(p-\lambda)]w(\varepsilon^\nu z)}{1+Bw(\varepsilon^\nu z)}, \tag{2.8}$$

From the equality (2.8), we get

$$\frac{\left(f^p_{j,k}(\alpha_1;\, z)\right)'}{f^p_{j,k}(\alpha_1;\, z)} - \frac{1}{z} = -\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{p+[pB+(A-B)(p-\lambda)]w(\varepsilon^\nu z)}{z\,(1+Bw(\varepsilon^\nu z))} - \frac{1}{z}.$$

Integrating this equality, we get

$$\begin{aligned}\log&\left\{\frac{f^p_{j,k}(\alpha_1;\, z)}{z}\right\}\\ &= \int_0^z\left(-\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{p+[pB+(A-B)(p-\lambda)]w(\varepsilon^\nu\zeta)}{\zeta\,(1+Bw(\varepsilon^\nu\zeta))} - \frac{1}{\zeta}\right)d\zeta,\\ &= \int_0^{\varepsilon^\nu z}\left(-\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{p+[pB+(A-B)(p-\lambda)]w(t)}{t\,(1+Bw(t))} - \frac{1}{t}\right)dt,\end{aligned}$$

or equivalently,

$$\begin{aligned}&f^p_{j,k}(\alpha_1;\, z)\\ &= z\exp\left\{\int_0^{\varepsilon^\nu z}\left(-\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{p+[pB+(A-B)(p-\lambda)]w(t)}{t\,(1+Bw(t))} - \frac{1}{t}\right)dt\right\}.\end{aligned}$$

This completes the proof of Theorem 2.2. □

Theorem 2.3. *Let $f \in \mathcal{S}^p_{j,k}(\alpha_1;\, A,\, B;\, \lambda)$, then we have*

$$\begin{aligned}&H^p(\alpha_1)f(z)\\ &= \int_0^z \exp\left\{\int_0^{\varepsilon^\nu\zeta}\left(-\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{p+[pB+(A-B)(p-\lambda)]w(t)}{t\,(1+Bw(t))} - \frac{1}{t}\right)dt\right\}\\ &\qquad\cdot\left(-\frac{p+[pB+(A-B)(p-\lambda)]w(\zeta)}{1+Bw(\zeta)}\right)d\zeta\end{aligned} \tag{2.9}$$

where $w(z)$ is analytic in $\mathcal{U}$ and $w(0) = 0$, $\mid w(z)\mid < 1$.

3. Inclusion Properties of the Class $\mathcal{S}^p_{j,k}(\alpha_1;\ A,\ B;\ \lambda)$

We need the following Lemmas in the sequel.

Lemma 3.1. *[4]Let $\beta\,(\beta \neq 0)$ and γ be complex numbers and also let ϕ be convex and univalent in $\mathcal{U}$, with $\phi(0) = 1$ and $Re\,[\beta\phi(z) + \gamma] > 0$ $(z \in \mathcal{U})$. If $p(z)$ is analytic in $\mathcal{U}$ with $p(0) = \phi(0)$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \phi(z) \quad \Longrightarrow \quad p(z) \prec \phi(z), \quad (z \in \mathcal{U}).$$

Lemma 3.2. *[5]Let $\beta\,(\beta \neq 0)$ and γ be complex numbers and also let ϕ be convex univalent in $\mathcal{U}$, with $\phi(0) = 1$ and $Re\,[\beta\phi(z) + \gamma] > 0$. Also let $q(z) \prec \phi(z)$ If $p(z)$ is analytic in $\mathcal{U}$ with $p(0) = \phi(0)$, then*

$$p(z) + \frac{zp'(z)}{\beta q(z) + \gamma} \prec \phi(z) \quad \Longrightarrow \quad p(z) \prec \phi(z), \quad (z \in \mathcal{U}).$$

Theorem 3.3. *If*

$$\Re\left\{(\alpha_1 + p) - p\left(\frac{p + [pB + (A-B)(p-\lambda)]z}{1 + Bz}\right)\right\} > 0, \tag{3.1}$$

$$(-1 < B < A \leq 1;\ 0 \leq \lambda < p,\ p \in \mathbb{N}),$$

then $\quad \mathcal{S}^p_{j,k}(\alpha_1 + 1;\ A,\ B;\ \lambda) \subset \mathcal{S}^p_{j,k}(\alpha_1;\ A,\ B;\ \lambda)$.

Proof. Let $f \in \mathcal{S}^p_{j,k}(\alpha_1 + 1;\ A,\ B;\ \lambda)$ and set

$$h(z) = -\frac{z\,(H^p(\alpha_1)f(z))'}{p\,f^p_{j,k}(\alpha_1;\ z)}, \quad k(z) = -\frac{z\left(f^p_{j,k}(\alpha_1;\ z)\right)'}{p\,f^p_{j,k}(\alpha_1;\ z)}, \tag{3.2}$$

we observe that $h(z)$ and $k(z)$ are analytic is $\mathcal{U}$ with $h(0) = k(0) = 1$. Then by applying (1.5) in $h(z)$, we obtain

$$h(z)\,f^p_{j,k}(\alpha_1;\ z) = -\frac{\alpha_1}{p}\,H^p(\alpha_1 + 1)f(z) + \frac{(\alpha_1 + p)}{p}H^p(\alpha_1)f(z). \tag{3.3}$$

Differentiating both sides of equation (3.3) with respect to z,

$$zh'(z) + \left((\alpha_1 + p) + \frac{z\left(f^p_{j,k}(\alpha_1;\ z)\right)'}{f^p_{j,k}(\alpha_1;\ z)}\right) h(z)$$

$$= -\frac{\alpha_1}{p}\frac{z\left(H^p(\alpha_1+1)f(z)\right)'}{f^p_{j,k}(\alpha_1;\, z)}. \tag{3.4}$$

Using the relation between (1.5) and (1.10),

$$\begin{aligned} z(f^p_{j,k}(\alpha_1;\, z))' &+ (\alpha_1+p)f^p_{j,k}(\alpha_1;\, z) \\ &= \frac{\alpha_1}{k}\sum_{\nu=0}^{k-1}\varepsilon^{-\nu j p}\left(H^p(\alpha_1+1)f(\varepsilon^\nu z)\right) \\ &= \alpha_1\, f^p_{j,k}(\alpha_1+1;\, z). \end{aligned} \tag{3.5}$$

Using (3.5) in (3.4), we have

$$h(z) + zh'(z)\left[(\alpha_1+p) + \frac{z\left(f^p_{j,k}(\alpha_1;\, z)\right)'}{f^p_{j,k}(\alpha_1;\, z)}\right]^{-1}$$

$$-\frac{z\left(H^p(\alpha_1+1)f(z)\right)'}{p\, f^p_{j,k}(\alpha_1+1;\, z)}.$$

From the definition of $\mathcal{S}^p_{j,k}(\alpha_1+1;\, A,\, B;\, \lambda)$, we have

$$h(z) + \frac{zh'(z)}{\alpha_1+p-p\,k(z)} \prec \frac{p+[pB+(A-B)(p-\lambda)]z}{1+Bz}. \tag{3.6}$$

In view of the Lemma 3.2, the assertion of the Theorem would follow once we prove that $k(z) \prec \frac{p+[pB+(A-B)(p-\lambda)]z}{1+Bz}$ $(z\in\mathcal{U})$.

It follows from $k(z)$ and (3.5) that

$$(\alpha_1+p) - p\,k(z) = -\alpha_1\frac{f^p_{j,k}(\alpha_1+1;\, z)}{p\, f^p_{j,k}(\alpha_1;\, z)}. \tag{3.7}$$

By logarithmical differentiation of equation (3.7), we get

$$k(z) + \frac{zk'(z)}{(\alpha_1+p)-p\,k(z)} = -\frac{z\left(f^p_{j,k}(\alpha_1+1;\, z)\right)'}{p\, f^p_{j,k}(\alpha_1+1;\, z)}. \tag{3.8}$$

Using Theorem 2.1 in the equality (3.8), we have

$$k(z) + \frac{zk'(z)}{(\alpha_1+p)-p\,k(z)} \prec \frac{p+[pB+(A-B)(p-\lambda)]z}{1+Bz} \tag{3.9}$$

In view of (3.1) and (3.9), we deduce from Lemma 2.1 that

$$k(z) \prec \frac{p + [pB + (A - B)(p - \lambda)]z}{1 + Bz}.$$

This implies that

$$\mathcal{S}^{p}_{j,\,k}(\alpha_1 + 1;\, A,\, B;\, \lambda) \subset \mathcal{S}^{p}_{j,\,k}(\alpha_1;\, A,\, B;\, \lambda). \qquad \square$$

Next we prove an inclusion property associated with a certain integral transform.

Theorem 3.4. *Let $f \in \mathcal{M}_p$ and $F = L_c[f]$, where $L_c[f]$ is defined by*

$$L_c[f] = \frac{c - p}{z^c} \int_0^z t^{c-1} f(t)\, dt, \;\; (f \in \mathcal{M}_p). \tag{3.10}$$

If $f \in \mathcal{S}^{p}_{j,\,k}(\alpha_1;\, A,\, B;\, \lambda)$, then $F \in \mathcal{S}^{p}_{j,\,k}(\alpha_1;\, A,\, B;\, \lambda)$.

Proof. From definition of F and the linearity of the operator $H^p(\alpha_1)f$, we have

$$z(H^p(\alpha_1)F(z))' = (c - p)H^p(\alpha_1)f(z) - c\, H^p(\alpha_1)F(z). \tag{3.11}$$

From (3.11), we have

$$(c - p)f^{p}_{j,\,k}(\alpha_1;\, z) = cF^{p}_{j,\,k}(\alpha_1;\, z) + z\left(F^{p}_{j,\,k}(\alpha_1;\, z)\right)' \tag{3.12}$$

If we let

$$w(z) = -\frac{z\left(F^{p}_{j,\,k}(\alpha_1;\, z)\right)'}{p\, F^{p}_{j,\,k}(\alpha_1;\, z)},$$

then w is analytic in $\mathcal{U}$ and $p(0) = 1$. From (3.12), we observe that

$$c - p\, w(z) = (c - p)\frac{f^{p}_{j,\,k}(\alpha_1;\, z)}{F^{p}_{j,\,k}(\alpha_1;\, z)}. \tag{3.13}$$

Differentiating both sides of (3.13) with respect to z, we obtain

$$w(z) + \frac{zw'(z)}{c - p\, w(z)} = -\frac{z\left(f^{p}_{j,\,k}(\alpha_1;\, z)\right)'}{p\, f^{p}_{j,\,k}(\alpha_1;\, z)},$$

By Theorem 2.1, we have

$$w(z) + \frac{zw'(z)}{c - p\, w(z)} \prec \frac{p + [pB + (A - B)(p - \lambda)]z}{1 + Bz}$$

which on using Lemma 2.1 implies $w(z) \prec \frac{p+[pB+(A-B)(p-\lambda)]z}{1+Bz}$. Now suppose that

$$q(z) = -\frac{z\,(H^p(\alpha_1)F(z))'}{p\,F^p_{j,k}(\alpha_1;\, z)},$$

then $q(z)$ is analytic in $\mathcal{U}$, with $q(0) = 1$, and it follows from (3.11) that

$$F^p_{j,k}(\alpha_1;\, z)\,q(z) = -\frac{(c-p)}{p}H^p(\alpha_1)f(z) + \frac{c}{p}H^p(\alpha_1)F(z). \tag{3.14}$$

Differentiating both sides of (3.14), we get

$$zq'(z) + (c - pw(z))\,q(z) = -(c-p)\frac{z\,(H^p(\alpha_1)f(z))'}{p\,F^p_{j,k}(\alpha_1;\, z)}. \tag{3.15}$$

Now, from (3.13) and (3.15), we can deduce that

$$q(z) + \frac{zq'(z)}{c - p\,w(z)} = -\frac{z\,(H^p(\alpha_1)f(z))'}{p\,F^p_{j,k}(\alpha_1;\, z)}$$

$$\prec \frac{p + [pB + (A-B)(p-\lambda)]z}{1 + Bz}.$$

Hence an application of Lemma 3.2 yeilds $q(z) \prec \frac{p+[pB+(A-B)(p-\lambda)]z}{1+Bz}$, which shows that $F \in \mathcal{S}^p_{j,k}(\alpha_1;\, A,\, B;\, \lambda)$. □

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