CERTAIN CLASSES OF MEROMORPHIC MULTIVALENT FUNCTIONS WITH RESPECT TO \((j, k)\)-SYMMETRIC POINTS

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Abstract: The author introduces new class of \(p\)-valent meromorphic multivalent analytic function with respect \((j, k)\)-symmetric points. Integral representation and inclusion relations for functions in these classes are obtained.

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1. Introduction

Let \(\mathcal{M}_p\) denote the class of functions of the form

\[
f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p}, \quad (p \geq 1),
\]

which are analytic in the punctured open unit disk

\[U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \setminus \{0\}.
\]
For the functions $f(z)$ of the form (1.1) and

$$g(z) = z^{-p} + \sum_{n=1}^{\infty} b_n z^{-n-p},$$

the Hadamard product (or convolution) of $f$ and $g$ is defined by $(f \ast g)(z) = z^{-p} + \sum_{n=1}^{\infty} a_n b_n z^{-n-p}$.

For complex parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ ($\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- = 0, -1, -2, \ldots; j = 1, \ldots, s$), we define the generalized hypergeometric function $\genfrac{}{}{0pt}{}{q}{s} F_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by

$$\genfrac{}{}{0pt}{}{q}{s} F_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_q)_n z^n}{(\beta_1)_n \ldots (\beta_s)_n n!} \quad (q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathcal{U}),$$

where $(x)_k$ is the Pochhammer symbol defined, in terms of the Gamma function $\Gamma$, by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & \text{if } k = 0 \\ x(x+1)(x+2) \ldots (x+k-1) & \text{if } k \in \mathbb{N} = \{1, 2, \ldots\}. \end{cases}$$

Corresponding to a function $\mathcal{G}_{q,s}^p(\alpha_1, \beta_1; z)$ defined by

$$\mathcal{G}_{q,s}^p(\alpha_1, \beta_1; z) := z^{-p} \genfrac{}{}{0pt}{}{q}{s} F_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z),$$

we consider the operator $H^p(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s) f : \mathcal{U} \to \mathcal{U}$ which is defined by means of Hadamard product (or convolution):

$$H^p(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s) f(z) = \mathcal{G}_{q,s}^p(\alpha_1, \beta_1; z) \ast f(z).$$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$H^p(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s) f(z) = z^{-p} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_q)_n a_n z^{-n-p}}{(\beta_1)_n \ldots (\beta_s)_n n!}. \quad (1.4)$$

For convenience, we write

$$H^p(\alpha_1) = H^p(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s).$$
It can be easily seen from (1.3) that
\[ z(H^p(\alpha_1)f(z))' = \alpha_1 H^p(\alpha_1 + 1)f(z) - (\alpha_1 + p)H^p(\alpha_1)f(z). \] (1.5)

The linear operator \( H^p(\alpha_1)f(z) \) was introduced and investigated by Liu and Srivastava [3].

Let \( k \) be an arbitrarily fixed integer not smaller than 2, then the set \( G_k \) of all roots of \( k \)-th degree of unity has the form
\[ G_k = \{ \varepsilon^0, \varepsilon^1, \ldots, \varepsilon^{k-1} \} \] (1.6)
where \( \varepsilon = \exp(2\pi i/k) \).

A function \( f \in M_p \) is said to be \((j, k)\)-symmetrical if for each \( z \in \mathcal{U} \)
\[ f(\varepsilon z) = \varepsilon^{-pj} f(z), \] (1.7)

\( (k = 1, 2, \ldots; j = 0, 1, 2, \ldots (k - 1)). \)

The family of \((j, k)\)-symmetrical functions will be denoted by \( \mathcal{F}_k \). We observe that \( \mathcal{F}_2^1, \mathcal{F}_2^0 \) and \( \mathcal{F}_k^1 \) are well-known families of odd functions, even functions and \( k \)-symmetrical functions respectively.

Also let \( f_{j, k}(z) \) be defined by the following equality
\[ f_{j, k}(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} f(\varepsilon^\nu z) \varepsilon^{-\nu pj}, \] (1.8)

\((f \in M_p; k = 1, 2, \ldots; j = 0, 1, 2, \ldots (k - 1)). \)

It is obvious that \( f_{j, k}(z) \) is linear from \( \mathcal{U} \) into \( \mathcal{U} \). The notion of \((j, k)\)-symmetric functions was introduced and studied by P. Liczberski and J. Polubiński in [2].

If \( \nu \) is an integer, then the following identities follow directly from (1.8):
\[ f_{j, k}(\varepsilon^\nu z) = \varepsilon^{-\nu pj} f_{j, k}(z), \]
\[ f'_{j, k}(\varepsilon^\nu z) = \varepsilon^{-\nu pj-\nu} f'_{j, k}(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{\nu+\nu pj} f'(\varepsilon^\nu z), \] (1.9)
\[ f''_{j, k}(\varepsilon^\nu z) = \varepsilon^{-\nu pj-2\nu} f''_{j, k}(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{2\nu+\nu pj} f''(\varepsilon^\nu z). \]
Throughout this paper, we assume that $p, k \in \mathbb{N}, \varepsilon = \exp(2\pi i/k)$ and

$$f_{j,k}^p(\alpha_1; z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{\nu j} p \left( H^p(\alpha_1) f(\varepsilon^\nu z) \right) = z^{-p} + \cdots,$$  

(1.10)

($f \in \mathcal{M}_p; k = 1, 2, \ldots; j = 0, 1, 2, \ldots (k - 1)$).

Clearly, for $j = k = 1$, we have $f_{j,k}^p(\alpha_1; z) = H^p(\alpha_1) f(z)$.

Motivated by the concept introduced by K. Sakaguchi in [6], recently several subclasses of analytic functions with respect to $k$-symmetric points were introduced and studied by various authors (see [7, 8, 9, 10]). In this paper, new class of functions in $\mathcal{M}_p$ with respect to $(j, k)$-symmetric points is introduced.

For fixed parameters $A, B$ and $\lambda$ ($-1 \leq B < A \leq 1; 0 \leq \lambda < p; p \in \mathbb{N}$), we say that a function $f(z) \in \mathcal{M}_p$ is in the class $S_{j,k}^p(\alpha_1; A, B; \lambda)$ of meromorphically $p$-valent functions in $\mathcal{U}$, if it also satisfies the inequality:

$$\left| \frac{z(H^p(\alpha_1)f(z))'}{f_{j,k}^p(\alpha_1; z)} + p \right| < 1, \quad (z \in \mathcal{U}).$$  

(1.11)

If we let $j = k = 1$ in (1.11), the class $S_{j,k}^p(\alpha_1; A, B; \lambda)$ reduces to the function class $Q_{p,q,s}(\alpha_1; A, B; \lambda)$ introduced and studied by Aouf [1].

2. Integral Representation

**Theorem 2.1.** If $f \in S_{j,k}^p(\alpha_1; A, B; \lambda)$, then

$$f_{j,k}(z) \in Q_{p,q,s}(\alpha_1; A, B; \lambda).$$

**Proof.** Let $f \in S_{j,k}^p(\alpha_1; A, B; \lambda)$, then equivalently the condition (1.11) can be rewritten as have

$$\frac{z(H^p(\alpha_1)f(z))'}{f_{j,k}^p(\alpha_1; z)} = -p + [pB + (A - B)(p - \lambda)] w(z) \frac{1}{1 + Bw(z)}$$  

(2.1)

where the function $w(z)$ is either analytic or meromorphic in $\mathcal{U}$, with $w(0) = 0$. If we replace $z$ by $\varepsilon^\nu z$ in (2.1), then (2.1) will be of the form

$$\frac{\varepsilon^\nu z (H^p(\alpha_1)f(\varepsilon^\nu z))'}{f_{j,k}^p(\alpha_1; \varepsilon^\nu z)} = -p + [pB + (A - B)(p - \lambda)] w(\varepsilon^\nu z) \frac{1}{1 + Bw(\varepsilon^\nu z)}$$  

(2.2)
Using (1.8) in (2.2), we get
\[ \frac{\varepsilon^{\nu + \nu pj} z (H^p(\alpha_1) f(\varepsilon^\nu z))'}{f^p_{j,k}(\alpha_1; z)} = - p + [pB + (A - B)(p - \lambda)] w(\varepsilon^\nu z) \quad (2.3) \]

\[ \frac{\varepsilon^{\nu + \nu pj} z (H^p(\alpha_1) f(\varepsilon^\nu z))'}{f^p_{j,k}(\alpha_1; z)} = - p + [pB + (A - B)(p - \lambda)] w(\varepsilon^\nu z) \]

\[ \frac{\varepsilon^{\nu + \nu pj} z (H^p(\alpha_1) f(\varepsilon^\nu z))'}{f^p_{j,k}(\alpha_1; z)} = - p + [pB + (A - B)(p - \lambda)] w(\varepsilon^\nu z) \]

Let \( \nu = 0, 1, 2, \ldots, k - 1 \) in (2.3) respectively and summing them, we get
\[ \sum_{\nu=0}^{k-1} \varepsilon^{\nu + \nu pj} z (H^p(\alpha_1) f(\varepsilon^\nu z))' \]
\[ \frac{\varepsilon^{\nu + \nu pj} z (H^p(\alpha_1) f(\varepsilon^\nu z))'}{f^p_{j,k}(\alpha_1; z)} = - p + [pB + (A - B)(p - \lambda)] w(\varepsilon^\nu z) \]

Or equivalently,
\[ \frac{\varepsilon^{\nu + \nu pj} z (H^p(\alpha_1) f(\varepsilon^\nu z))'}{f^p_{j,k}(\alpha_1; z)} = - p + [pB + (A - B)(p - \lambda)] w(\varepsilon^\nu z) \]

That is, \( f_{j,k}(z) \in Q_{p,q,s}(\alpha_1; A, B; \lambda) \).

**Theorem 2.2.** Let \( f \in S^p_{j,k}(\alpha_1; A, B; \lambda) \), then we have
\[ f^p_{j,k}(\alpha_1; z) = z \exp \left\{ \int_0^{\varepsilon^\nu z} \left( - \frac{1}{k} \sum_{\nu=0}^{k-1} p + [pB + (A - B)(p - \lambda)] w(t) \right) \frac{1}{t} \right\} dt \]

where \( f_{j,k}(z) \) defined by equality (1.8), \( w(z) \) is analytic in \( U \) and \( w(0) = 0, | w(z) | < 1 \).

**Proof.** Let \( f \in S^p_{j,k}(\alpha_1; A, B; \lambda) \). In view of (1.11), we have
\[ \frac{z (H^p(\alpha_1) f(z))'}{f^p_{j,k}(\alpha_1; z)} = - p + [pB + (A - B)(p - \lambda)] w(z) \]

where \( w(z) \) is analytic in \( U \) and \( w(0) = 0, | w(z) | < 1 \). Substituting \( z \) by \( \varepsilon^\nu z \) in the equality (2.5) respectively \( (\nu = 0, 1, 2, \ldots, k - 1, \varepsilon^k = 1) \), we have
\[ \frac{\varepsilon^\nu z (H^p(\alpha_1) f(\varepsilon^\nu z))'}{f^p_{j,k}(\alpha_1; \varepsilon^\nu z)} = - p + [pB + (A - B)(p - \lambda)] w(\varepsilon^\nu z) \]

\[ \frac{\varepsilon^\nu z (H^p(\alpha_1) f(\varepsilon^\nu z))'}{f^p_{j,k}(\alpha_1; \varepsilon^\nu z)} = - p + [pB + (A - B)(p - \lambda)] w(\varepsilon^\nu z) \]

\[ \frac{\varepsilon^\nu z (H^p(\alpha_1) f(\varepsilon^\nu z))'}{f^p_{j,k}(\alpha_1; \varepsilon^\nu z)} = - p + [pB + (A - B)(p - \lambda)] w(\varepsilon^\nu z) \]
Using (1.8), equation (2.6) can be rewritten in the form
\[
\frac{z^{\nu_j + \nu_{pj}} (H^p(\alpha_1) f(\nu_j z))'}{\tilde{f}_{j,k}(\alpha_1; z)} = -\frac{p + [pB + (A - B)(p - \lambda)]w(\nu_j z)}{1 + Bw(\nu_j z)}.
\]
(2.7)

Let \(\nu = 0, 1, 2, \ldots, k - 1\) in (2.7) respectively and summing them we get,
\[
\frac{z \left( \tilde{f}_{j,k}(\alpha_1; z) \right)'}{\tilde{f}_{j,k}(\alpha_1; z)} = \frac{-1}{k} \sum_{\nu=0}^{k-1} p + [pB + (A - B)(p - \lambda)]w(\nu_j z)}{1 + Bw(\nu_j z)} - \frac{1}{z}.
\]
(2.8)

From the equality (2.8), we get
\[
\frac{\left( \tilde{f}_{j,k}(\alpha_1; z) \right)'}{\tilde{f}_{j,k}(\alpha_1; z)} - \frac{1}{z} = \frac{-1}{k} \sum_{\nu=0}^{k-1} p + [pB + (A - B)(p - \lambda)]w(\nu_j z)}{1 + Bw(\nu_j z))} - \frac{1}{z}.
\]

Integrating this equality, we get
\[
\log \left\{ \frac{\tilde{f}_{j,k}(\alpha_1; z)}{z} \right\} = \int_0^z \left( -\frac{1}{k} \sum_{\nu=0}^{k-1} p + [pB + (A - B)(p - \lambda)]w(\nu_j \zeta)}{\zeta (1 + Bw(\nu_j \zeta))} - \frac{1}{\zeta} \right) d\zeta,
\]
\[
= \int_0^z \left( -\frac{1}{k} \sum_{\nu=0}^{k-1} p + [pB + (A - B)(p - \lambda)]w(t)}{t (1 + Bw(t))} - \frac{1}{t} \right) dt,
\]
or equivalently,
\[
\tilde{f}_{j,k}(\alpha_1; z) = z \exp \left\{ \int_0^{\nu_j z} \left( -\frac{1}{k} \sum_{\nu=0}^{k-1} p + [pB + (A - B)(p - \lambda)]w(t)}{t (1 + Bw(t))} - \frac{1}{t} \right) dt \right\}.
\]

This completes the proof of Theorem 2.2.

**Theorem 2.3.** Let \(f \in S_{j,k}^p(\alpha_1; A, B; \lambda)\), then we have
\[
H^p(\alpha_1)f(z) = \int_0^z \exp \left\{ \int_0^{\nu_j \zeta} \left( -\frac{1}{k} \sum_{\nu=0}^{k-1} p + [pB + (A - B)(p - \lambda)]w(t)}{t (1 + Bw(t))} - \frac{1}{t} \right) dt \right\} \frac{p + [pB + (A - B)(p - \lambda)]w(\zeta)}{1 + Bw(\zeta)} d\zeta
\]
(2.9)

where \(w(z)\) is analytic in \(U\) and \(w(0) = 0, \ |w(z)| < 1\).
3. Inclusion Properties of the Class
\[ S^p_{j, k}(\alpha_1; A, B; \lambda) \]

We need the following Lemmas in the sequel.

**Lemma 3.1.** [4] Let \( \beta (\beta \neq 0) \) and \( \gamma \) be complex numbers and also let \( \phi \) be convex and univalent in \( U \), with \( \phi(0) = 1 \) and \( \text{Re}[\beta \phi(z) + \gamma] > 0 \) (\( z \in U \)). If \( p(z) \) is analytic in \( U \) with \( p(0) = \phi(0) \), then
\[
p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \phi(z), \quad (z \in U).
\]

**Lemma 3.2.** [5] Let \( \beta (\beta \neq 0) \) and \( \gamma \) be complex numbers and also let \( \phi \) be convex univalent in \( U \), with \( \phi(0) = 1 \) and \( \text{Re}[\beta \phi(z) + \gamma] > 0 \). Also let \( q(z) \prec \phi(z) \) if \( p(z) \) is analytic in \( U \) with \( p(0) = \phi(0) \), then
\[
p(z) + \frac{zp'(z)}{\beta q(z) + \gamma} \prec \phi(z) = \Rightarrow p(z) \prec \phi(z), \quad (z \in U).
\]

**Theorem 3.3.** If
\[
\Re \left\{ (\alpha_1 + p) - p \left( \frac{p + [pB + (A - B)(p - \lambda)]z}{1 + Bz} \right) \right\} > 0, \quad (3.1)
\]
\[
(-1 < B < A \leq 1; 0 \leq \lambda < p, p \in \mathbb{N}),
\]
then \( S^p_{j, k}(\alpha_1 + 1; A, B; \lambda) \subset S^p_{j, k}(\alpha_1; A, B; \lambda) \).

**Proof.** Let \( f \in S^p_{j, k}(\alpha_1 + 1; A, B; \lambda) \) and set
\[
h(z) = -\frac{z}{p} \left( H^p(\alpha_1; z) \right) \quad k(z) = -\frac{z}{p} \left( f^p_{j, k}(\alpha_1; z) \right),
\]
we observe that \( h(z) \) and \( k(z) \) are analytic in \( U \) with \( h(0) = k(0) = 1 \). Then by applying (1.5) in \( h(z) \), we obtain
\[
h(z) f^p_{j, k}(\alpha_1; z) = -\frac{\alpha_1}{p} H^p(\alpha_1 + 1)f(z) + \frac{\alpha_1 + p}{p} H^p(\alpha_1) f(z). \quad (3.3)
\]

Differentiating both sides of equation (3.3) with respect to \( z \),
\[
zh'(z) + \left( (\alpha_1 + p) + \frac{z}{f^p_{j, k}(\alpha_1; z)} \right) h(z)
\]
Using the relation between (1.5) and (1.10),

\[ z(f^p_{j,k}(\alpha_1; z))' + (\alpha_1 + p)f^p_{j,k}(\alpha_1; z) \]

\[ = \frac{\alpha_1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu j} p \left( H^p(\alpha_1 + 1) f(\varepsilon^\nu z) \right) \]

\[ = \alpha_1 f^p_{j,k}(\alpha_1 + 1; z). \]

Using (3.5) in (3.4), we have

\[ h(z) + z h'(z) \left( \alpha_1 + p \right) + \frac{z}{p f^p_{j,k}(\alpha_1; z)} \left[ z \left( f^p_{j,k}(\alpha_1; z) \right)' \right]^{-1} \]

\[ - \frac{z(H^p(\alpha_1 + 1)f(z))'}{p f^p_{j,k}(\alpha_1 + 1; z)}. \]

From the definition of \( S^p_{j,k}(\alpha_1 + 1; A, B; \lambda) \), we have

\[ h(z) + \frac{z h'(z)}{\alpha_1 + p - p k(z)} \prec \frac{p + [pB + (A - B)(p - \lambda)]z}{1 + Bz}. \]

In view of the Lemma 3.2, the assertion of the Theorem would follow once we prove that \( k(z) \prec \frac{p + [pB + (A - B)(p - \lambda)]z}{1 + Bz} \) \( (z \in \mathcal{U}) \).

It follows from \( k(z) \) and (3.5) that

\[ (\alpha_1 + p) - p k(z) = -\alpha_1 \frac{f^p_{j,k}(\alpha_1 + 1; z)}{p f^p_{j,k}(\alpha_1; z)}. \]

By logarithmical differentiation of equation (3.7), we get

\[ k(z) + \frac{z k'(z)}{(\alpha_1 + p) - p k(z)} = -\frac{z \left( f^p_{j,k}(\alpha_1 + 1; z) \right)'}{p f^p_{j,k}(\alpha_1 + 1; z)}. \]

Using Theorem 2.1 in the equality (3.8), we have

\[ k(z) + \frac{z k'(z)}{(\alpha_1 + p) - p k(z)} \prec \frac{p + [pB + (A - B)(p - \lambda)]z}{1 + Bz} \]
In view of (3.1) and (3.9), we deduce from Lemma 2.1 that
\[
k(z) \prec \frac{p + [pB + (A - B)(p - \lambda)]}{1 + Bz}z.
\]
This implies that
\[
S_{j,k}^p(\alpha_1 + 1; A, B; \lambda) \subset S_{j,k}^p(\alpha_1; A, B; \lambda).
\]

Next we prove an inclusion property associated with a certain integral transform.

**Theorem 3.4.** Let \( f \in \mathcal{M}_p \) and \( F = L_c[f] \), where \( L_c[f] \) is defined by
\[
L_c[f] = \frac{c - p}{z^c} \int_0^z t^{c-1} f(t) \, dt, \quad (f \in \mathcal{M}_p).
\]
\((3.10)\)

If \( f \in S_{j,k}^p(\alpha_1; A, B; \lambda) \), then \( F \in S_{j,k}^p(\alpha_1; A, B; \lambda) \).

**Proof.** From definition of \( F \) and the linearity of the operator \( H^p(\alpha_1)f \), we have
\[
z(H^p(\alpha_1)F(z))' = (c - p)H^p(\alpha_1)f(z) - cH^p(\alpha_1)F(z).
\]
\((3.11)\)

From (3.11), we have
\[
(c - p)f^p_{j,k}(\alpha_1; z) = cF^p_{j,k}(\alpha_1; z) + z \left( F^p_{j,k}(\alpha_1; z) \right)'.
\]
\((3.12)\)

If we let
\[
w(z) = \frac{-z \left( F^p_{j,k}(\alpha_1; z) \right)'}{p f^p_{j,k}(\alpha_1; z)},
\]
then \( w \) is analytic in \( \mathcal{U} \) and \( p(0) = 1 \). From (3.12), we observe that
\[
c - p w(z) = (c - p) \frac{f^p_{j,k}(\alpha_1; z)}{F^p_{j,k}(\alpha_1; z)}.
\]
\((3.13)\)

Differentiating both sides of (3.13) with respect to \( z \), we obtain
\[
w(z) + \frac{zw'(z)}{c - p w(z)} = -\frac{z \left( f^p_{j,k}(\alpha_1; z) \right)'}{p f^p_{j,k}(\alpha_1; z)},
\]
By Theorem 2.1, we have
\[
w(z) + \frac{zw'(z)}{c - p w(z)} \prec \frac{p + [pB + (A - B)(p - \lambda)]}{1 + Bz}z
\]
which on using Lemma 2.1 implies \( w(z) \prec \frac{p+|pB+(A-B)(p-\lambda)|z}{1+Bz} \). Now suppose that

\[
q(z) = -\frac{z (H^p(\alpha_1)F(z))'}{p F^p_{j,k}(\alpha_1; z)},
\]

then \( q(z) \) is analytic in \( U \), with \( q(0) = 1 \), and it follows from (3.11) that

\[
F^p_{j,k}(\alpha_1; z) q(z) = -\frac{(c-p)}{p} H^p(\alpha_1)f(z) + \frac{c}{p} H^p(\alpha_1)F(z). \quad (3.14)
\]

Differentiating both sides of (3.14), we get

\[
zq'(z) + (c-pw(z)) q(z) = -(c-p) \frac{z (H^p(\alpha_1)f(z))'}{p F^p_{j,k}(\alpha_1; z)}. \quad (3.15)
\]

Now, from (3.13) and (3.15), we can deduce that

\[
q(z) + \frac{zq'(z)}{c-p w(z)} = -\frac{z (H^p(\alpha_1)f(z))'}{p F^p_{j,k}(\alpha_1; z)} \prec \frac{p+|pB+(A-B)(p-\lambda)|z}{1+Bz}.
\]

Hence an application of Lemma 3.2 yeilds \( q(z) \prec \frac{p+|pB+(A-B)(p-\lambda)|z}{1+Bz} \), which shows that \( F \in S^p_{j,k}(\alpha_1; A, B; \lambda) \).

\[\square\]

References


